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Trace Anomaly Redefined in a Convention for Pontryagin Equivalent to a Generalized Wick

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Abstract

The lack of Unitarity is sought after, and is first resolved by an extraction from a unite-scale diffeomorphic transformation. The same result can second be found independently and is based on an orbital-wise Pfaffian differential satisfying a Conformal geodesic.

Such a fundamental reason is borne out in the multiple methods for the Pontryagin Chiral Fermions density anomalies resolutions being either zero or imaginary results, were, then fore, contradictory or randomly correct outcomes due to the eigenvalue non-separable sorting.

Confirming then an equivalent (1st as necessary and 2nd as sufficient) condition for Unitarity is via a regularization for the zero component of the Dirac Matrix γ^0, and a generalization of the Wick rotation, whilst both above hypotheses (may directly) be contouring around the Einstein Gravity.

§1- Introduction

The existence of an imposition for Unitarity and the separation of the eigenvalues were the crucial requirements for the proofs concerning the Gravitational Trace Anomaly Chiral Pontryagin term calculation. Neither is seen problematic if a sort of a regularization of γ^0 is involved under a unit module diffeomorphism, which confirms through a generalized Wick rotation as derived from a Pfaffian orbital under Conformal metric.

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That necessitated the need for both the “volume” and the “surface” terms, that both can be explicit functions of time, to be resolved and also sufficiently to get down by one more dimension.
Since any external line does not have anomaly (as easily countered by direct duality), there remains the non-trivial (containing the integration weighting metric) task of internal lines propagators, which are mere description of an internal wave propagation.
By an effective and local Hamiltonian, under the Fourier Transform, they may be acting under interferences and more precisely coherence should be well-defined such that the Hilbert space needs so a justification of existence and independence, under these arbitrary waves.
That either at the periphery of time, as an infinity, whose duality matching is proven as necessity be chosen at the surface terms. Or at the decaying mixed structures which, since sufficiently existent as chiral under a combined flavor freedom, are (Wick) decaying at various rotated space-time orbitals.
One has to note that even though the Einstein-Hilbert Gravity could be implicitly embedded in the metric of the generalized Weyl term, all the derivation down did not use in any way such a Gravity, except in the case when the proved Wick operation occurred in the time-transverse isotropy.
However, it was able always to provide either a diffeomorphism that assures, if imposed, Unitarity, or a Wick rotated decay (from a form of a Bianchi identity) whose Hermiticity processes that entity into being physical then unitary.
As a conclusion Quantum non-anomalous Gravity can be warrantied if followed the cited conditions.

§2- From a Missing Unitary Mapping into a Conventional Re-definition

1- A Reminder

Recalling the conventional definition of the trace anomaly, via functional derivatives of the energy-momentum tensor in a field with respect to the metric, being such

$T_{\mu\nu}(x) = 2/\sqrt{|g|}\,\frac{dS}{dg^{\mu\nu}}$.

That would follow after the introduction of the conformal transformation $g_{\mu\nu} \to e^{2\sigma(x)} g_{\mu\nu}$.

For its inverse, with an infinitesimal value of the parameter $\sigma(x)$: $g^{\mu\nu} \to [1 - 2\sigma(x)] g^{\mu\nu}$,

$$\delta S = \frac{1}{2} \int d^4x \sqrt{|g|} T_{\mu\nu} \delta g^{\mu\nu} = - \int d^4x \sqrt{|g|} \sigma(x) T^{\mu}_{\mu} \qquad (1)$$

Then, and for an arbitrary $\sigma(x)$, the invariance of S under the above conformal transformation requires that the trace of the energy-momentum tensor has to be $T^{\mu}_{\mu} = 0$.

This so far classical traceless identity is broken by quantum effects beyond tree and on shell levels, such

$$0 \neq \langle T^{\mu}_{\mu} \rangle \equiv A \qquad (1)'$$

Where the defined quantity A is called the trace or conformal anomaly, [1].

On dimensional grounds and in four-dimension, the most general form for the trace anomaly was found to be, [2],

$$\langle T^{\mu}_{\mu} \rangle = aG + bR^2 + b' \Box R + cF + e\epsilon^{\alpha\beta\gamma\delta} R_{\alpha\beta\mu\nu} R^{\mu\nu}_{\gamma\delta} \qquad (2)$$

Where the Gauss-Bonnet term $G = R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} + 4R^{\alpha\beta} R_{\alpha\beta} + R^2$ yields the Euler invariant, and $F = R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} - 2R^{\alpha\beta} R_{\alpha\beta} + \frac{1}{3} R^2$ is the square of the Weyl tensor.

The last term is the Parity-odd Pontryagin density, [3].

In the case of chiral fermions being added to the system, due to its Parity-odd symmetry properties, and at that lowest composition level, only the Pontryagin term contributes to the anomaly that is then why it is labeled as of type I.

2- A Non-Anomalous Diffeomorphic Action Leading to Unitarity

It is proposed here a resolution to that anomaly that however appears to encounter a dilemma which originates as is shown below from the degeneracy of its eigenvalues in the decomposition.

One is then looking for the symmetries which contain the above variation that caused such an anomaly. And as the considered action is gravitational, one has to distinguish between flat and curved space-time metrics, [4].

For Flat space-time, with a scaling $x \to x^c = e^{\sigma} x$, so then for an arbitrarily chosen wave function $\phi(x) \to \phi^c(x) = e^{\sigma\Delta}\phi(x)$, with Δ being a canonical mass dimension brought by

Conformal $\sqsupset$ Scale + Rotations + Boosts +Translations

That gets $\sigma(x)$ as a function of the quadri-norm of the vector x.

For Curved Space-time.

Conformal: Same as above when reduced infinitesimally into flat space-time which under the existence of the Killing invariants.

Weyl : Besides the functional transformation, there should be a metric transformation such that $g_{\mu\nu}^c(x) = e^{2\sigma} g_{\mu\nu}(x)$

Note here that the above diffeormorphism $x \to x^c$ map will not cause any anomaly since it remains at the classical level, and that is easily seen from its exact Lorentzian pairing up with the metric. But at the quantum level, it may be picked up some phases which may be too eliminated by remarking that such phases can be paired up as opposite under degenerate eigenvalues.

However now, acting in plus by the derivative on both the left and the right sides makes it in need of regularization since these derivatives have to pass through the chiral composition which is known to be made from effective operators so the need for regularizations starting from $g_{\mu\nu}(x) \equiv \frac{\partial y_\mu}{\partial x^\nu}$ and ending with $T_{\mu\nu}(x) = 2/\sqrt{|g|}\frac{\delta S}{\delta g^{\mu\nu}}$.

Due that the result is

$$0 \neq \langle T_\mu^\mu \rangle = \langle g^{\mu\nu} T_{\mu\nu} \rangle \neq g^{\mu\nu} \langle T_{\mu\nu} \rangle \qquad (2)'$$

It has to be eliminated the common eigenvalues as since are due to diffeomorphism.

That can be done either by subtracting, [5, 6], following what was adopted by [2], as

$g^{\mu\nu}\langle T_{\mu\nu}\rangle - \langle g^{\mu\nu}T_{\mu\nu}\rangle \equiv A_{reg}$.

Or by 'diagonalizing' in a fully non-degenerate space.

Which is simpler, and clarifies its subtilities. So that what is done and for that purpose the degeneracy here is unconventionally non-trivial.

3- The Regularization of γ^0

a- Saving Chirality and Unitarity

The degeneracy in the metric space is treated in next, see also Appendix I.

It is based on finding the binding condition for any metric to be unitary and non-degenerate. That would be on the 4-norm of a rescaling of g.

What, is such needed, is a Jacobian for a common scaling variation to both of x and y in

$$g_{\mu\nu} = \frac{\partial y_\mu}{\partial x^\nu} \to f(y_\mu) f^{-1}(x^\nu) \frac{\partial y_\mu}{\partial x^\nu} = f^2 g_{ij} + f^2 (g_{0j} - g_{i0}) - f^2 g_{00} \qquad (3a)$$

So a metric with $F \equiv f^2$, while being fully symmetric can be deduced for two sides scaling as

$$ds'^2 = Fdt^2 + Fdx^2 = Fds^2 \qquad (3b)$$

Now, the above form of the metric is encountered as a solution for the orbital variation along a metric in a Gauss-Bonnet Gravity, [7].

Then, in the case the eigenvalues are searched, such a configuration leads merely to an operator's acting change along a modular form orbifold (or in its simplest form a torus).

Since the norm of f is less or equal to one that keeps overall infinitely acting operators convergent.

Plus, since in the above map only the zero components get opposed by sign, so there is a temporal twist (or a negative spatial twist). To break that degeneracy (of the metric), it is sufficient to vary, here a rescaling on the one side-coordinate supposed to be the zero one.

After absorbing F then being rescaled by a negative $-\lambda$ as

$$Fds^2 \to -\lambda dt'^2 + dx^{2^2} \to [(-\lambda - 1) + 1]dt'^2 + dx^2 \to \left(-1 + \frac{1}{\lambda+1}\right) dt'^2 + \frac{1}{\lambda+1} dx^2$$

As $|F| \leq 1$ was arbitrary, $\left|\frac{1}{\lambda+1}\right| \leq 1$ can be re-identified with F, so one got a dispersive map as

$$ds'^2 = Fds^2 + Fdt^2 \rightarrow \; ds"^2 = -(1-F)dt^2 + Fdx^2 \qquad \text{(3b)'}$$

So, as was already used above as a property of elimination for the redundant eigenvalues, but with the supposed broken space, one has then to use the Jacobian not of f, but of

$$\Delta = \begin{vmatrix} [\varepsilon_t(f^2-1)]^{\frac{1}{2}} & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & f \end{vmatrix} \neq 0 \qquad (4)$$

The coefficient $\varepsilon_t = \pm$ is considered to express in the case of correlations between two space-time structures, existent in close neighborhoods the need of any possible extra time twists.

As proved in the Appendix I the case $\Delta = 1$ is the case of Unitary operators however with $\varepsilon_t = -1$.

And an operator $\mathcal{T}^{3-\mathrm{space}}(\mathrm{F}, \mathrm{F}^*, g_{ii}) \mapsto \frac{1\pm i\sqrt{3}}{2}\frac{g_{ii}}{\sqrt{3}}$.

If the problem is considered for the 4-spinors of Dirac, the action is given

$$S = \int \sqrt{g}\bar{\Psi} i \overset{/}{\overleftrightarrow{D}} \Psi \Rightarrow W = -ilog \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi \, e^{iS} = -ilogdet\sqrt{g} i \overset{/}{\overleftrightarrow{D}} \qquad \text{(5a)}$$

Standard diagonalization in compact manifolds or any unitarily equivalent manifold

$$\sqrt{g} i \overset{/}{\overleftrightarrow{D}} \Psi_n = \lambda_n \Psi_n \Rightarrow det\sqrt{g} i \overset{/}{\overleftrightarrow{D}} = \prod_n \lambda_n \qquad \text{(5a)'}$$

The use of the operator $\mathrm{T}^{3-\mathrm{space}}$ will have the impact of pairing the eigenvalues between F and space-wise inversion F^*.

So if $\Psi \rightarrow \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix}$. The 2nd space is that of the right chirality however with opposite eigenvalues.

Then, by Inverting the Time, the Parity Becomes Odd (5b)

Therefore, the procedure conserves the Dirac character for the spinors and it is specifically a plain regularization for its zero index Gamma matrix.

That logic can be confirmed by another means using the Atiyah-Singer index theorem for the Polyakov strings under the Liouville action, [11,17]. More theoretical justifications and also their direct implications are developed in the next subparagraph.

One can proceed into any of the usual regularizations and whose original result was advent by, [8]. That refutes the claim of Ref. [4], that the regularization of the Dirac fermions has no P-odd terms. As since the problem in their case originated from the fact that W was ill defined in $(1/2, 0)$ spin space as it goes to $(0,1/2)$, and it was sufficient to regularize Ψ_L and $W = -i \log \det i\sigma. D$ to lead into δW^{Weyl}.

Also, the imposition of the Unitarity as an external condition is not necessary as claimed by [5], what is necessary, however, is providing the conditions of its diffeomorphic variations in such way to verify the above entities, or sufficiently looking for a down completed generalized Wick rotation.

b- γ^0 Regularization

Then, as due to that in the Weyl representation $\overline{\Psi}$ contains $\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, the above diffeomorphic rescaling is merely a regularization of γ^0 in Weyl spaces, as can be verified.

In fact, when changing the representation from the Weyl to the Dirac-Pauli one, γ^0 changes into $\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ which has spontaneously the minus sign for the adjoint spinors.

However, any dimensional operator including the non-chiral Dirac-Pauli in the Lagrangian has its space as doubled as well its added Hermitian conjugate, so the definite regularizing diffeomorphism will have being acted by twice, then only and only one minus sign remains.

Note about the regularization in non-gravity environment as it uses the Ward identity, [12], that instead involves functional integral forms with matrix elements so can be handled also as path integrals. The matrices can act on vertices and propagators.

The vertices if spinorial contain their spin polarizations which are acted on then trivially.

The propagators are non-trivial when neither Lorentzian nor Euclidean; a way which was suggested in the Wicked path integral, [13], as they are

$$\Delta(P) = \frac{i}{E^2 - p^2 - m^2 + i\epsilon} \cong \frac{i}{E^2(1+i\epsilon) - p^2 - m^2}$$

With the not quite Minkowskian metric is $\eta_\epsilon = (1 + i\epsilon, 1,1,1)$, whose determinant is $\sqrt{-det\eta_\epsilon} = \sqrt{1 + i\epsilon}$. That becomes for, $\epsilon = 0$, $\eta_\epsilon \to \eta_L$ as Lorentzian and $\epsilon = 2i$, $\eta_\epsilon \to \eta_E$ as Euclidean.

This method dealing with operators defined on metrics with signatures somewhere between η_E and η_L, can be well extended after a trivial rescaling into either η_E or η_L. If not by the generalized Wick rotation described below which turned out as a regularization by the unitary operator $\mathcal{T}$, too.

c- Theoretical Justifications and Consequences

To fully justify the above regularization, one has to find if the global domain of definition for the driven from Dirac spinors remains well-behaved.

What is concerned here and are seen from the whole spectrum:

Does the Hilbert space remain well defined.

That is true under the developed down conditions and in the Appendix I.

Plus is there a change in the duality property, or what remains self-dual and what defies it.

Here, the trick applies as low in dimension as the 1+1 string models which was found to be self dual with the permutation operator $\epsilon^{i_1 \cdots i_n}$ is becoming complex, [14], which under the canonical gauging, [15], claimed to be associated with the Color charge remains self-dual, except however not under space doubling since that leads to the pop of the Real Orthogonal Flavor group indices, [16].

What is noticed here is the similarity in the complex-real transition between their breaking self-duality and ours in breaking the corresponding determinants. Supposedly then from unitary into composite-symmetric or -unitary operators, famously eliminating triangular flavor anomalies. More clearly while remaining in the complex structure, that would be, and since the base space of start is finite and that has led to the self-duality, then (as that is the statement negation) that would be for its operators being made as adjoint and in an infinite base space as also noticed in the cited references, so any iteration would be redundant and odd then the outcome set tends to be fractal as mentioned in the Appendix I.

§3. Solving a Classical Chiral Orbit Moving up to the Surface

1- Working what-ever Connection with a Pfaffian Differential Element

An orbital variation under e.g. a fixed 3+1 space-time (in fact, that is a lift along the 3rd or z-direction, is supposed to be a moving away axis from a projected (x-y) - plane), is as

$$\theta^{k} \equiv \sum_{k'} e^{kk'} dx_{k'} = \sum_{k'} \left(\eta^{kk'} + h^{kk'}\right) dx_{k'} \qquad (6)$$

Where the tensor $\eta^{kk'}$ $k = 0,1,2,3$ is the usual Minkowskian 4-metric, while h is supposed to be a variation (most probably remaining small) from η.

Despite that θ^{k} is the same as the Vielbein (a generalization of the 4-dimensional Vierbein to an arbitrary dimension, here it could be very well three), to which associates a connection such $\omega_{k}{}^{k'} = \omega_{k_l}{}^{k'} dx^{l}$ (which we choose the notation for dimensional generality), the following resolution goes without the need of the latest except at verifying end conditions.

The condition for θ to be integrable requires in its definition, if the surface term being localized at the end region of x_3 is yet to be folded, as since it is a matrix form made from two-space vectors or some constraint tensor, so following the form's equations

$$d\theta^{k} = \sum_{k'',l} \frac{\partial e^{kk''}}{\partial x_l} dx_{k''} \wedge dx_l = \sum_{k'',l} \frac{\partial h^{kk''}}{\partial x_l} dx_{k''} \wedge dx_l$$

$$\Longrightarrow 0 = \theta^{k} \wedge d\theta^{k} = \sum_{k',k'',l} \left(\eta^{kk'} + h^{kk'}\right) \frac{\partial h^{kk''}}{\partial x_l} dx_{k'} \wedge dx_{k''} \wedge dx_l \qquad (6)'$$

One remarks that the above equation remains true even without the k' and k'' summations, so

$$\overset{\sum_k}{\Longrightarrow} \frac{\partial}{\partial x_l}\left(h^{k'k''} + \frac{1}{2}\sum_k h^{kk'} h^{kk''}\right) dx_{k'} \wedge dx_{k''} \wedge dx_l \longmapsto d\,\frac{H_l^{k'k''}}{2} \wedge dx_{k'} \wedge dx_{k''} = 0 \qquad (6)''$$

Which leads to the definition of $H_l^{k'k''}$, but restricted to two variables considered as the surface terms, so that in plus it satisfies

$$H_l^{k'k''} \equiv 2h^{k'k''} + \sum_k h^{kk'} h^{kk''} = c_l x_l + c_{l'} x_{l'} \qquad l, l' \text{ are different from } k', k'' \qquad (7)$$

This equation's solution has the look of an orbit including the remaining variable is then confirming the surface term under correlations, required in the App. II-a.

Therefore, the metric defines as

$$g^{kk'} = e^{kk''}e^{k''k'} = \left(\eta^{kk'} + h^{kk''}\eta^{k''k'} + \eta^{kk'}h^{k''k'} + h^{kk''}h^{k''k'}\right)$$

$$= \eta^{kk'} + h^{kk'} + h^{k'k} + h^{kk''}h^{k''k'} = \eta^{kk'} + H^{kk'}$$

And vice versa since the vierbein is invertible that leads also it being a square-root of the metric, which is clearly sorted out under the (co-)Homological Fiber Bundle diagrams description, [17].

In the product inside $\theta \wedge d\theta$, due completeness it can be assumed that the tensor h is either symmetric or anti-symmetric.

Anti-symmetric h with $h^{kk'} + h^{k'k} = 0$ will make the metric a second order in $|h| \ll 1$ can be rendered however diagonal, but being parts of the orbits the particles will still pop up after an integration is manifested, however shifted non-commutatively by parallel transports despite being absorbed in the curvature.

The case of symmetric with

$$\frac{\partial h^{k'k''}}{\partial x_l} dx_{k'} \wedge dx_{k''} = 0$$

Has a resolvable equation at the boundary similar to (7), then

$$\sum_K \epsilon_{k'k''l} \frac{\partial\left(h^{k'k}h^{kk''}\right)}{\partial x_l} = 0 \;\; \Longrightarrow V_r^{k'}.\left(\nabla \wedge V_l^{k''}\right) = 0 \Leftrightarrow \nabla.\left(V_r^{k'} \wedge V_l^{k''}\right) = 0 \Longrightarrow \text{Two solutions:}$$

$$V_l^{k''} \parallel \text{and } \neq V_r^{k'} \text{ for } k' \neq k'' \qquad (7\text{-a})$$

Where each of the three vectors is such that if it has its row and line associated according to

$$\left.\overrightarrow{V_r^{k'}}\right)_k = h^{k'k} \text{ Or } \left.\overrightarrow{V_l^{k''}}\right)_k = h^{kk''}$$

Then, under the duality needed to eliminate anomalies, described above, a necessary and sufficient condition is to have instead of the inclusive Or is to have a Decisive And.

That is, a minimal solution is

$$k' = k \text{ And } V_l^{k''} = 0, \text{ no off-diagonal elements } h^{k'k''} = 0 \qquad (7\text{-a})'$$

The solution, with $h^{k'k} \equiv h_1^{k'k}(x)$, since $h_2^{k_2'k_2}(y)\delta y_{k_2'}\delta y_{k_2} = h_1^{k'k}(x)\delta x_{k_1'}\delta x_{k_1}$, as $\frac{\delta x_{k_1'}}{\delta x_{k_1}}$ becomes different from $\frac{\delta y_{k_2'}}{\delta y_{k_2}}$ so, is a distortion for the motion.

More, using what was obtained, [7 1st], as such

$$R^{\mu\nu} = \frac{1}{2}\left(\Box H^{\mu\nu} - \partial^\mu \partial^{\mu'} H^{\mu'\nu} - \partial^\nu \partial^{\mu'} H^{\mu'\mu} + \partial^\mu \partial^\nu H^{\rho\rho}\right)$$

$$= \frac{1}{2}(\Box H^{\mu\nu} - \partial^\mu \partial^\nu H^{\nu\nu}) + \frac{1}{2}(\partial^\mu \partial^\nu H^{\rho\rho} - \partial^\nu \partial^\mu H^{\mu\mu})$$

Where the repetitive index sum is valid for non-triple ones only, plus of noting the addition of two Ricci tensors $R_1^{\mu\nu} \equiv \frac{1}{2}(\Box H^{\mu\nu} - \partial^\mu \partial^\nu H^{\nu\nu})$ and $R_2^{\mu\nu} \equiv \frac{1}{2}(\partial^\mu \partial^\nu H^{\rho\rho} - \partial^\nu \partial^\mu H^{\mu\mu})$

From the freedom over the surface orbits, the metric variation is picked such whose off-diagonal elements are $\mu \neq \nu$; $H^{\mu\nu} = 0 \Rightarrow g^{\mu\nu} = 0$.

However, $R^{\mu\nu}$ is still be getting off-diagonal elements as

$$\mu \neq \nu \rightarrow R^{\mu\nu} = \frac{1}{2}\partial^\mu \partial^\nu\left(\sum_\rho H^{\rho\rho} - H^{\nu\nu} - H^{\mu\mu}\right) = \frac{1}{2}\partial^\mu \partial^\nu \left(\sum_{\substack{\rho\neq\mu \\ \neq\nu}} H^{\rho\rho}\right)$$

$$\mu = \nu \rightarrow R^{\mu\mu} = R_1^{\mu\mu} + R_2^{\mu\mu}$$

$$R_1^{00} = -\frac{1}{2}\partial^i \partial^i H^{00} \qquad \text{And} \qquad R_1^{ii} = \frac{1}{2}\left(\partial^0 \partial^0 - \sum_{j\neq i} \partial^j \partial^j - 2\,\partial^i \partial^i\right)H^{ii}$$

$$R_2^{00} = \frac{1}{2}\partial^0 \partial^0 \sum_j H^{jj} \quad \text{And } R_2^{ii} = \frac{1}{2}\partial^i \partial^i (H^{00} - \sum_{j\neq i} H^{jj}) \tag{7-b}$$

So, it can be written to leading order in H, as $g_{\mu\nu} \sim \eta_{\mu\nu}$ in the presence of$\times$ $R^{\mu\nu}$

$$R = g_{\mu\nu}\left(R_1^{\mu\nu} + R_2^{\mu\nu}\right) = g_{00}R_1^{00} + g_{ii}R_1^{ii} + g_{00}R_2^{00} + g_{ii}R_2^{ii}$$

$$\left(g_{00}R_1^{00} + g_{ii}R_1^{ii}\right) = R_1^{00} - R_1^{ii} \qquad \overset{H\to 0}{=} \frac{1}{2}\left(-\partial^i \partial^i H^{00}\right) - \frac{1}{2}\left[\partial^0 \partial^0 H^{ii} - \partial^j \partial^j H^{ii}\right]$$

$$\left(g_{00}R_2^{00} + g_{ii}R_2^{ii}\right) = R_2^{00} - R_2^{ii} \qquad \overset{H\to 0}{=} \frac{1}{2}\partial^0 \partial^0 \sum_j H^{jj} - \frac{1}{2}\left[\partial^i \partial^i\left(H^{00} - \sum_{j\neq i} H^{jj}\right)\right] \tag{7-b)'$$

$$\Longrightarrow R^2 = R_1^2 + R_2^2 + 2R_1R_2 =$$

$$\left(R_1^{00} - R_1^{ii}\right)^2 + \left(R_2^{00} - R_2^{ii}\right)^2 + 2(R_1^{00} - R_1^{ii})(R_2^{00} - R_2^{i'i'}) \tag{7-c}$$

$$R_{\mu\nu}R^{\mu\nu} = R_{100}\, R_1^{00} + R_{1ii}R_1^{ii} + 2R_{100}R_2^{00} + \; 2R_{1ii}R_2^{ii} + R_{200}\, R_2^{00} + R_{2ii}R_2^{ii}$$

$$+\frac{1}{4}\partial_\mu\,\partial_\nu\left(\sum_{\substack{\rho\neq\mu\\ \neq\nu}} H_\rho{}^\rho\right)\partial^\mu\,\partial^\nu\left(\sum_{\substack{\rho\neq\mu\\ \neq\nu}} H_\rho{}^\rho\right) \tag{7-c)'}$$

1^{st} line of $R_{\mu\nu}R^{\mu\nu} = (R_1^{00} + R_2^{00})^2 + \left(R_1^{ii} + R_2^{ii}\right)^2 = R_{00}^2 + R_{ii}^2$

One can note the appearances of perfect squares which is even more apparent when using

$$R^2 - 2R_{\mu\nu}R^{\mu\nu} = \left(R_1^{00} - R_1^{ii}\right)^2 - 2R_{100}\,R_1^{00} - 2R_{1ii}R_1^{ii} + \left(R_2^{00} - R_2^{ii}\right)^2 - 2R_{200}\,R_2^{00} - 2R_{2ii}R_2^{ii}$$

$$+2\left(R_1^{00} - R_1^{ii}\right)\left(R_2^{00} - R_2^{ii}\right) - 4R_{100}R_2^{00} - 4R_{1ii}R_2^{ii} - \frac{1}{2}\partial^\mu\,\partial^\nu\left(\sum_{\substack{\rho\neq\mu\\ \neq\nu}} H^{\rho\rho}\right)\partial^\mu\,\partial^\nu\left(\sum_{\substack{\rho\neq\mu\\ \neq\nu}} H^{\rho\rho}\right)$$

$$\leftrightarrow -\left(R_1^{00} + R_1^{ii}\right)^2 - \left(R_2^{00} + R_2^{ii}\right)^2 - 2\left(R_1^{00} + R_1^{ii}\right)\left(R_2^{00} + R_2^{ii}\right)$$

$$+\sum_i \frac{1}{2}\partial^0\,\partial^i\Big(\sum_{\substack{\rho\neq 0\\ \neq i}} H^{\rho\rho}\Big)\,\partial^0\,\partial^i\Big(\sum_{\substack{\rho\neq 0\\ \neq i}} H^{\rho\rho}\Big) - \sum_{i\neq j}\frac{1}{2}\partial^i\,\partial^j\Big(\sum_{\substack{\rho\neq i\\ \neq j}} H^{\rho\rho}\Big)\,\partial^i\,\partial^j\Big(\sum_{\substack{\rho\neq i\\ \neq j}} H^{\rho\rho}\Big)$$

What is resolvable as it leads to the already manifested conformal equation $R^2 - 3R_{\mu\nu}R^{\mu\nu} = 0$

$$0 = R^2 - 3R_{\mu\nu}R^{\mu\nu} = -\left(R^{00} + R^{ii}\right)^2 - (R_{00}^2 + R_{ii}^2)$$

$$-\frac{3}{4}\sum_{i\neq j}\partial^i\,\partial^j\Big(\sum_{\substack{\rho\neq i\\ \neq j}} H^{\rho\rho}\Big)\,\partial^i\,\partial^j\Big(\sum_{\substack{\rho\neq i\\ \neq j}} H^{\rho\rho}\Big) + \frac{3}{4}\sum_i \partial^0\,\partial^i\Big(\sum_{\substack{\rho\neq 0\\ \neq i}} H^{\rho\rho}\Big)\,\partial^0\,\partial^i\Big(\sum_{\substack{\rho\neq 0\\ \neq i}} H^{\rho\rho}\Big) \tag{8}$$

That is to be expected due to the ratio of isotropic 2-surfaces out of 3-spaces in eq. (7) above.

And that is essential for the shuffle of the eigen-frequencies if needed there.

Then if it has been defined the surface terms and found be different from the time coordinate say then x_{i_2} and x_{i_3} so they follow linear equation (7).

Then, if it is searched a resolution in eq. (8) with an isotropy between the time and one of the surface variable. Since, the remaining two being independent of time and appearing with a negative sign, a convenient rescaling in time in the 4th terms brings it with the 3rd terms to eliminate.

Leading to the equations

$$R^{00} = R^{ii} = 0 \tag{8)'}$$

Choose one as the other can be checked be true

$$0 = -\frac{1}{2}\sum_i \partial^i\,\partial^i\,H^{00} + \frac{1}{2}\partial^0\,\partial^0\sum_i H^{ii} \tag{8)”}$$

2- The Generalized Wick Rotation

In the frequency space, the above equation can be written as

$0 = -\partial_1 \partial_1 H^{00} - \partial_2 \partial_2 H^{00} - \partial_3 \partial_3 H^{00} + \partial_0 \partial_0 (H^{11} + H^{22} + H^{33}) = -\partial_1 \partial_1 H^{00} + \partial_0 \partial_0 H^{11}$ (8-a)”

So now a generalized Wick rotation can be working by picking an internal SU(3) invariance under a rescaling of the above orbital definitions using the freedom over $x_l = l = 1,2$ in (h^{11}, h^{22}) .

Now one has to choose the left and the right unitary variations such as

$$(dt, dx_1) \to \left(dt^L = \frac{1}{2}dt + i\frac{\sqrt{3}}{2}dx_1, dx_{123}^L = \frac{1}{2}dx_1 - \frac{i\sqrt{3}}{2}dt\right) \quad (8\text{-a})'''$$

$$(dt, dx_1) \to (dt^R = \frac{1}{2}dt - i\frac{\sqrt{3}}{2}dx_1, dx_{123}^R = \frac{1}{2}dx_1 + (i\sqrt{3})/2\ dt) \quad (8\text{-b})'''$$

So one can proceed, also in the Fourier projection space, as

$$0 = -\left(k_1^{0L}k_1^{0R} - 3\omega_0^L\omega_0^R\right)H^{00} + \left(\omega_1^L\omega_1^R - 3k_1^{1L}k_1^{1R}\right)H^{11} \quad (9)$$

Which is the side correlated identity advertised above.

To deduce from the above duality the new expression for the generalized Wick’s frequencies (or Hamiltonian) squared

$$0 = -\left({k_1^0}^2 - 3\omega_0^2\right)H^{00} + \left(\omega_1^2 - 3{k_1^1}^2\right)H^{11} \quad (9)'$$

One confirms the above result in view of the factors since also they meet the used 3-isotropy. Since it has been proved that multiplying a differential element by the unit modular factors F or F* will not change its unitarity properties.

Note:

Under $\sigma = 1$, see Appendix I for notation, it is clear from the Pfaffian resolution that the surface term can extend to one more dimension so the anomaly is resolved by changing the Wick to the two equivalent spatial directions plus time, while leaving the longitudinal direction as real.

To deduce that such case in eq. (8) spreads over all of the 4-d space-time so verifies an Einsteinian Gravity otherwise the appearing conformal property is very superficial, recalling so a sub-conformal holography.

Appendices:

I- An Eikonal Mapping for Metrics

Starting from the most elementary change, with $\lambda \equiv \frac{1-F}{F}$, that is

$$ds''^2 = ds'^2 - dt^2 = \varepsilon(F-1)d\tau^2 + Fd\chi^2 \tag{I-1a}$$

An isotropifying map operation with the time component, is done with $(3-\sigma)$ −space-like directions. Then, that mapping, $s(t,x) \rightarrow s''(\tau = \varepsilon t'', \chi = x'')$ is rendered a $(4-\sigma)$ dimensional vector, but with any an additional acting as by an inversion operator $\varepsilon = \pm$ as $\varepsilon(t', x'_{iso})$, is done as true all along on one side of the scalar product so via a vierbein sandwiching non-trivially only the anti-symmetric permutations in the spectral representation so the double derivative action would be proportional to the representation itself.

Mathematically, this is an affine form for the Friedrichs extension, [9], conserving then any Unitarity if proved existent. That is through its re-defined Hilbert spaces, and as e.g. those implicitly dealt with, alike for the module-kink-cusp links exposed in [10], rescaling it convex-wisely.

By kerneling these modular forms it leads into a Jacobian, non-zero positive and bounded by one.

So defining $F = f^2$,

$$0 \neq \begin{vmatrix} [\varepsilon_t(f^2-1)]^{\frac{1}{2}} & 0 & 0 & \\ 0 & f & & \\ & & \dots & 0 \\ 0 & & 0 & f \end{vmatrix} = [\varepsilon_t(f^2-1)]^{\frac{1}{2}} f^{3-\sigma} \leq 1 \Leftrightarrow 0 \neq \varepsilon_t(f^2-1)f^2(f^{2-\sigma})^2 \leq 1 \tag{I-1b}$$

In solving such a system, there should exist a simply connected set where that isotropy can be applied. Then, σ represents the reduction in spatial degree of freedom under the kind of isotropy as compared to the surface term. Also,

$$0 \neq \varepsilon_t(f^2-1)f^2(f^{2-\sigma})^2 \Rightarrow f \neq 0 \text{ And } f \neq 1$$

While $f = 0$ is the trivial identity transformation, $f = 1$ generates a specific co-dimension where the time isotropy is broken.

Resolving, then, for orbital symmetric length isotropy, i.e. by setting $\sigma = 2$ so $x \equiv r$,

$$\varepsilon_t(F-1)F = 1 \Leftrightarrow \varepsilon_t F^2 - \varepsilon_t F - 1 = 0 \Leftrightarrow \omega^2 = 1 + 4\varepsilon_t \Longrightarrow F = \frac{1 \pm \sqrt{1+4\varepsilon_t}}{2} \qquad \text{(I-2)}$$

The case $\varepsilon_t = 1$ leads to the solution $\mathrm{F} = \frac{1\pm\sqrt{5}}{2}$ which is the Golden Ratio representing the emergence of Fractals within the system. While the case $\varepsilon_t = -1$, leads to the solution $\mathrm{F} = \frac{1\pm \mathrm{i}\sqrt{3}}{2}$ as it is associated with the time reversal.

The operation of orbital skipping can be applied as a scaling product with its conjugate-like such

$$-FF'^{*} = -\frac{1-3}{4} = \frac{1}{2} \text{ A Counter to Pile-ups} \qquad \text{(I-2)'}$$

This upper limit result, has to be extracted to represent an eigenvalues' transfer from a 3-space isotropic 2-disk into a 3-sphere rolling, view it is normalized such that

$\mathcal{T}^{3-\text{space}}(\mathrm{F},\mathrm{F}^*) \propto \frac{F g_{ii}}{\sqrt{3}} \; or \; \frac{\mathrm{F}^* g_{ii}}{\sqrt{3}}$.

So however the regularizations impose what is being equivalent to a time ordering T for F and F^* as

$$\mathcal{T}(\mathrm{F}_i,\mathrm{F}_i^*) \propto \frac{T(F_i,\mathrm{F}_i^*)}{\sqrt{3}} = \frac{F}{\sqrt{3}} \; if \; t < t^* \; and \; \frac{F^*}{\sqrt{3}} \; if \; t > t^* \qquad \text{(I-3a)}$$

And, now, whose imaginary satisfies the Unitarity semi-equation at an internal loop variable time

$$-2Im\mathcal{T} = \mathcal{T}\mathcal{T}^* \Leftrightarrow -2Im\left[\sum_{i=1}^{3}\mathcal{T}(\mathrm{F}_i,\mathrm{F}_i^*)\right] = \sum_{i=1}^{3}\mathcal{T}(\mathrm{F}_i,\mathrm{F}_i^*)\mathcal{T}^*(\mathrm{F}_i,\mathrm{F}_i^*) + \sum_{i\neq j}^{3}\mathcal{T}(\mathrm{F}_i,\mathrm{F}_i^*)\mathcal{T}^*\left(\mathrm{F}_j,\mathrm{F}_j^*\right),$$

But, $i \neq j$ has a unique ordering so the minus sign showing in Eq. (6a) and due to the time reversal does not show again so $\mathcal{T}(\mathrm{F}_i,\mathrm{F}_i^*)\mathcal{T}^*\left(\mathrm{F}_j,\mathrm{F}_j^*\right) = -\mathcal{T}\left(\mathrm{F}_j,\mathrm{F}_j^*\right)\mathcal{T}^*(\mathrm{F}_i,\mathrm{F}_i^*)$. Also,

$$\sum_{i=1}^{3}\mathcal{T}(\mathrm{F}_i,\mathrm{F}_i^*)\mathcal{T}^*(\mathrm{F}_i,\mathrm{F}_i^*) \propto 3\frac{\mathrm{F}}{\sqrt{3}}\frac{\mathrm{F}\prime^*}{\sqrt{3}} + 3\frac{\mathrm{F}\prime^*}{\sqrt{3}}\frac{\mathrm{F}}{\sqrt{3}} = -\frac{1}{2} - \frac{1}{2} = -1 \;\text{ And } -2Im\left[\sum_{i=1}^{3}\mathcal{T}(\mathrm{F}_i,\mathrm{F}_i^*)\right] = -1 \qquad \text{(I-3b)}$$

The sphere map acting on the 3-metric via F and F^*, is thus proved an equality

$$\mathcal{T}^{3-\text{space}}(\mathrm{F},\mathrm{F}^*,g_{ii}) = \frac{T(F_i,\mathrm{F}_i^*)g_{ii}}{\sqrt{3}}$$

Therefore, the operation satisfy the Optical Theorem and is Unitary.

The case of $\sigma = 1$ is discussed end of §3.

II. a- The Necessary Lorentzian Surface Term

The simplest commonly example is the Dirac equation on a covariant Curved Space, whose details were worked in [18].

So one can deduce a convenient representation for $\gamma^{\mu}(x) = b_a{}^{\mu}(x)\gamma^a$ where $b_a{}^{\mu}$ is the vierbein as the metric is $g^{\mu\nu} \equiv b_a{}^{\mu} b_b{}^{\nu} \eta^{ab}$, while $\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2\eta^{\mu\nu}$. More, the connection is

$\Gamma_{\mu} = -\frac{1}{4}\gamma_a\gamma_b b^a{}_{\nu} g^{\nu\lambda} b^a_{\lambda;\mu} + iqA$

A can be set to zero, since arbitrary in this context.

The equation, with a covariant derivatives $\nabla_{\mu} \equiv \partial_{\mu} - \Gamma_{\mu}$ via Γ_{μ} the Spinor affine connection, is

$$\left(\gamma^{\mu}(x)\nabla_{\mu} + m\right)\psi(x) = 0 \qquad \text{(II-a1)}$$

Where $\gamma^{\mu}(x)$ is the coordinate dependent Dirac matrices whose covariant derivative is given by

$\nabla_{\mu}\gamma_{\nu}(x) = \partial_{\mu}\gamma_{\nu}(x) - \Gamma^{\lambda}_{\mu\nu}\gamma_{\lambda}(x) - \Gamma_{\mu}\gamma_{\nu}(x) + \gamma_{\nu}(x)\Gamma_{\mu} = 0$

Where here $\Gamma^{\lambda}_{\mu\nu}$ is the Christopher symbol, it differs from the Gauge invariance correction Γ_{μ}, noted also ω_{μ} as it deals with tensor with tensors.

The Hamiltonian defined as $i\frac{\partial}{\partial t}\psi = H\psi$ should be regulated as

$$\frac{i}{2}\left(\frac{\overrightarrow{\partial}}{\partial t} - \frac{\overleftarrow{\partial}}{\partial t}\right)\psi = H_{reg}\psi \Rightarrow 2H_{reg} = \left(2H + i\overleftarrow{\left(\frac{\partial}{\partial t}\right)}\right) \qquad \text{(II-a2)}$$

To include the time twist in it. One can see this equation in the Hilbert Space of ψ's as

$$\left(\psi_j, H_{reg}\psi_i\right) = \frac{i}{2}\int d^3x < \psi_j{}^{\dagger}(x,t)[\gamma^0(x)]^{-1}\frac{\partial}{\partial t}[\gamma^0(x)\psi_i(x,t)]$$

$$-\frac{\partial}{\partial t}\{\psi_j{}^{\dagger}(x,t)[\gamma^0(x)]^{-1}\}\gamma^0(x)\psi_i(x,t) > \qquad \text{(II-a2)'}$$

One can deduce that the Hamiltonian becomes Hermitian, since in the space of frequencies the time derivative transforms into frequencies while $[\gamma^0(x)]^{-1}\gamma^0(x) = 1$.

However, it has been dropped above the surface term $\int d^3x\, \partial/\partial t\left[\psi_j{}^{\dagger}(x,t)\psi_i(x,t)\right]$.

b- The Expansion' al Popping up of an Independent Neutrino Flavor Wave Vector

One can proceed by a simplifying illustration but that a physical implication on the Flavor Physics.

Recalling that the amplitude of one neutrino generation say c out of ν_a $a = e, \mu, \tau$ is given to first order, by an expression $\hat{a}_{cc} = \sum_{ab}(M_{cc})_{ab}(a_L)_{ab}$ when expanded, [19], as such

$$\tilde{a}_{cc} = \sum_{\substack{a\neq c\\ b\neq c}}(M_{cc})_{ab}(a_L)_{ab} + (M_{cc})_{ec}(a_L)_{ec} + (M_{cc})_{\mu c}(a_L)_{\mu c} + (M_{cc})_{c\tau}(a_L)_{c\tau} \quad \text{(II-b1)}$$

While using in plus of $\sum_{a'} U^*_{a'a}U_{a'b} = \delta_{ab}$,

$$(M_{cc})_{ab} = \sum_{a'b'} \tau_{a'b'}\, U^*_{a'c}U_{a'a}U^*_{b'b}U_{b'c}$$

Where the matrix amplitude τ derives from the plane waves of oscillations taken at the boundary $t \to 0$ or $t \ll \frac{1}{E_{scale}}$.

Since in the above it was derived that the boundary should contain at least a relation between two boundary variables, which leads that the radial distance can be parameterized in terms of time. So

$$\tau_{a'b'} = \begin{cases} \exp(-iE_{b'}t) & E_{a'} = E_{b'} \\ \frac{\exp(-iE_{a'}t)-\exp(-iE_{b'}t)}{-i(E_{a'}-E_{b'})t} & E_{a'} \neq E_{b'} \end{cases} \quad \text{(II-b2)}$$

Since, a kink in the region subject to an energy $E_{a'} = E_{b'}$, eliminates.

We are interested in $E_{a'} \neq E_{b'}$.

The fact of arbitrariness of the difference of energy in the case of the oscillation leads to take one of the energies to be zero, say $E_{b'} \equiv 0$.

Plus, due a doubling that can occur in the cusp case only, which and so $E_{a'}t \ll 1$. One has then,

$$\tau_{a'b'} \to \frac{1}{2i}(E_{a'} + E_{b'})t \Rightarrow$$

$$(M_{cc})_{ab} \approx \sum_{a'b'}\left(1 + \frac{1}{2i}E_{a'}t\right)U^*_{a'c}U_{a'a}U^*_{b'b}U_{b'c} = \sum_{a'}\left(1 + \frac{1}{2i}E_{a'}t\right)U^*_{a'c}U_{a'a}\delta_{bc} \quad \text{(II-b1)'}$$

One sees that the summation over b' is totally decoupled and it cannot be reduced (to a Kronicker δ), unless the orthonormal eigen proper basis indexed by b has no mixing to the orthonormal eigen proper basis indexed by b'.

Therefore, the flavor indexing is independent from any external currents that may link it then through Color or Charge, as seen in §2.

With the existence of Wave Vector under that the radial parameterization is directly related to the time through a linear relation leads to that the velocity is well-defined and is independent therefore its Helicity is non-zero and unique, then its Chiral nature.

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