

Research Article

# Constructing Longlence in the Galerkin-Regularized Nonlinear Schrödinger and Complex Ginzburg-Landau Systems

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(Quasi-)periodic solutions are constructed analytically for Galerkin-regularized or Fourier truncated nonlinear Schrödinger (GrNLS) systems and numerically for those of complex Ginzburg-Landau (GrCGL). Compared to the simultaneous results of other Galerkin-regularized hydrodynamics-type systems, new GrNLS features include the existence of nontrivial monochromatic solutions or condensates (independent of the truncation) and of quasi-periodic tori with and without additional on-torus invariants. Numerical tests find that instability leads such solutions to nontrivial longlence states with remarkable solitonic longons amidst disordered weaker components, corresponding to presumably whiskered tori. The possibility of nontrivial GrCGL longlence states is also discussed for motivation.

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## 1. Introduction

A wide spectrum of multidisciplinary processes in Nature, ranging from hydrodynamics, optics to Bose-Einstein condensates (BEC), can be well modeled by the nonlinear Schrödinger (NLS) or Gross-Pitaevskii (GP) equation. In the transition from classical soliton theory to its quantum counterpart, particularly in the Hamiltonian framework, the NLS equation is often regarded as simpler and more fundamental than the Korteweg-de Vries (KdV) equation<sup>[1]</sup>. The two models also present different nonlinear physics, with no KdV but possible NLS finite-time or asymptotic blow-up (c.f., Ref.<sup>[2]</sup> and references therein for the focusing cubic case focused here.)

Building on previous studies of periodic patterns in Galerkin-regularized or Fourier-truncated (Gr) systems, such as the Burgers-Hopf (BH), compacton-and-peakon (CP), and KdV equations<sup>[3]</sup>, as well as the untruncated even-odd alternating Korteweg-de Vries (aKdV) equation<sup>[4]</sup>, we analyze the Gr-system with cubic nonlinearity for the order parameter (GrNLS or GrGP). The common features include the loss of some of the invariants, sometimes infinitely many (and the integrability) for some of the systems, leaving a few rugged ones, the regularization of the structures (singular for BH and CP), and the admittance of new travelling waves, (quasi-)periodic orbits and pseudo-periodic (statistically) stable “longulent” state or “longulence”; the latter is characterized by solitonic “longons” accompanied by the less-ordered (in general, chaotic) component and corresponds to a presumably whiskered torus. [For clarity, here we should require the

nontrivial longulent state or longulence

to be also with significant truncation effect; so, neither the state without a disordered component nor that with solitonic structure(s) but also already with convergence to the full-/untruncated-system one satisfies this criterion.] The different scenario here is that, unlike those systems where torus-specific or on-torus invariants (varying outside the torus, thus not rugged) have to be introduced to construct quasi-periodic solutions to support the *a-posteriori* Kolmogorov-Arnold-Moser (KAM) argument for the longulent states, now GrNLS rugged invariants are sufficient to augment quasi-periodic orbits. Such aspects of Gr-systems resemble those of the traditional integrable systems, which is why below we have to discuss the relevant backgrounds of the latter.

The complex Ginzburg-Landau (CGL) equation is coefficient-complexified, or, with small imaginary part(s), a perturbation resulting in damping and (autonomous) driving to NLS<sup>[5]</sup>. Actually, in our context, CGL also supports solitonic structures of great physical importance (e.g., recently the application of Kerr and Nozaki-Bekki solitons in optics<sup>[6]</sup>) and presumably high-dimensional whiskered tori<sup>[7][8]</sup>, with also chaotic dynamics somewhat trackable<sup>[9]</sup> and mimicking aspects of fluid turbulence<sup>[10]</sup>. So, it is more helpful to consider GrNLS in the broader context of GrCGL<sup>[11][12][13]</sup> <sup>[14]</sup> which will also be remarked on, with preliminary analyses and numerical tests to motivate attacking the much more challenging problem.

A lot of relevant studies having already been discussed in Refs.<sup>[3][4]</sup>, we only briefly mention the most relevant here. Concerning our periodic problem, associated to NLS, probably the most remarkable phenomena are the fractalization and quantum revival, associated to the Talbot effect<sup>[15][16][17]</sup>, and

solitons (see, e.g., Refs.<sup>[18][19]</sup> and references therein for traditional Hamiltonian theory and most recent developments on soliton gas.) For the associated spectral theory of similar systems integrable by inverse scattering transform, see, e.g., Ref.<sup>[20]</sup> and references therein for infinite-gap theory, and, Ref.<sup>[21]</sup> and references therein for recent advancements of unified transform/Fokas method over the classical inverse scattering transform for periodic problems. The classical studies of periodic and quasi-periodic NLS solutions can be found in, e.g., Refs.<sup>[22][23][24]</sup>. Finally, related to the instability in our numerical tests, note that, even for the infinite-line problem, the modulational-instability-stage problem is nontrivial (see, e.g., the recent different results of Zakharov-Gelash<sup>[25]</sup> and Biondini-Mantzavinos<sup>[26]</sup>.)

Can we learn or borrow wisdoms from such progresses for studying nonintegrable soliton gases<sup>[27]</sup> or general conservative dynamics? The most closely relevant models, at least formally, are of course their Galerkin regularizations which preserve important parts of the mode interaction structure, including some conservation laws<sup>[3][28]</sup>. [A sidenote about the terminology: for the real variable  $u$  dealt with previously<sup>[3]</sup>, conjugate Fourier coefficients  $\hat{u}_{\pm k}$  work together to form a “mode”; now, the (Gr)NLS variable is complex, with independent “modes” of wavenumbers  $\pm k$ . So, to avoid confusion, we will try to resist using such a notion.] Although the Lax pair structure is absent, we aim to recover and extend certain elements, such as by combining Fourier expansion with truncation, and exploring other analytical approaches. For example, the introduction of torus-specific invariants<sup>[3][4]</sup> represents an attempt to achieve a breakthrough, potentially leading to longlived states that are closer to the exact solutions used as the initial data or are even corresponding precisely to the whiskered tori.

Using GrNLS or GrGP with sufficient number of the eigen modes of the harmonic oscillator potential to probe the quasi-integrability of the full dynamics<sup>[29]</sup> and the higher-dimensional GrGP thermalization aspect (e.g., Ref.<sup>[30]</sup> and references therein) belong to different lines of research. Our distinct work is organized as follows. Sec. 2 finds analytically the exact GrNLS travelling-wave and (quasi-)periodic multi-frequency solutions, emphasizing the critical sets specified by rugged and torus-specific invariants; Sec. 3 discovers the universal longlived states numerically, with remarks including preliminary considerations on the (non)persistence of GrNLS longons against the GrCGL perturbation; and, finally, Sec. 4 naturally extends the discussion of GrCGL itself, including the expectation of quasi-periodic tori and longliveness, and, the challenge to construct them.

## 2. The problem and solutions

We start with the Hamiltonian formulation<sup>[1]</sup> of the  $2\pi$  periodic NLS problem directly in the Fourier ( $k$ ) space, as Gardner did for KdV<sup>[31]</sup>, deferring the Poisson structure in physical ( $x$ ) space to the point when needed.

Let  $\Psi(x, t) = \sum_n \hat{\Psi}_n(t) e^{inx}$  (where  $i^2 = -1$ ) with  $2\pi$   $x$ -period solve the NLS equation (2.3) below with  $g = 0$ :

$$\dot{q}_k = \frac{\partial \mathcal{H}}{\partial p_k}, \quad \dot{p}_k = -\frac{\partial \mathcal{H}}{\partial q_k}, \quad (2.1)$$

with  $q_k = \hat{\Psi}_k$ ,  $p_k = i \hat{\Psi}_k^*$  for the Fourier coefficient  $\hat{\Psi}_k$  and its conjugate  $\hat{\Psi}_k^* = (\widehat{\Psi^*})_{-k}$  of each wavenumber  $k$ , and,

$$\mathcal{H} = \sum_n \left( n^2 |\hat{\Psi}_n|^2 \mp \sum_{k+l-j=n} \hat{\Psi}_k \hat{\Psi}_l \hat{\Psi}_j^* \hat{\Psi}_n^* \right). \quad (2.2)$$

The upper sign (“−” here) is for the focusing case, and the lower (“+” here) for defocusing: we will eventually focus on the focusing case.

For Galerkin-regularized or Fourier-truncated  $\psi = \sum_{|n| \leq K} \hat{\psi}_n(t) e^{inx}$  ( $\Leftrightarrow \Psi$ , through a simple “hard cutoff” as a pseudo-differential operation in the language of analysis<sup>[3]</sup>) and “well-prepared” initial data  $\psi(0)$  with  $\hat{\psi}_m(0) = 0 \forall m > K$ <sup>1</sup>, the GrNLS system involves a Galerkin function/force  $g$  with the effect of projecting the dynamics on to the space of  $|k| \leq K$ ,

$$i \partial_t \psi + \partial_{xx} \psi \pm 2|\psi|^2 \psi = g; \quad (2.3)$$

$$i \hat{\psi}_n - n^2 \hat{\psi}_n \pm 2 \sum_{k+l-j=n} \hat{\psi}_k \hat{\psi}_l \hat{\psi}_j^* = \hat{g}_n; \quad (2.4)$$

$$\hat{g}_m = \begin{cases} \pm 2 \sum_{k+l-j=m} \hat{\psi}_k \hat{\psi}_l \hat{\psi}_j^* & \text{for } K < |m|, \\ 0 & \text{otherwise.} \end{cases} \quad (2.5)$$

It is seen that the above Hamiltonian formulation, with

$$H = \sum_{|k| \leq K} \left( k^2 |\hat{\psi}_k|^2 \mp \sum_{\substack{|i|, |l|, |j| \leq K \\ i+l-j=k}} \hat{\psi}_i \hat{\psi}_l \hat{\psi}_j^* \hat{\psi}_k^* \right) \Leftrightarrow \mathcal{H},$$

still applies, similar to the GrKdV case<sup>[3][28][31][32]</sup>, and the other two invariants,

$$M_{-1} = \sum_{|k| \leq K} |\hat{\psi}_k|^2 \quad \text{and} \quad M_0 = \sum_{|k| \leq K} k |\hat{\psi}_k|^2, \quad (2.6)$$

are still conserved: Gardner<sup>[31]</sup> actually used the finite-mode case for the intermediate stage in proving some results — see Ref.<sup>[28]</sup> for physical-space analysis, or, Ref.<sup>[32]</sup> for a direct calculation of Fourier mode interactions to show the preservation of GrBH  $H$ . No other NLS invariants can be found to be preserved by GrNLS. Such (non)conservation laws can be argued similarly to the analysis for GrBH in Ref.<sup>[28]</sup>, but it is actually more straightforwardly seen in  $k$ -space, following Gardner<sup>[31]</sup>, which is one of the reason for us to emphasize the  $k$ -space formulation in the above.

Note that, unlike in quadratic interaction with  $\hat{g}_m \equiv 0$  for  $|m| > 2K$ <sup>[31]</sup>, we now have  $\hat{g}_m \equiv 0$  for  $|m| > 3K$ .

### 2.1. One- and two-frequency/phase solutions

With the cubic nonlinearity, a remarkable apparent difference between the current quartic interaction to the triadic one for the quadratically nonlinear systems is that a  $\hat{\psi}_k$  can have (changes by the) interaction with itself. So, the simplest nontrivial/nonzero GrNLS solution to Eq. (2.4) is that occupying only a single wavenumber  $k = S$  and satisfying

$$\dot{\hat{\psi}}_S = S^2 \hat{\psi}_S \mp 2|\hat{\psi}_S|^2 \hat{\psi}_S, \quad (2.7)$$

also solving the untruncated NLS, the well-known monochromatic wave or condensate. We look for those slightly more nontrivial than the one occupying only the wavenumber 0. Assuming  $\hat{\psi}_S = A e^{i\theta_S(t)}$  with constant  $A = \alpha S$ , we get

$$\hat{\psi}_S = A e^{-i(1 \mp 2\alpha^2)S^2 t} \text{ or } \psi = A e^{iS[x - (1 \mp 2\alpha^2)St]} \quad (2.8)$$

where  $\theta_S(0) = 0$  is taken for simplicity: the focusing GrNLS waves become stationary for  $\alpha = 1/\sqrt{2}$ .

[It deserves to reiterate that the above GrNLS solution occupying a single wavenumber and solving also the original NLS equation is due to the odd-order nonlinearity, which is not the case for (Gr)KdV, (Gr)BH and (Gr)CP<sup>[31]</sup> with even-order nonlinear terms.]

Extremizing GrNLS Hamiltonian  $H$  constrained by  $M_{-1}$  and  $M_0$ , with the respective (real) lagrangian multiplier  $-\lambda_{-1}$  and  $-\lambda_0$ , leads to

$$\frac{\delta H}{\delta \psi^*} - \lambda_{-1} \frac{\delta M_{-1}}{\delta \psi^*} - \lambda_0 \frac{\delta M_0}{\delta \psi^*} = 0 : \quad (2.9)$$

$$\dot{\hat{\psi}}_k = k^2 \hat{\psi}_k \mp 2 \sum_{\substack{|k|, |m|, |l|, |j| \leq K \\ j+l=m+k}} \hat{\psi}_j \hat{\psi}_l \hat{\psi}_m^* = (\lambda_{-1} + \lambda_0 k) \hat{\psi}_k, \quad (2.10)$$

resulting in the solutions  $\hat{\psi}_k = \hat{\psi}_{k0} e^{-i(\lambda_{-1} + \lambda_0 k)t}$ . Initial  $\hat{\psi}_{k0}$  is determined by the equality of the middle and right-hand sides, and the time dependence by the left- and right-hand sides. Such a special-

solution approach resembles the integrable cases for finite-band solutions<sup>[24][33][34]</sup>.

Travelling waves may be realized by Eq. (2.10) with  $\lambda_{-1} = 0$ . When occupying multiple wavenumbers, the solutions with  $\lambda_{-1} \neq 0$  to Eq. (2.10), or even with more general  $\omega_K(k)\hat{\psi}_k$  on the right hand side, are generally not travelling waves. We now consider solutions occupying only wavenumbers of  $k = \pm S$ . With whatever dispersion function  $\omega_K(k)$ , we have two frequencies for respectively the two modes of  $\pm S$ .

As we have seen and can be further expected, our analyses for focusing and defocusing cases are formally the same. So, from now on we restrict ourselves to the focusing case to be more focused, leaving the possible other interesting aspect of defocusing GrNLS aside for the time being.

Then, because the modes occupying  $k = \pm S$  excite only others of  $k = \pm 3S$ , we obtain, for instance, the solutions occupying  $k = \pm S$  for  $S \leq K \leq 3S - 1$ , with  $|\hat{\psi}_{\pm S}|^2 = (S^2 \pm 3\lambda_0 S - \lambda_{-1})/6$ . The latter is not always guaranteed to be nonnegative by arbitrary combinations of  $\pm S$ ,  $\lambda_{-1}$  and  $\lambda_0$ , thus indicating nontriviality of the existence of such solutions specifically and of those occupying more wavenumbers in general. The solutions read:

$$\begin{aligned} \psi = & \frac{\sqrt{S^2 + 3\lambda_0 S - \lambda_{-1}}}{\sqrt{6}} e^{i[Sx - (\lambda_{-1} + \lambda_0 S)t + \theta_+]} \\ & + \frac{\sqrt{S^2 - 3\lambda_0 S - \lambda_{-1}}}{\sqrt{6}} e^{-i[Sx + (\lambda_{-1} - \lambda_0 S)t + \theta_-]}. \end{aligned} \quad (2.11)$$

Such  $\psi$ , composed of two travelling-wave components with initial phases  $\theta_{\pm}$ , by itself is not for travelling waves in general, being quasi-periodic when  $\lambda_{-1}$  and  $\lambda_0$  are rationally independent/incommensurate. The reduction with  $\lambda_0 = 0$  and  $\theta_+ = -\theta_- = \theta_0$  for standing or rotating waves reads

$$\psi = \frac{2\sqrt{S^2 - \lambda_{-1}} \cos(Sx)}{\sqrt{6}} e^{-i\lambda_{-1}t} e^{i\theta_0}. \quad (2.12)$$

Note that these GrNLS, but not NLS, non-travelling-wave solutions are still periodic in time.

The form of Eq. (2.11) is referred by some authors to a "travelling wave", which to our point of view is not accurate, due to the change of the wave form by the dispersion (different phase velocities for the  $\pm S$ -components). This is by itself only a matter of terminology, but concerning the quasi-periodicity nature of the solution, it deserves to be pointed out to avoid possible confusion: for example, Ref. <sup>[5]</sup> claimed for generalized NLS that more invariants other than  $M_{-1}$ ,  $M_0$  and  $H$  should be used for

constructing quasi-periodic solutions, while here the above GrNLS solutions can already be quasi-periodic (in Ref.<sup>[35]</sup>, the author actually used the simple critical points of the spectrum).

## 2.2. Additional torus-specific invariants

In general, we can have GrNLS exact solutions with modes occupying any amount of wavenumbers with the above mentioned  $\omega_K$  containing accordingly the parameters to quantify the corresponding frequency components. We however do not know any other generic global rugged invariants for defining the critical set of some combined functional as we did in Sec. 2.1 to realized such solutions.

As in Ref.<sup>[3]</sup>, we can introduce torus-specific invariants to construct such high-tori. A natural choice is the extension of Eq. (2.6),

$$M_\tau = \sum_{|k| \leq K} k^{\tau+1} |\hat{\psi}_k|^2, \quad (2.13)$$

now with  $\tau = 1$  and the associated

$$\frac{\delta H}{\delta \psi^*} - \lambda_{-1} \frac{\delta M_{-1}}{\delta \psi^*} - \lambda_0 \frac{\delta M_0}{\delta \psi^*} - \lambda_1 \frac{\delta M_1}{\delta \psi^*} = 0, \quad (2.14)$$

and, the right-hand side of Eq. (2.10) replaced with  $(\lambda_{-1} + \lambda_0 k + \lambda_1 k^2) \hat{\psi}_k$  for a 3-frequency solution set. We can find special solutions to such a GrNLS system. Probably the simplest are those of modes occupying only modes of  $k = 0$  and  $\pm S$ , in which case, with further simplification by restricting to real initial  $\psi$ , we can solve the algebraic equation and obtain, for instance,  $\hat{\psi}_k = \hat{\varphi}_k e^{-i(\lambda_{-1} + \lambda_0 k + \lambda_1 k^2)t}$  with  $\hat{\varphi}_k = 0$  except for

$$\begin{cases} \hat{\varphi}_S = \frac{\sqrt{2\lambda_{-1} - \lambda_0 S - \lambda_1 S^2 - S^2}}{\sqrt{30}} e^{i\theta_0}, \\ \hat{\varphi}_0 = \frac{\sqrt{\lambda_{-1} + 2\lambda_0 S + 2S^2 + 2\lambda_1 S^2}}{\sqrt{10}}, \\ \hat{\varphi}_{-S} = \frac{\sqrt{2\lambda_{-1} - \lambda_0 S - \lambda_1 S^2 - S^2}}{\sqrt{30}} e^{-i\theta_0}. \end{cases} \quad (2.15)$$

The phase parameter  $\theta_0$  can be arbitrary. Such a solution describes the combination of a standing wave of vanishing wavenumber and two travelling waves of wavenumbers  $\pm S$ , interacting to excite modes of  $k = \pm 2S$ , thus also valid for  $|k|$  truncated up to  $K = 2S - 1$ .

As shown for the GrBH case<sup>[3]</sup>, the above  $M_1$  Poisson commute with  $H$  (thus invariant) and  $\lambda_{-1} M_{-1} + \lambda_0 M_0$  on the torus, and like the example demonstrated there, we also seem to find the above solutions closer to the final longulent states, from the observation of the numerical test below.

In the current context, the above torus-specific invariant bears some similarity to the “test functional”  $\mathcal{F}$  of Ref.<sup>[51]</sup> for the Melnikov method, so we restate the result and remark on the differences below.

For comparison, we follow closely Ref.<sup>[51]</sup> for the symbolic convention and terminologies; see, e.g., Ref. <sup>[11]</sup> for more background on the complete theory of the infinitely many NLS invariants (or “local functionals/integrals of motion”).

The evolution of a functional  $\mathcal{F}$  under the NLS ( $\Psi$ ) flow obeys

$$d\mathcal{F}/dt = \{\mathcal{F}, \mathcal{H}\} = -i \int_0^{2\pi} \frac{\delta\mathcal{F}}{\delta\Psi} \frac{\delta\mathcal{H}}{\delta\Psi^*} - \frac{\delta\mathcal{F}}{\delta\Psi^*} \frac{\delta\mathcal{H}}{\delta\Psi} dx \quad (2.16)$$

the right-hand side of which indicates the Poisson structure which is preserved by the GrNLS flow with the corresponding  $F$  and  $H$  defined by the truncated  $\psi$ , as mentioned earlier.  $\mathcal{M}_{-1}$  and  $\mathcal{M}_0$  are preserved by GrNLS for the reason similar to the KdV or Burgers-Hopf case<sup>[31][28][31][32]</sup>, as mentioned before. Presumably any (higher-order) NLS invariant  $\mathcal{M}$  other than  $\mathcal{M}_{-1}$  and  $\mathcal{M}_0$  (with the corresponding  $M$  redefined by  $\psi$ ) are not supposed to be still preserved by GrNLS, also similar to KdV or Burgers-Hopf.

Now, for the tori defined by Eq. (2.14), we can use the latter to replace  $M_1$  in computing  $\{M_1, H\}$  which then is seen to vanish, thus the invariance of  $M_1$  on this torus, because the other three integrals mutually Poisson commute. Similarly,  $M_1$  Poisson commutes with  $\lambda_0 M_0 + \lambda_{-1} M_{-1}$  (but not necessarily with  $M_0$  or  $M_{-1}$ .) It can be checked that, in general, without the constraint of Eq. (2.14),  $M_1$  is not invariant.

From the above explanation, we see that the “test” functional  $\mathcal{F}$  used in Ref.<sup>[51]</sup> to establish the persistence criteria has some similarity with our torus-specific invariant but is obviously of different nature, for not used for defining the tori and for the requirement to Poisson commute with the other three functionals.

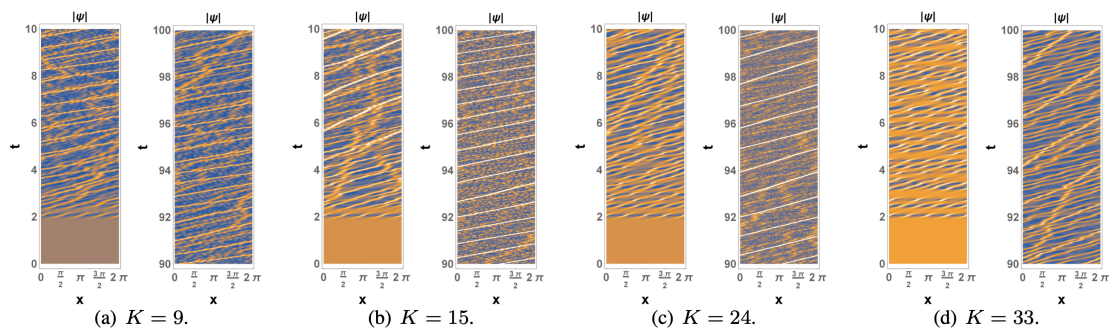
### 3. Longlence

As mentioned, Eq. (2.7) of the monochromatic wave or condensate solves also NLS, which means that the truncation is not relevant for the solution itself. However, with (modulational) instability, the final states should depend on the truncation threshold  $K$ . For  $K \rightarrow \infty$ , if the solution for the NLS is well-behaved (with no singular behaviors such as clapse or blow-up), then the GrNLS solution should



converge to that of NLS (a kind of “focusing NLS persistence to the Galerkin perturbation”). [The well-known pseudo-conformal transformation formally turns into finite-time blowup which does not necessarily always happen in the dynamics.] In general, the focusing (Gr)NLS, like (Gr)BH and (Gr)CP in Ref.[3], do not have such convergence or persistence property. Also, with an additional well-designed potential, commonly as in the so-called GP equation of BEC, convergence of GrNLS to NLS can also happen, which is the case in the work of Bland et al.[29] for the defocusing/repulsive case, who use the Galerkin approximation with the harmonic-oscillator eigenmodes rather than our Fourier modes. Not surprisingly, they found dynamics close to full GP, which is similar to one of our cases below. Working nevertheless with finite  $K$ , we will not consider the issue of convergence any more except for one apparent case.

### 3.1. GrNLS longulence



**Figure 1.**  $\alpha = 1, S = 3$ : lighter colors indicate larger values in all figures, coded per panel.

Fig. 1 presents the carpets/contours of the GrNLS fields starting from Eq. (2.7) at  $t = 0$ , showing the evolution into a stable (statistically, with respect to the small scale weaker dis-ordered component) pseudo-periodic state (after around  $t = 2$ ) with longons, which is the case even for the case of  $K = 1$  (not shown). Different  $S$  and  $K$  are quantitatively different, but with the same qualitative scenario characterized by solitonic longons amidst disordered components, appearing to converge to NLS with smaller speeds of the dominant larger- $K$ -GrNLS longons (for such a case with presumably no blow-up): of course, due to the finiteness of the computations, we can not absolutely exclude the possibility of blow-up after an extremely or even infinitely long time (and thus no convergence).

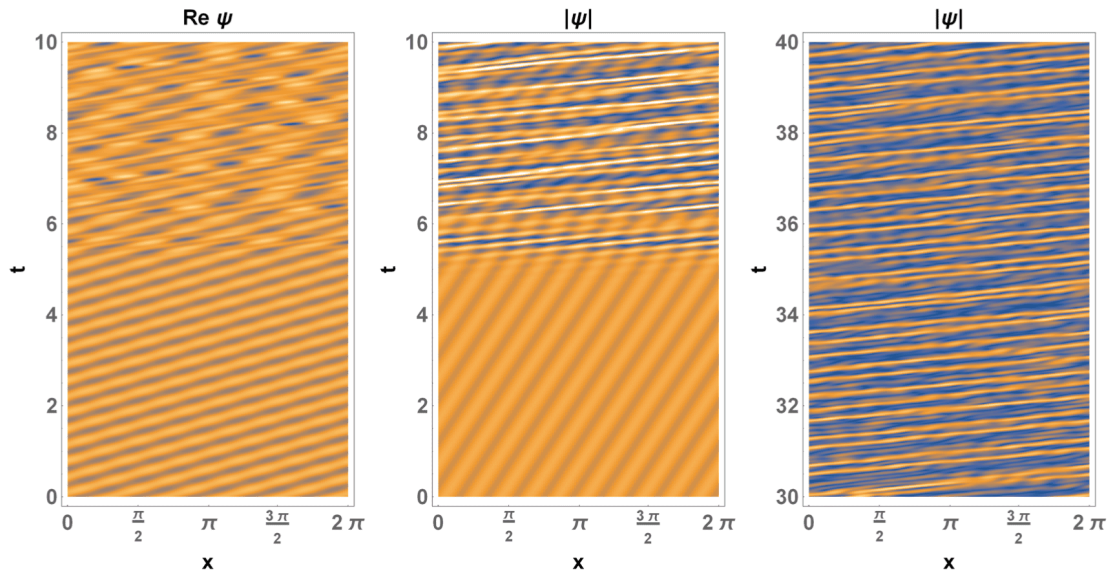
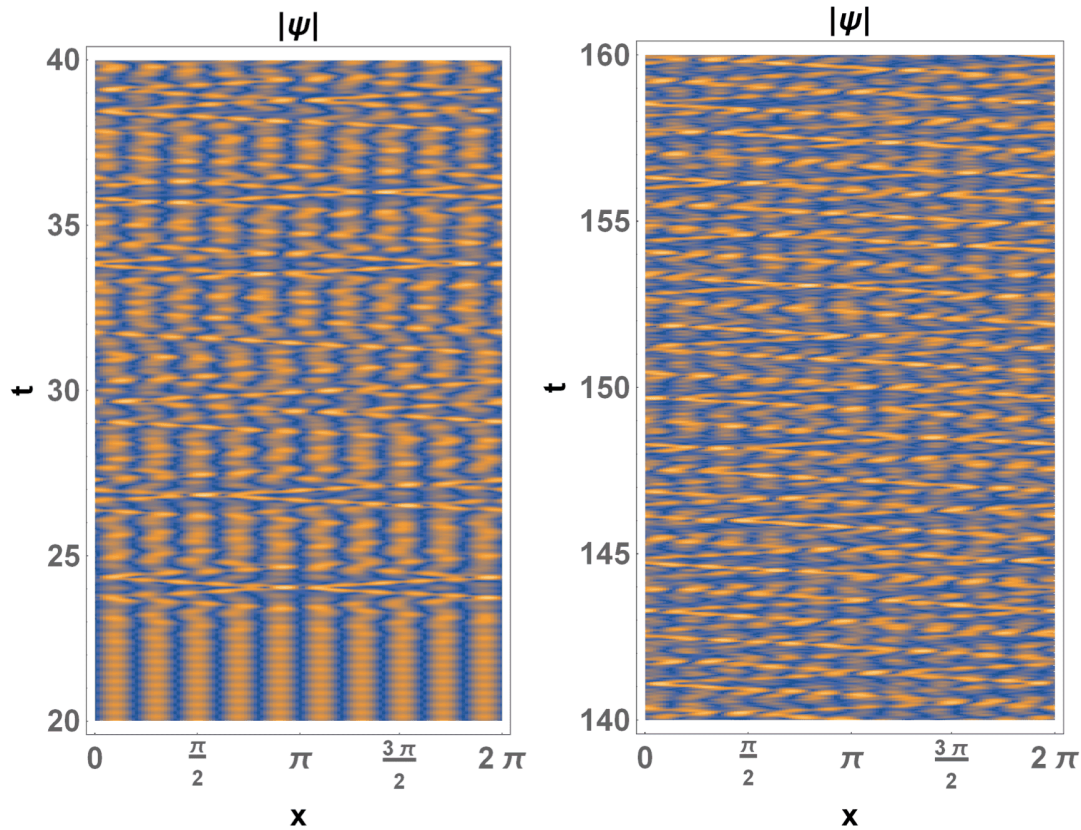


Figure 2.  $S = 5$ ,  $K = 9$ ,  $\lambda_0 = 11.3 + \pi$  and  $\lambda_{-1} = \frac{\sqrt{6}}{3.7}$ .

Fig. 2 for the case of two-frequency (2.11), including the real part of  $\psi$  for visualization of the 2-frequency character of the early exact solution, is similar, with no essential differences between periodic and quasi-periodic, satisfying the Diophantine condition or not, cases, as we see in the GrBH results<sup>[3]</sup> (thus other parameterizations with, say,  $\lambda_0 = 13 + \sqrt{2}$  and  $\lambda = 2/3$  are not shown in Fig. 2).



**Figure 3.**  $\theta_0 = \pi/2$ ,  $S = 5$ ,  $K = 9$ ,  $\lambda_{-1} = \lambda_1 = -1$  and  $\lambda_0 = -2.1\lambda_{-1}S$ .

Finally, we present results corresponding to Eq. (2.15) in Fig. 3 which, like the results in Ref. with additional on-torus invariants for GrBH, may be observed from the pattern to indicate that the well-developed longlence is closer to the exact solution, compared to Figs. 1 and 2: like in the latter but with slightly more careful observation, we still see similar solitonic longons in the well-developed stage (right panel).

All the numerical results indicate a kind of universal attractor characterized by solitonic longons among less-ordered components, which should be underlined by some high-dimensional whiskered tori, but some details and points should be emphasized and further clarified, thus the following extended remarks.

Even the quasi-periodic solutions with rugged invariants are not sufficient to have enough stability, resulting in far different longlentic states, which may be a further indication of the relevance of additional on-torus invariants. So, as in Ref.<sup>[3]</sup> and tried in Fig. 3, it is possible to similarly generate

invariant tori with much closer longlence developed, if not stable. In all our numerical tests of multi-frequency tori, except for minor “improvements” concerning stability, rational independence or the Diophantine condition of the frequencies does not appear to have essential effects on the developments of longlence (thus other similar results corresponding to Fig. 3 not shown). The indication seems that the (universal) longlulent state or the torus is the only stable attractor or that the initial data prepared as such are all close enough to the basin of the longlulent attractor.

A caveat is that  $M_1$  has not been chosen to Poisson commute with both  $M_{-1}$  and  $M_0$ . For given parameters such as  $S$  and  $K$ , there is of course the problem of what the right choice of the on-torus invariants is, which might benefit from other broader issues such as the (Gr)CGL quasi-periodic tori to which we will come back. Systematic improvements might be made by appropriate choice of more torus-specific invariants, but we so far do not really have a good theory, for lack of a mathematical structure, say, as for the NLS<sup>[1]</sup>. The conjecture follows again<sup>[3]</sup>: the “right” choice of the torus-specific invariants should be such as to have the total number be the degrees of freedom of the truncated system that a new kind of “pseudo-integrability” be established, in the sense of specifying precisely a longlulent state.

A natural further question is the (non)persistence of the (solitonic) GrNLS longons to GrCGL perturbation, which however appears an even more difficult step than that from Li et al.<sup>[36]</sup> to Cruz-Pacheco et al.<sup>[5]</sup>. Nevertheless, this is a problem associated to the above pseudo-integrability issue, and some tentative numerical experiments have also been performed. So, let’s go a bit further for motivations.

### 3.2. No GrCGL nontrivial longlence?

For the CGL equation,  $\hat{i} \partial_t \Phi + C \partial_{xx} \Phi + 2G|\Phi|^2 \Phi = 0$  where  $C = 1 + \hat{i} \eta$  and  $G = 1 + \hat{i} \epsilon$ , with possibly an additional term to be picked up later. The one-wavenumber-occupation GrCGL equation reads

$$\hat{i} \hat{\phi}_S = CS^2 \hat{\phi}_S + 2G|\hat{\phi}_S|^2 \hat{\phi}_S, \quad (3.1)$$

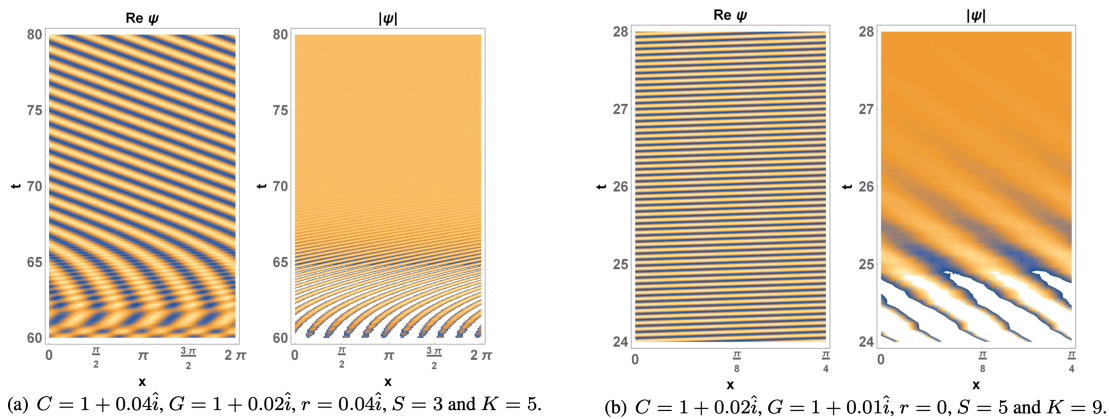
and the solution similar to Eq. (2.8) is

$$\psi = \alpha S e^{\hat{i} S [x - (C - 2\alpha^2 G) S t]}. \quad (3.2)$$

When  $Im(C) = 2\alpha^2 Im(G)$ , or  $\eta/\epsilon = 2\alpha^2$ , we have the travelling wave solution with wave speed  $c = [Re(C) - 2\alpha^2 Re(G)]S$  or  $c = (1 - 2\alpha^2)S$  as for the GrNLS, and, again also solves the untruncated CGL. No nontrivial (statistically) stable GrCGL longlulent states have been found in our numerical

experiments starting from this solution, with everything else as in the GrNLS (Fig. 1), neither from the (quasi-)periodic (2.11) and (2.15). As we mentioned, it is possible that with more appropriate GrNLS torus-specific invariants, we could have the final longlence very close to the initial one; then, it could be that persistence to the GrCGL perturbation takes place under suitable conditions.

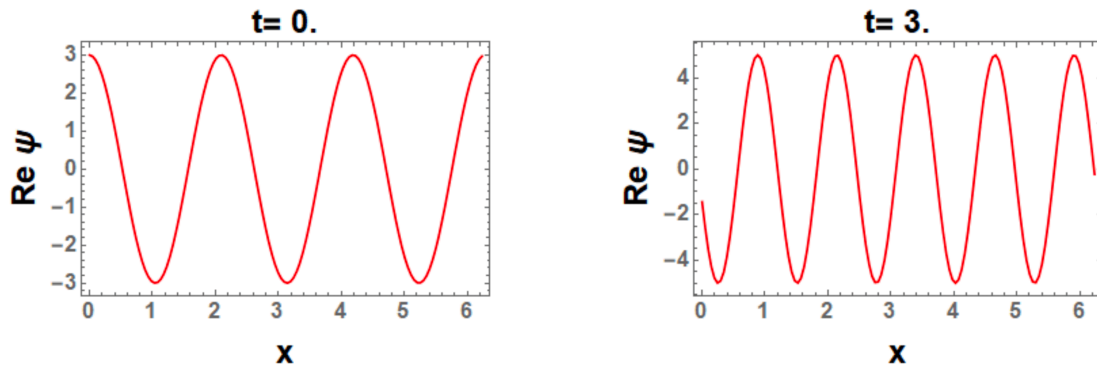
However, from weakly-nonlinear dynamics point of view, it appears important to have the linear damping or forcing be appropriately balanced, also linearly. With constant  $r$  for such an additional perturbation term  $\hat{i}r\Phi$ , thus balance formally taking place only at a single scale (wavelength), it is possible but nontrivial for the existence of (quasi-)periodic solutions, since multi-scales are excited by nonlinearity. The necessity of the linear term is the usual case in physics<sup>[6]</sup>, and, in mathematical treatments only special choice of the parameters have been found possible to have the CGL quasi-periodic solutions<sup>[5][8][7][35]</sup>. Note that Luce<sup>[14]</sup> used  $r$  as an independent and increasing parameter to enhance the forcing, and thus presumably the nonlinearity, which raises the complexity of the dynamics through more homoclinic explosions and other bifurcations.



**Figure 4.**  $\theta_0 = \pi/2, \lambda_{-1} = \lambda_1 = -1, \lambda_0 = -2.1\lambda_{-1}S$ .

For GrCGL, the corresponding  $\hat{i}r\phi$  then may not be really necessary, since the Galerkin regularization term  $g$  can make the nonlinearity strong, thus even more nontrivial balance could happen. When the inertial manifold property is in control and the truncation wavenumber is large, things become kind of trivial because of the convergence to the full CGL, in which case the truncation effect is a small perturbation. Indeed, we found in various numerical tests the convergence to “clean” periodic solutions, in absence of any signature of disorder, with and without the  $\hat{i}r\phi$  term from the

corresponding cases with everything else the same as used for GrNLS. For example, Fig. 4, for cases corresponding to Fig. 3 starting from Eq. (2.15), shows convergence to constant-amplitude travelling-wave solutions which do not satisfy the *nontrivial longlence* criteria given in the introductory discussion: the travelling waves are of shortest respective wavelengths, i.e.,  $|k| = K$  and of the form (3.2) found earlier (verified by checking the relations between the respective wave amplitudes and speeds) but not of the linearized dynamics (plotting only for the smaller region in the case of  $S = 5$  and  $K = 9$  is to avoid the artificial Moré patterns.) Note that, unlike the KAM results in Refs.<sup>[7]</sup><sup>[8]</sup>, we did not start from the solutions of the linearized system, thus not of that perturbative nature.



**Figure 5.** Initial data (left) prepared with  $\alpha = 1$ ,  $C = 1 + 2\hat{i}$ ,  $G = 1 + \hat{i}$ ,  $r = 0$ ,  $S = 3$  in Eq. (3.2), and a final condensate wave profile (right).

Results similar to Fig. 4 for GrCGL with large  $\eta$ ,  $\epsilon$ , with or without (large)  $r$ , are also found, with faster convergence. For example, Fig. 5 presents the transition of condensate from  $k = S$  to  $k = K$  purely by instability in the pseudo-spectral computation, subjecting only to roundoff error perturbation as in all previous cases, with the final state return to the monochromatic wave (3.2) of the same  $\alpha = 1$  but of different wave number. [Periodic solutions of such large  $K$ 's were never reported, to the best of our knowledge (for instance, Luce<sup>[14]</sup> studied bifurcation and chaos of low-dimensional dynamics).]

Other results, such as the convergence to the condensate at  $K$  with  $\alpha \neq 1$ , with other initial data, have also been collected. No satisfying theory for these observations are available so far (see below), which definitely is motivating a specific more systematic study. Having not found quasi-periodic or longulent GrCGL states though, we have no reason to exclude the existence. Combing with the chaotic

aspect<sup>[14]</sup> and references therein), we tend to believe that nontrivial GrCGL longulent states are still possible, deserving further remarks.

## 4. Expectation

Good understanding of relevant GrCGL dynamics beyond the small perturbation to GrNLS can be beneficial for the latter, by learning from the differences, say, as is the purpose here. So, we proceed by noting that the CGL damping and forcing may balance on particular orbits/tori in such a way that some on-torus invariants present and longulence emerges.

Although explicit multi-frequency CGL solutions have not been found analytically, there are suggestions of the existence of quasi-periodic (whiskered) tori<sup>[5][7][8][35]</sup>. Note that the formulas from the KAM method [Eqs. (6) and (3.22) of, respectively, Refs.<sup>[7][8]</sup>] are with whiskered components, qualitative or asymptotic, and, are only for particular choices of parameters. What's more, the term proportional to  $\Phi$  (also physically important<sup>[6]</sup>) is crucial in their KAM method, although it balances only one scale. In the GrCGL case with truncation  $g$ , this term however appears not as needed. Nevertheless, we have not yet been able to construct GrCGL nontrivial longulent states from the corresponding GrNLS data. New techniques are needed. The Lyapunov-function approach seems promising, with however caveats: for instance, obviously,  $\int u^{2n} dx$  for any integer  $n > 0$  is a Lyapunov function for  $u_t - u^2 u_x = -u$  controlling real  $u$ , and the pattern selection was not found to minimize the one written down in Ref.<sup>[37]</sup> for the slightly more complex dynamics.

As already mentioned, the on-torus-invariant trick shares some similarity with part of the Melnikov method used by Cruz-Pacheco, Levermore and Luce<sup>[5]</sup>, and we may hopefully expect further combination for more powerful techniques. For general (Gr)CGL with coefficients neither appropriate for a perturbative treatment<sup>[5]</sup> nor so special to have the nearly (quasi-)periodic solution for the corresponding linearized system<sup>[7]</sup>, it is not impossible still to simultaneously set up the right multiple on-torus invariants, correspondingly an appropriate Lyapunov function, to construct specific invariant (whiskered) tori for nontrivial longulence: trivially,  $M_{-1}$ ,  $M_0$  and  $H$  are constants in the late stages of the evolutions presented in Fig. 4.

If the GrCGL quasi-periodic or longulent states can be found as for GrNLS and other hydrodynamic-type Gr-systems<sup>[3]</sup>, then we are closer to the *a-posteriori* KAM scheme that would assure the existence of (whiskered) tori close by (e.g., de la Llave and collaborators' recent works, including Ref.<sup>[38]</sup> on



partial differential equation and Ref.<sup>[39]</sup> on maps, and references therein). Preliminary numerical experiments of the hydrodynamic-type Gr-systems studied in Ref.<sup>[3]</sup> indicate the persistence of those longlulent states against appropriate (small) forcing and damping, which, being more technically subtle (with differences between independent and autonomous/GrCGL forcing) and deserving a more specific study though, appears to support the existence of nontrivial GrCGL longlulence.

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## Footnotes

<sup>1</sup> “Ill-prepared” initial data  $\psi(0)$ ,  $\exists m > K$ ,  $\hat{\psi}_m(0) \neq 0$  are not of our interest here. Also, we resist introducing further regularizations, such as the truncations on the quadratic interactions  $|\psi\psi|^*$  or  $\psi^2$  before forming the cubic  $|\psi|^2\psi^*$ .

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