Research Article

# Unification of the Fluid Flow Field, the Electromagnetic Field-Strength Tensors, and the Field Dynamic Equations

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**Either mechanical waves or electromagnetic waves propagate in space at a finite speed. The wave propagation speed and particle flow velocity form a four-vector. Using the four-vector, we can obtain the fluid flow field and the electromagnetic field strength tensor. Both tensors share the same mathematical structure and can be unified into a single mathematical frame. The field dynamic equations, either for fluid flow or for the charged particle motions, are a combination of the translational and the rotational motion. The rotational motion behaves as wave properties. The strength tensor contraction (inner product) forms a hypersurface. It is an indefinite quadratic form (saddle-shaped surface) that can take positive or negative values to reveal the dominating moving types. The electromagnetic waves are located at the saddle points, and the strength tensor for electromagnetic waves is zero. In general, the motion in the field obeys the weak form of Newton's action and reaction law, namely, the field has an induced secondary flow, due to the interactions between vorticity and velocity for fluid flow or the magnetic field and the charge flow for the electromagnetic field. It is found that this approach is equivalent to the Euler-Lagrangian method, which is expressed by $D_t\vec{p} = \nabla\mathcal{L}$. Both methods will produce the same field dynamic equations.**

## 1. Introduction

Fluid flow fields represent the velocity in space and dynamics of a flow. The flow depends on the pressure (density) and physical properties of the fluid. Electromagnetic fields describe the behavior of electric and magnetic fields in space, influenced by charges and currents.

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In fluid flow, the sources are forces (e.g., pressure gradients) driving the fluid to move. In the electromagnetism field, the sources are charges and currents, which generate electric and magnetic fields.

Though fluid flow and electromagnetic fields differ significantly in their physical interpretation, they share similarities in their mathematical structure.

Both the fluid field, $(p, \vec{v}, \vec{\omega})$, and the electromagnetic field, $(\rho, \vec{E}, \vec{B})$, are described by scalar and vector fields that vary in space and time.

They are mathematically represented by functions $\mathbf{F}(\vec{r}, t)$ where $\mathbf{F}$ is the scalar or vector field (e.g., pressure (volumetric energy density), $\vec{v}$, and $\vec{\omega}$ for the fluid flow field, or charge density, $\vec{E}$ and $\vec{B}$ for the electromagnetic field).

Both fields exhibit rotational behavior described by the curl operator of the field. In electromagnetism, the curl of the electric field $(\nabla \times \vec{E})$ and the magnetic field $(\nabla \times \vec{B})$ describes how they circulate due to time-varying fields or currents, while the curl of the velocity field in fluid, $(\nabla \times \vec{v})$, represents the vorticity field (rotational motion in the flow field). Namely, both fields contain certain "circulating" quantities that behave as wave propagation fields. In other words, fluid waves describe pressure oscillations that propagate in the form of sound waves at the sound wave propagation speed of $c_m$, while electromagnetic waves represent the oscillating $\vec{E}$ and $\vec{B}$ fields that propagate in space at the speed of light, c. Both fields propagate in space at a finite wave speed.

Thus, in order to describe the flow field or the electromagnetic field correctly and completely, the wave propagation speed cannot be ignored. In this article, the wave propagation speed and medium flow velocity will constitute a four-vector. Based on this four-vector, we will give out the field strength tensor and the field dynamic equations, both for the fluid flow field and the electromagnetic field.

## 2. Fluid Flow Field Strength Tensor and Equations

In 3D space, given a flow field, there is a distribution of some physical quantity, such as pressure and velocity vector, at each point in this physical space. The contravariant four-velocity and four-momentum can be defined as:

$$v^{\mu} = [c_m \quad u \quad v \quad w] = [c_m \quad \vec{v}] \tag{1}$$

and


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$$A^{\mu} = \begin{bmatrix} \frac{p}{c_m} & \rho_m u & \rho_m v & \rho_m w \end{bmatrix} = \begin{bmatrix} \frac{p}{c_m} & \vec{p} \end{bmatrix} \tag{2}$$

where $c_m$ is the pressure wave propagation speed, u, v, and w are velocities in Cartesian coordinates, $\rho_m$ is the mass density, and $p$ is the thermodynamic pressure measured in the field when the fluid is flowing (volumetric energy density).

The product, $\rho_m \vec{v}$, gives the rate at which mass flows through a unit area in a given direction.

The four-position vector is defined as:

$$x^{\mu} = [c_m t, \vec{x}] = [\,c_m t \quad x \quad y \quad z\,] \tag{3}$$

In this article, we use the (- + + +) metric signature.

$$\eta_{\mu\nu} = diag(-,+,+,+) \tag{4}$$

Thus, the contravariant four-gradient is defined as:

$$\partial^{\mu} = -\vec{e}_0 \partial^0 + \vec{e}_i \partial^i = \begin{bmatrix} -\frac{1}{c_m}\partial_t & \partial_x & \partial_y & \partial_z \end{bmatrix} \tag{5}$$

and the covariant four-velocity in Minkowski space is:

$$v_{\mu} = \eta_{\mu\nu} v^{\nu} = [-\,c_m \quad u \quad v \quad w] \tag{6}$$

With these definitions, the contravariant flow field strength tensor can be defined as:

$$F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu} \tag{7}$$

Accordingly, it is an antisymmetric rank-2 tensor field — in Minkowski space. The field tensor's anti-symmetry guarantees that $F^{00} = F^{11} = F^{22} = F^{00} = 0$. It is traceless and possesses only six independent non-zero components.

By the definition of $F^{\mu\nu}$ and using the four-momentum of eq. (2), algebraic evaluation yields, for $\mu = 0$:

$$\begin{cases} F^{01} = -\frac{\partial \rho_m u}{c_m \partial t} - \frac{\partial}{\partial x}\left(\frac{p}{c_m}\right) = -\frac{1}{c_m}\left(\frac{\partial \rho_m u}{\partial t} + \frac{\partial p}{\partial x}\right) = -\frac{F^x}{c_m} \\ F^{02} = -\frac{\partial \rho_m v}{c_m \partial t} - \frac{\partial}{\partial y}\left(\frac{p}{c_m}\right) = -\frac{1}{c_m}\left(\frac{\partial \rho_m v}{\partial t} + \frac{\partial p}{\partial y}\right) = -\frac{F^y}{c_m} \\ F^{03} = -\frac{\partial \rho_m w}{c_m \partial t} - \frac{\partial}{\partial z}\left(\frac{p}{c_m}\right) = -\frac{1}{c_m}\left(\frac{\partial \rho_m w}{\partial t} + \frac{\partial p}{\partial z}\right) = -\frac{F^z}{c_m} \end{cases} \tag{8}$$

These terms can be written in a vector form as:

$$\vec{F}^{0} = -\frac{1}{c_m}\left(\frac{\partial \rho_m \vec{v}}{\partial t} + \nabla p\right) \tag{9}$$

For $\mu = 1$, the components are:

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$$\begin{cases} F^{10} = \frac{\partial A^0}{\partial x^1} - \frac{\partial A^1}{\partial x^0} = \frac{1}{c_m}\left(\frac{\partial p}{\partial x} + \frac{\partial \rho_m u}{\partial t}\right) = \frac{F^x}{c_m} \\ F^{12} = \frac{\partial A^2}{\partial x^1} - \frac{\partial A^1}{\partial x^2} = \frac{\partial \rho_m v}{\partial x} - \frac{\partial \rho_m u}{\partial y} = \omega^z \\ F^{13} = \frac{\partial A^3}{\partial x^1} - \frac{\partial A^1}{\partial x^3} = \frac{\partial \rho_m w}{\partial x} - \frac{\partial \rho_m u}{\partial z} = -\omega^y \end{cases} \tag{10}$$

With a similar approach, we can get other components for $\mu = 2$ and $\mu = 3$. Combining all the terms, the matrix representation of the contravariant field tensor is

$$F^{\mu\nu} = \begin{bmatrix} 0 & -F^x/c_m & -F^y/c_m & -F^z/c_m \\ F^x/c_m & 0 & \omega^z & -\omega^y \\ F^y/c_m & -\omega^z & 0 & \omega^x \\ F^z/c_m & \omega^y & -\omega^x & 0 \end{bmatrix} \tag{11}$$

The Minkowski inner product of the contravariant four-acceleration and the covariant four-velocity, eq. (6), in the field is zero:

$$F^{\mu\nu} v_\nu = 0 \tag{12}$$

This contraction yields the following equations:

$$\begin{cases} -\frac{\partial \rho_m u}{\partial t} - \frac{\partial p}{\partial x} + v\omega^z - w\omega^y = 0 \\ -\frac{\partial \rho_m v}{\partial t} - \frac{\partial p}{\partial y} + w\omega^x - u\omega^z = 0 \\ -\frac{\partial \rho_m w}{\partial t} - \frac{\partial p}{\partial z} + u\omega^y - v\omega^x = 0 \end{cases} \tag{13}$$

In a more compact vector form, it reads:

$$-\frac{\partial \rho_m \vec{v}}{\partial t} - \nabla p + \vec{v} \times \vec{\omega} = 0 \tag{14}$$

If it is divided by the wave propagation speed, we can get another variant of this equation:

$$-\frac{\partial \rho_m \vec{\beta}}{\partial t} - \nabla\left(\frac{p}{c_m}\right) + \vec{\beta} \times \vec{\omega} = 0 \tag{15}$$

where, $\vec{\beta}$ is the ratio of $\vec{v}$ to *c*. In fluid dynamics, we call it the local Mach number:

$$\vec{\beta} = \vec{M} = \frac{\vec{v}}{c_m} \tag{16}$$

Here, we define the induced vorticity field as:

$$\vec{\omega} = \nabla \times (\rho_m \vec{v}) \tag{17}$$

For example, the component $\omega_z$ of the induced vorticity is shown in Fig.1.

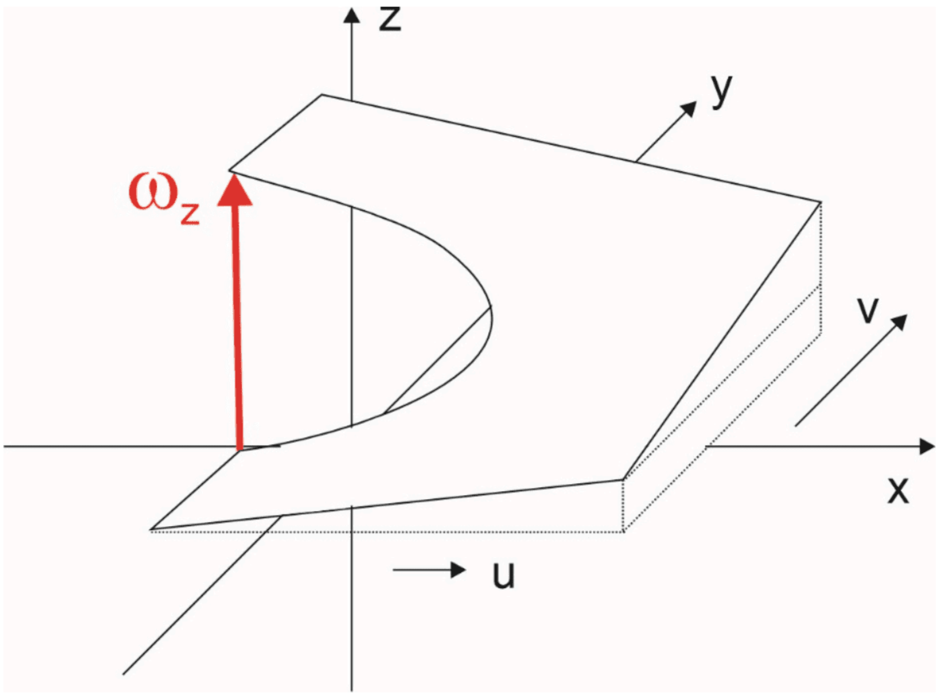


**Figure 1.** The induced vorticity component $\omega_z$.

Using the dot product rule of the vector calculus identity, we have

$$\vec{v} \times \vec{\omega} = \vec{v} \times \nabla \times (\rho_m \vec{v}) = \frac{1}{2} \nabla (\rho_m \vec{v} \cdot \vec{v}) - (\vec{v} \cdot \nabla)(\rho_m \vec{v}) \tag{18}$$

Thus, eq. (14) can be re-written as

$$\frac{\partial \rho_m \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)(\rho_m \vec{v}) = -\nabla p + \nabla \left( \frac{1}{2} \rho_m \vec{v} \cdot \vec{v} \right) \tag{19}$$

The last term in the RHS of eq. (19) is the volumetric kinetic energy density; thus,

$$\frac{D(\rho_m \vec{v})}{Dt} = \nabla(-p + T) \tag{20}$$

In eq. (20), if we define the Lagrangian density as

$$\mathcal{L} = T - p \tag{21}$$

Then, the eq. (20) can be written more compactly in Cartesian coordinates:

$$D_t \vec{p} = \nabla \mathcal{L} \tag{22}$$

Here, $D_t$ represents the material derivative of the linear momentum of $\vec{p}$.

The potential (pressure) energy density can be written as

$$p = \rho_m c_m^2 \tag{23}$$

Details can be found in the reference[1].

Then, the Lagrangian density can be rewritten as

$$\mathcal{L} = \frac{1}{2}\rho_m \vec{v}^2 - p = \frac{1}{2}\rho_m \vec{v}^2 - \rho_m c_m^2 \tag{24}$$

If the flow velocity and wave propagation speed c are defined as four vectors, the Lagrangian density is a hyperboloid quadratic form in four dimensions, or it is a four-dimensional saddle-shaped function:

$$\mathcal{L} = \frac{1}{2} x^T A x \tag{25}$$

where the four-vector is defined to be

$$x^T = [\, c_m \quad u \quad v \quad w \,] \tag{26}$$

and the metric tensor for Lagrangian density is

$$A = \frac{\rho_m}{2} \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{27}$$

If the gradient of the volumetric kinetic energy density is written as a matrix-velocity vector product in 3D space, the equation (19) will become:

$$\frac{\partial \rho_m \vec{v}}{\partial t} + \left(\vec{v} \cdot \nabla\right) \left(\rho_m \vec{v}\right) = -\nabla p + 2\mu \bar{\bar{S}} + \frac{1}{2} \vec{v} \times \vec{\omega} \tag{28}$$

where, $\bar{\bar{S}}$ is the engineering strain tensor (it is a symmetric tensor), $\mu$ is the dynamic viscosity of the fluid.

This is the modified Navier-Stokes equations, derived from the Euler-Lagrangian approach; details can be found in the reference[2]. It is confirmed that the Navier-Stokes equations are not complete. It ignores the last term in the RHS of eq. (28).

### *2.1. Velocity decomposition and field strength tensor contraction*

Any flow field will produce a vorticity field, as long as the corresponding shear strain tensor is not symmetric, for instance,

$$\frac{\partial \rho_m v}{\partial x} \neq \frac{\partial \rho_m u}{\partial y} \tag{29}$$

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such that

$$\omega_z = \frac{\partial \rho_m v}{\partial x} - \frac{\partial \rho_m u}{\partial y} \neq 0 \tag{30}$$

Rearranging eq. (14):

$$\frac{\partial \rho_m \vec{v}}{\partial t} = -\nabla p + \vec{v} \times \vec{\omega} \tag{31}$$

The velocity field can be decomposed into two parts:

$$\vec{v} = \vec{v}_t + \vec{v}_r \tag{32}$$

Then, the dynamic equation (31) can be decomposed into two parts:

$$\begin{cases} \frac{\partial \rho_m \vec{v}_t}{\partial t} = -\nabla p \\ \frac{\partial \rho_m \vec{v}_r}{\partial t} = \vec{v} \times \vec{\omega} = \frac{1}{2} \nabla \left( \rho_m \vec{v} \cdot \vec{v} \right) - \left( \vec{v} \cdot \nabla \right) \left( \rho_m \vec{v} \right) \end{cases} \tag{33}$$

where $\vec{v}_t$ represents the velocity produced by the negative gradient of potential energy (pressure field). The object will move in the direction of decreasing potential energy, which aligns with how forces "pull" or "push" objects. We can call this pressure gradient-produced velocity field “translational motion”. It is the first row/column of the matrix representation of the contravariant field tensor, eq. (11). Physically, the fluid parcel will stretch or shrink without rotational motion.

The term $\vec{v} \times \vec{\omega}$ represents a motion or force induced by the interaction of the velocity and vorticity field (because of the relative motion between fluid layers), the induced velocity is perpendicular to both directions. It describes how vorticity interacts with velocity to produce secondary flows and instabilities. We can call this velocity “vorticity field inducted motion”. In the contravariant field tensor, it is the effect of the gradient of the shear flow (relative motions between fluid layers) or vorticity field, $\vec{\omega}$, by the curl operation onto the velocity field, eq. (17). Physically, the fluid parcel will take a rotational motion.

The combination effect of eq. (33) obeys the weak form of Newton’s action and reaction law. Furthermore, from eq. (17), we can conclude that if the shear strain tensor is symmetric, the velocity field will not produce a vorticity field, $\vec{\omega} = 0$, which means if fluid layers have no shear flow, then the fluid parcel has no rotational motion. Then eq. (33) degenerates to the strong form of Newton’s action and reaction law, namely, the particle motion is derived only by the negative gradient of potential energy, as is shown in Fig. 2. The induced secondary flow by the vorticity field forces the boundary layer to become thicker and thicker along the x-direction.


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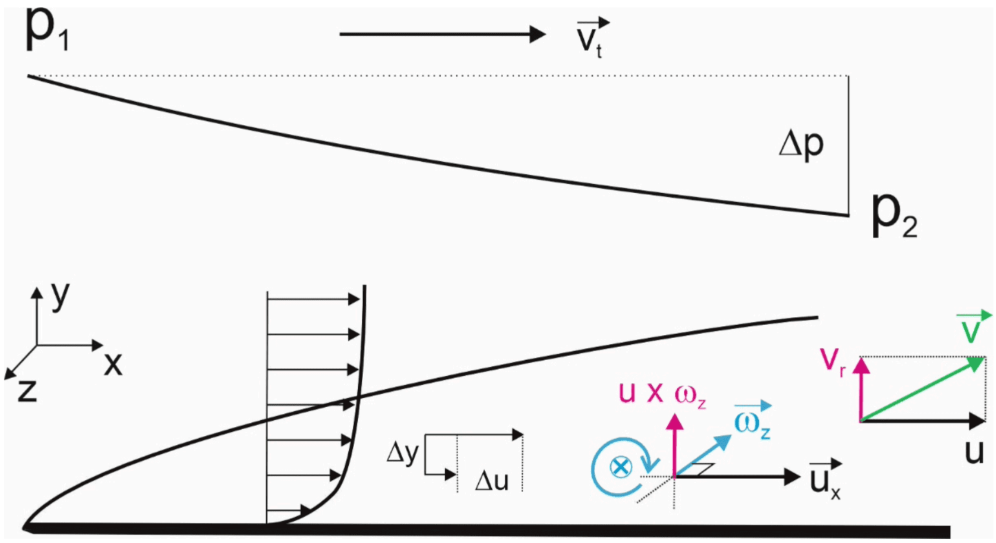


**Figure 2.** Shear flow produces a vorticity field; the interaction between vorticity and velocity induces a secondary flow; the boundary layer becomes thicker and thicker along the x-direction.

If we contract the fluid dynamics strength tensor, eq. (11), it reads:

$$F^{\mu\nu}F_{\mu\nu} = 2\left(\vec{\omega}^2 - \frac{\vec{F}^2}{c_m^2}\right) \tag{34}$$

Define a 6-dimensional vector $\vec{x}$, by concatenating $\vec{\omega}$ and $\frac{\vec{F}}{c_m}$:

$$\vec{x}^T = \begin{bmatrix} \vec{\omega} & \frac{\vec{F}}{c_m} \end{bmatrix}^T = \begin{bmatrix} \omega_x & \omega_y & \omega_z & \frac{F_x}{c_m} & \frac{F_y}{c_m} & \frac{F_z}{c_m} \end{bmatrix}^T \tag{35}$$

The contraction of the fluid dynamics strength tensor can be written as a quadratic form (it is a saddle-shaped function):

$$\frac{1}{2}F^{\mu\nu}F_{\mu\nu} = \vec{x}^T Q \vec{x} \tag{36}$$

where $Q$ is a $6 \times 6$ symmetric matrix:

$$Q = \begin{bmatrix} I_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & -I_{3\times3} \end{bmatrix} \tag{37}$$

It is concluded that:

- $F^{\mu\nu}F_{\mu\nu} > 0$: The induced vorticity field (antisymmetric shear flow) dominates the flow; eventually, the induced secondary flow may be stronger than the translational flow


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$\left(\|\vec{\omega}\| > \|\vec{F}\|/c_m\right)$;

- $F^{\mu\nu}F_{\mu\nu} < 0$: The translational motion field (stretching or shrinking of the fluid parcel) dominates the flow $\left(\|\vec{\omega}\| < \|\vec{F}\|/c_m\right)$;
- $F^{\mu\nu}F_{\mu\nu} = 0$: The equal magnitudes of the translational motion field and the vorticity field, $\left(\|\vec{\omega}\| = \|\vec{F}\|/c_m\right)$.

## 3. Electromagnetic Field Strength Tensor and Equations

In space, there are positively charged particles (positrons, protons, etc., labeled as $\rho_+$) and negatively charged particles (electrons, etc. labeled as $\rho_-$). We define a surplus of positive relative to the negative charges as the net charges in a position:

$$\rho = \rho_+ - \rho_- \tag{38}$$

In 4D space, the contravariant four-charge-flux can be defined as:

$$A^\mu = \frac{\mu_0}{4\pi}\begin{bmatrix}\rho c & \rho u & \rho v & \rho w\end{bmatrix} = \frac{\mu_0}{4\pi}\begin{bmatrix}\rho c & \vec{J}\end{bmatrix} \tag{39}$$

where, $\mu_0$ is the permeability in space; $\rho$ is charge volumetric density, measured in $\left[\frac{C}{m^3}\right]$; u, v, and w are charged particle flow velocity, $\left[\frac{m}{s}\right]$, c is photon propagation speed, measured in $\left[\frac{m}{s}\right]$.

In the SI unit, this charge flux vector is measured in Tesla per meter, $\left[\frac{T}{m}\right]$.

Similarly, the contravariant electromagnetic strength tensor can be defined as:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{40}$$

For $\mu = 0$, by the definition, using the four-charge-flux of eq. (39), we have:

$$\begin{cases} 4\pi F^{01} = -\dfrac{\partial \mu_0 J_x}{c\partial t} - \dfrac{\partial \mu_0 \rho c}{\partial x} = \dfrac{1}{c}\left(-\dfrac{\partial \mu_0 \rho u}{\partial t} - \dfrac{\partial \mu_0 \rho c^2}{\partial x}\right) \\ 4\pi F^{02} = -\dfrac{\partial \mu_0 J_y}{c\partial t} - \dfrac{\partial \mu_0 \rho c}{\partial y} = \dfrac{1}{c}\left(-\dfrac{\partial \mu_0 \rho v}{\partial t} - \dfrac{\partial \mu_0 \rho c^2}{\partial x}\right) \\ 4\pi F^{03} = -\dfrac{\partial \mu_0 J_z}{c\partial t} - \dfrac{\partial \mu_0 \rho c}{\partial z} = \dfrac{1}{c}\left(-\dfrac{\partial \mu_0 \rho w}{\partial t} - \dfrac{\partial \mu_0 \rho c^2}{\partial x}\right) \end{cases} \tag{41}$$

where the wave propagation speed c is assumed to be constant. Recalling the photon propagation speed definition:

$$c^2 = \frac{1}{\mu_0 \varepsilon_0} \tag{42}$$

Substituting this definition into eq. (38) yields:


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$$\begin{cases} 4\pi F^{01} = \frac{1}{c}\left(-\frac{\partial\mu_0\rho u}{\partial t} - \frac{1}{\mu_0\varepsilon_0}\frac{\partial\mu_0\rho}{\partial x}\right) = \frac{1}{c}\left(-\frac{\partial\mu_0 J_x}{\partial t} - \frac{1}{\varepsilon_0}\frac{\partial\rho}{\partial x}\right) = -\frac{E^x}{c} \\ 4\pi F^{02} = \frac{1}{c}\left(-\frac{\partial\mu_0\rho v}{\partial t} - \frac{1}{\mu_0\varepsilon_0}\frac{\partial\mu_0\rho}{\partial x}\right) = \frac{1}{c}\left(-\frac{\partial\mu_0 J_y}{\partial t} - \frac{1}{\varepsilon_0}\frac{\partial\rho}{\partial y}\right) = -\frac{E^y}{c} \\ 4\pi F^{03} = \frac{1}{c}\left(-\frac{\partial\mu_0\rho w}{\partial t} - \frac{1}{\mu_0\varepsilon_0}\frac{\partial\mu_0\rho}{\partial z}\right) = \frac{1}{c}\left(-\frac{\partial\mu_0 J_z}{\partial t} - \frac{1}{\varepsilon_0}\frac{\partial\rho}{\partial z}\right) = -\frac{E^z}{c} \end{cases} \quad (43)$$

Here, $\varepsilon_0$ is the permittivity in space, E is the electric field per unit area, $\left[\frac{N}{(C\cdot m^2)}\right]$, and it results from spatial derivatives of the charge density or rate of change of current flux at a given point.

The equation (43) can be written in vector form:

$$4\pi c\overrightarrow{F^0} = -\left(\frac{\partial\mu_0\overrightarrow{J}}{\partial t} + \frac{1}{\varepsilon_0}\nabla\rho\right) \quad (44)$$

The gradient of the charge density, $\frac{1}{\varepsilon_0}\nabla\rho$, represents how the charge density distribution changes in space. It may relate to the electric field produced by the non-homogeneous distribution of the charges, as shown in Fig.3. If a charged particle is located at the position $\overrightarrow{r}$, it will accelerate and produce a current; as a consequence, the charge current will produce a magnetic field.

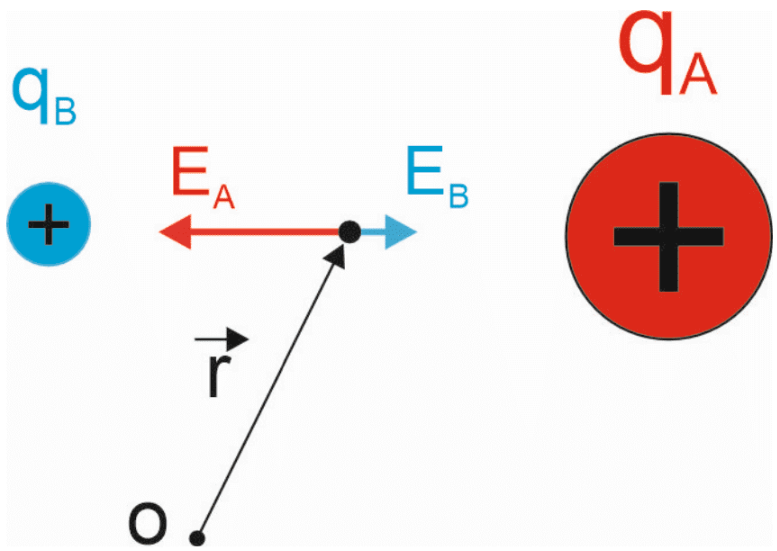


**Fig. 3.** The gradient of the charge distribution produces a net electric field in the space, which points to the direction of the negative gradient.

Taking a curl operator on this equation, it may relate to Faraday's law of induction, namely how a time-varying magnetic field corresponds to the curl of an electric field.

For $\mu = 1$, the components read:

$$\begin{cases} 4\pi F^{10} = \frac{1}{c}\left(\frac{1}{\varepsilon_0}\frac{\partial\rho}{\partial x} + \frac{\partial\mu_0\rho u}{\partial t}\right) = \frac{E^x}{c} \\ 4\pi F^{12} = \frac{\partial\mu_0\rho v}{\partial x} - \frac{\partial\mu_0\rho u}{\partial y} = B^z \\ 4\pi F^{13} = \frac{\partial\mu_0\rho w}{\partial x} - \frac{\partial\mu_0\rho u}{\partial z} = -B^y \end{cases} \tag{45}$$

The derivative of the current density with respect to the coordinates ($F^{12}$ *etc.*) describes how the current density and the induced magnetic effects vary spatially. As mentioned before, it quantifies how the magnetic field induced by current density changes in space, due to the relative motions between charges, as shown in Fig. 4. The SI unit is Tesla per unit area, $\left[\frac{T}{m^2}\right]$, which corresponds to the net magnitude of the magnetic field produced by relative charge motions per unit area.

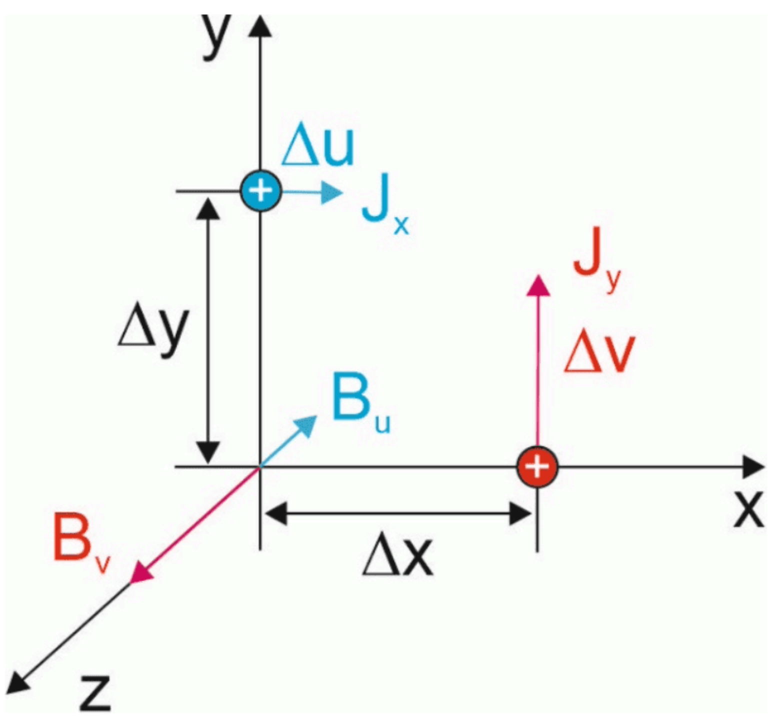


**Fig. 4.** Relative charge motion in space produces a magnetic field.

The other components for $\mu = 2$ and $\mu = 3$ can be calculated by the same approaches. The matrix representation of the contravariant electromagnetic strength tensor is

$$F^{\mu\nu} = \frac{1}{4\pi}\begin{bmatrix} 0 & -\frac{E^x}{c} & -\frac{E^y}{c} & -\frac{E^z}{c} \\ \frac{E^x}{c} & 0 & B^z & -B^y \\ \frac{E^y}{c} & -B^z & 0 & B^x \\ \frac{E^z}{c} & B^y & -B^x & 0 \end{bmatrix} \tag{46}$$

It should be noted here that this is the strength tensor in space at a point, produced by the charge spatial variation and charge fluxes locally in the vicinity of this point, by a first-order approximation (the first-order Taylor expansion), not by the charge per se at this point, but related to the charge at the point.

At this point, we define an electric charge with a Coulomb of q. The covariant four charge-flux-velocity is:

$$qv_\mu = q\eta_{\mu\nu}v^\nu = [-qc \quad qu \quad qv \quad qw] = q\left[-c \quad \overrightarrow{v}\right] \tag{47}$$

The contraction of the electromagnetic strength tensor with the covariant four charge-flux-velocity ( $F^{\mu\nu}(qv_\nu)$) in Minkowski space yields:

$$\begin{cases} -q\frac{\partial\mu_0\rho u}{\partial t} - \frac{q}{\varepsilon_0}\frac{\partial\rho}{\partial x} + qvB^z - qwB^y = 0 \\ -q\frac{\partial\mu_0\rho v}{\partial t} - \frac{q}{\varepsilon_0}\frac{\partial\rho}{\partial y} + qwB^x - quB^z = 0 \\ -q\frac{\partial\mu_0\rho w}{\partial t} - \frac{q}{\varepsilon_0}\frac{\partial\rho}{\partial z} + quB^y - qvB^x = 0 \end{cases} \tag{48}$$

Recalling the photon propagation speed definition of eq. (42). It can be written more compactly in a vector form:

$$-q\frac{\partial\mu_0\rho\vec{v}}{\partial t} - q\nabla(\mu_0\rho c^2) + q\vec{v}\times\vec{B} = 0 \tag{49}$$

The SI unit is volumetric energy density, $[\frac{J}{m^3}]$. This contraction is the Lorentz force law for the combination of electric and magnetic force on a charged point particle of q in the field.

Similarly, another variation of the equation reads:

$$-q\frac{\partial\mu_0\rho\vec{\beta}}{\partial t} - q\nabla(\mu_0\rho c) + q\vec{\beta}\times\vec{B} = 0 \tag{50}$$

if it is divided by the light wave speed of c: $\vec{\beta} = \frac{\vec{v}}{c}$.

The magnetic field density per unit area is defined as

$$\vec{B} = \nabla\times(\mu_0\rho\vec{v}) \tag{51}$$

Assuming the permeability, $\mu_0$, is constant in space, then eq. (49) can be rewritten as:

$$q\mu_0\left[-\frac{\partial\rho\vec{v}}{\partial t} - \nabla(\rho c^2) + \ \vec{v}\times\nabla\times(\rho\vec{v})\right] = 0 \tag{52}$$

Using the vector calculus identity of eq. (18), we have


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$$q\mu_0\left[-\frac{\partial\rho\vec{v}}{\partial t}-(\vec{v}\bullet\nabla)(\rho\vec{v})-\nabla(\rho c^2)+\frac{1}{2}\nabla(\rho\vec{v}\bullet\vec{v})\right]=0 \tag{53}$$

This equation can be rearranged as:

$$\frac{\partial q\mu_0\rho\vec{v}}{\partial t}+(\vec{v}\bullet\nabla)(q\mu_0\rho\vec{v})=\nabla\left(\frac{1}{2}q\mu_0\rho\vec{v}^2-q\mu_0\rho c^2\right) \tag{54}$$

The RHS in the parentheses can be defined as the volumetric Lagrangian density for the electromagnetic field:

$$\mathcal{L}=\frac{1}{2}q\mu_0\rho\vec{v}^2-q\mu_0\rho c^2 \tag{55}$$

Similarly, the first term can be regarded as "charge kinetic energy density", and the second term as "charge potential energy density"; then, eq. (54) can be written as:

$$\frac{D(q\mu_0\rho\vec{v})}{Dt}=\nabla\mathcal{L}=\nabla(T_c-p_c) \tag{56}$$

If we define the mass density of the charged particle as $\rho_m$, $[\frac{kg}{m^3}]$, the Lagrangian density of the charged particle should include the kinetic energy density:

$$\mathcal{L}=\frac{1}{2}\rho_m\vec{v}^2+\frac{1}{2}q\mu_0\rho\vec{v}^2-q\mu_0\rho c^2 \tag{57}$$

which can be written more compactly as

$$\mathcal{L}=\frac{q\mu_0\rho}{2}\left[\left(1+\frac{\rho_m}{q\mu_0\rho}\right)\vec{v}^2-2c^2\right] \tag{58}$$

This forms a four-dimensional quadratic form:

$$\mathcal{L}=\frac{1}{2}x^T g_{\mu\nu}x \tag{59}$$

where $x^T$ is the contravariant four-velocity vector of the charged particle:

$$x^T=[c\quad u\quad v\quad w]^T \tag{60}$$

Here, the metric tensor for the Lagrangian density is

$$g_{\mu\nu}=q\mu_0\rho\begin{bmatrix}-2 & 0 & 0 & 0\\ 0 & 1+\frac{\rho_m}{q\mu_0\rho} & 0 & 0\\ 0 & 0 & 1+\frac{\rho_m}{q\mu_0\rho} & 0\\ 0 & 0 & 0 & 1+\frac{\rho_m}{q\mu_0\rho}\end{bmatrix} \tag{61}$$

### *3.1. Electromagnetic tensor contraction and photon gases*

Similar to the fluid flow field tensor, the first row/column of the electromagnetic strength tensor represents the force on charged particles produced by the negative gradient of charge density. The other terms are the induced magnetic field strength (similar to the vorticity field strength) because of the relative motion between the charged particles.

Similar to the fluid flow vorticity field, provided that the motion velocity of the charged particles is not symmetric, it will produce a magnetic field, e.g.,

$$B^z = \frac{\partial \mu_0 \rho v}{\partial x} - \frac{\partial \mu_0 \rho u}{\partial y} \tag{62}$$

On the condition that the charged particle flow field at a point is symmetric, then the induced magnetic field can be zero, for example,

$$u = v; \text{ and } B^z = \mu_0 \left( \frac{\partial \rho v}{\partial x} - \frac{\partial \rho u}{\partial y} \right) = 0 \tag{63}$$

We form the inner product (contraction) of the electromagnetic strength tensor of eq. (46) as a scalar function; it yields:

$$F^{\mu\nu} F_{\mu\nu} = \frac{\mu_0}{8\pi^2} \left( \frac{\vec{B}^2}{\mu_0} - \varepsilon_0 \vec{E}^2 \right) = \frac{1}{8\pi^2} \left( \vec{B}^2 - \frac{\vec{E}^2}{c^2} \right) \tag{64}$$

By comparison with eq. (34), it can be seen that here the scalar value $F^{\mu\nu} F_{\mu\nu}$ measures the relative magnitudes of the electric and magnetic fields.

Similarly, the inner product of the electromagnetic strength tensor can be expressed as a 6-dimensional hypersurface quadratic form:

$$8\pi^2 F^{\mu\nu} F_{\mu\nu} = \vec{x}^T Q \vec{x} \tag{65}$$

Here, the 6-dimensional vector is defined as:

$$\vec{x}^T = \begin{bmatrix} \vec{B} & \frac{\vec{E}}{c} \end{bmatrix}^T = \begin{bmatrix} B_x & B_y & B_z & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \end{bmatrix}^T \tag{66}$$

It can be seen that,

- $F^{\mu\nu} F_{\mu\nu} > 0$: the magnetic field dominates ($\|B\| > \frac{1}{c}\|E\|$), the charge flux, ($\mu_0 \rho \vec{v}$), effect dominates;
- $F^{\mu\nu} F_{\mu\nu} < 0$: the electric field dominates ($\|B\| < \frac{1}{c}\|E\|$), the charge flux effect is weaker than the electric field.

- The function has zero points when $\|\vec{B}\| = \|\frac{\vec{E}}{c}\|$.

According to the definition of eq. (38), if the magnitudes of the negative charge dominate in a region:

$$\rho = \rho_+ - \rho_- < 0 \tag{67}$$

The electromagnetic strength tensor still holds but changes its sign. This strength tensor for a negative charge-dominated region behaves like a gravitational field, where particles attract each other.

We know that photon gases (electromagnetic waves) carry energy and transport it from one region of space to another at the speed of light. The total electromagnetic wave energy stored per unit volume [[3], p.398], $\left(\frac{J}{m^3}\right)$, in a region is

$$u = \frac{1}{2}\frac{\vec{B}^2}{\mu_0} + \frac{1}{2}\varepsilon_0\vec{E}^2 = \frac{1}{2\mu_0}\left(\vec{B}^2 + \frac{\vec{E}^2}{c^2}\right) \tag{68}$$

The first term of the RHS is the magnetic field energy density, and the second term is the electric field energy density.

Using the relation of B=E/C and the wave speed relation of eq. (42), we can re-write the total energy density of the electromagnetic waves, either by B field or by E field, as

$$\frac{\vec{B}^2}{\mu_0} = \varepsilon_0\vec{E}^2 = u \tag{69}$$

Namely, the energy density associated with the B field equals that due to the E field, and each contributes half to the total energy. In other words, the two terms in the RHS of eq. (68) are equal to each other $\left(\vec{B}^2 = \frac{\vec{E}^2}{c^2}\right)$. Substituting this relation into eq. (64), for the electromagnetic waves yields:

$$F^{\mu\nu}F_{\mu\nu} = 0; \quad \vec{x}^T Q\vec{x} = 0 \tag{70}$$

The eq. (64) forms a hypersurface function (a saddle-shaped function), which means the electromagnetic wave is located at a saddle point or minimax point on the saddle-shaped surface, where $F^{\mu\nu}F_{\mu\nu} = 0$.

If the positively and negatively charged particles form a homogenous mixture in a region, or the space is occupied with electron dipoles, accordingly, the positive and negative charge densities are equal to each other:

$$\rho_+ = \rho_-; \quad \rho = \rho_+ - \rho_- = 0 \tag{71}$$

If the Gaussian integral surface embraces a complete electron dipole, and the electron dipole is observed as a complete object, the integral form of Gauss's law reads:

$$\iiint_R \left( \nabla \cdot \vec{E} - \frac{\rho}{\varepsilon_0} \right) = 0 \quad \text{and} \quad \nabla \cdot \vec{E} = 0 \tag{72}$$

As a consequence, the region of space is considered as having no charges. Moreover, if the charge flow velocity in space is random and isotropic, $u = v = w = c$, (wavefront in three dimensions), then the electromagnetic strength tensor automatically degenerates to

$$F^{\mu\nu} = 0 \quad \text{and} \quad F^{\mu\nu} F_{\mu\nu} = 0 \tag{73}$$

That means the equal magnitudes of the E field and the B field in a relativistic sense, as in electromagnetic waves. It will be regarded that in this region there are no "charges" and no "current" ($J_x = J_y = J_z$), so that the space is seen as a "vacuum space".

Based on the aforementioned arguments, it seems we have reason to hypothesize that photon gases may behave as homogenous mixtures of electric dipoles with equal negative and positive charge quantities; they travel in space at the speed of light c, relative to the Lab frame. If the photon gas passes through a strong magnetic field, it will show the magneto-optic Faraday effect; eventually, the photons can be split and produce positrons[4][5][6][7].

Therefore, for electromagnetic waves, we call it a "vacuum" space:

$$\begin{cases} \vec{E} = 0 \quad \text{and} \quad \nabla \cdot \vec{E} = 0 \\ \vec{B} = 0 \quad \text{and} \quad \nabla \cdot \vec{B} = 0 \end{cases} \tag{74}$$

# 4. Scope of the Application and Discussions

Either in the fluid flow field tensor, eq. (11), or in the electromagnetic strength tensor, eq. (46), we do not specify the initial velocities. The differential equations consider only the relative motion (first-order Taylor approximation through the first derivative of the flow field).

It is assumed that the mass particle velocity (or charged particle) should be much smaller than the wave propagation speed of c, either in the flow field or in the electromagnetic field:

$$\vec{v} \ll c \quad \text{or} \quad c \to \infty \tag{75}$$

Here, $c \to \infty$ means the field is incompressible. Details and arguments can be found in the references[1][2].


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The reason for this assumption is that we did not consider the relativity effect. The fluid or electromagnetic field is assumed to be a quasi-incompressible fluid model. If the particle flow velocity approaches the wave propagation speed, the relative effect cannot be ignored.

Actually, the Lorentz factor represents a volume compression factor (length contraction) because of the relative motion for the compressible flow model[1].

If we consider the volume compression effect for a compressible model, the mass volumetric density becomes

$$\rho_m = \gamma\rho_0 \tag{76}$$

where the $\rho_0$ is the mass density when the particle is at rest in the Lab. frame, $\gamma$ is the Lorentz factor or volume compression factor:

$$\gamma = \frac{1}{\sqrt{1-\vec{\beta}^2}} \tag{77}$$

If we take the Taylor expansion of the Lorentz factor:

$$Taylor(\gamma) = 1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \ldots \tag{78}$$

Eq. (76) can be approximated to be

$$\rho_m c^2 = \gamma\rho_0 c^2 \approx \rho_0 c^2 + \frac{1}{2}\rho_0 v^2 + v^2\left[\frac{3\rho_0}{8}\left(\frac{v}{c}\right)^2 + \frac{5\rho_0}{16}\left(\frac{v}{c}\right)^4 + \ldots\right] \tag{79}$$

The first term is the potential energy density at rest relative to the Lab. frame, the second term is the linear kinetic energy density. The third and thereafter terms include the wave propagation speed of c, to consider the volume compression effect because of the relative motion.

If we pick up only the first two terms and ignore the other terms, namely, we ignore the volume compression effect as an approximation; we assume the fluid density and charge density are only a function of space and time, not a function of the wave propagation speed of c and the flow velocity $\vec{v}$.

$$\rho_m = \rho_m(t,x,y,z) \; and \; \rho = \rho(t,x,y,z) \tag{80}$$

As is classical Newtonian mechanics. In a Newtonian mechanics frame, we assume the wave propagation speed is infinitely great. Thereby, the field is incompressible; any disturbance in the field will instantaneously propagate into the whole field without any time lag, no matter how big the field is and how far two points are from each other. Thus, in Newtonian mechanics, we ignore the wave's momentum and energy propagation in the field.

If the photon gas is assumed to be an elastic compressible fluid, the elastic compression bulk modulus can be defined as[1]:

$$B_{photon} = \rho c^2 = \frac{\rho}{\mu_0 \varepsilon_0} \tag{81}$$

## 5. Conclusions

Either mechanic waves or electromagnetic waves carry energy and momentum and propagate in the field at a finite speed. In order to describe the field dynamics correctly and completely, the wave propagation speed term cannot be ignored. The wave speed and particle flow velocity form a four-vector. Based on the four-vector, the field strength tensor can be derived. It is an antisymmetric rank-2 field tensor. In matrix representation, the first row/column is the “translation motion”; other terms are the “rotational motion”. The fluid dynamics strength tensor and electromagnetic field tensor share an essential similarity in their mathematical structure, though they differ significantly in their physical interpretation and governing principles. In the general case, any flow field will produce a vorticity field, as long as the corresponding shear strain is not symmetric. Similarly, a non-symmetric charged particle flow field will produce a magnetic field. The non-symmetry is an intrinsic property of the field. The contraction of the field tensor gives a hypersurface function; it can be positive, negative, or zero, depending on whether “translation motion” or “rotational motion” dominates. The field tensor is zero for electromagnetic waves, which are located at the saddle point of a hypersurface quadratic form, because the energy density associated with the B field equals that due to the E field. It seems that photon gases may be hypothesized as electric dipoles with equal negative and positive charge quantities. Experimental results show that the photon can be split and produce a positron in a strong magnetic field. The Minkowski inner product of the contravariant four-acceleration and the covariant four-velocity gives the fluid dynamic equations, while the contraction of the electromagnetic field strength tensor with the four-velocity of a charged particle results in the Lorentz force law for the electromagnetic field. It is the combination of electric and magnetic force on the charge q in the field. From the dynamic equations, we can deduce the volumetric Lagrangian density for the fields, either for fluid dynamics or for electromagnetic dynamics. It is concluded that the dynamic equations obtained by this approach are equivalent to the Euler-Lagrangian approach. Finally, both approaches will give the same field equations.


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