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# From complex to real numbers: A reverse detour for solving polynomial equations using complex numbers

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## Abstract

Solving polynomial equations by starting with complex numbers appears counter-intuitive particularly when real roots of equations are sought after. However, when attempting to solve polynomial equations such as cubic equations, complex numbers finally appear in the solution even if the roots are all real numbers. Cardan's solution as such proceeds from real to complex numbers. This paper demonstrates that by starting with complex numbers, it is possible to arrive at the solution that eventually appears in real number form. In effect, such a procedure follows a reverse detour, i.e., from complex to real numbers. In addition, certain factors are simple when expressed in complex forms. This paper presents methods of solving quadratic, cubic, and quartic equations using complex numbers. The formulation of the method and application of the formulae based on the roots of complex numbers is simple and intuitive to follow. Examples are provided for the application of the methods for solving polynomial equations of degrees less than five. The method shows the power of using complex number arithmetic in solving equations despite the fact that the solution can be a real number.

**Keywords:** Polynomial equations, Polynomial functions, Quartic equations, Mathematics, Algebra, Roots of equations.

## 1. Introduction

The techniques of solving polynomial equations including quadratic and cubic equations have been recorded with the Babylonians around 2000 BC. The algebraic solution to cubic and quartic equations was successfully established during the Renaissance period (1450-1630). Scipione Del Ferro (1465-1526) found the solution for the cubic equation in reduced form but his solution was kept secret (Conner, 1956). Tartaglia also developed the solution to the cubic equation which was also not published but only told to Cardano. Girolamo Cardano (1501-1576) published the first public method of solving cubic equations crediting Del Ferro for the method. Francois Viete (1540-1603) also similarly established a method for solving cubic equations using two-step transformation involving one variable only rather than the two variables involved in Cardano's method. The original solutions for cubic equations by both Cardano and Viete are not exactly intuitive and look somehow magical discoveries. Later attempts at more explicit and intuitive approaches have been forwarded (Mukundan, 2010). Simplifications of the solution using derivatives have also been used (Abesheck Das, 2014; Tiruneh, 2020).

Joseph Luis Lagrange (1736-1813) used a combination of symmetric functions that are enough to specify the polynomial equations in reduced form and thus solve them. Lagrange's solution



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as such implicitly used the Fourier transform though the Fourier transform was not yet established during that time (Jansen, 2009). Lagrange's method is also said to be a precursor to the Group theory credited to Evariste Gallois (1811-1832).

The solution to quartic polynomial equations was first established by Ferrari (1522-1565). However, since the method involves solving a resolvent cubic equation, Ferrari's method became public only when the method for solving cubic equations was established (Dickson, 1920). Rene Descartes (1596-1650) and a number of other mathematicians also suggested methods of solving quartic polynomial equations (Dickson, 1914). The occurrence of repeating roots in quartic equations could be apparent when the resolvent cubic has also a repeating root (Neumark, 1965). Leonard Euler (1707-1783) made use of the fact that the sum of the four roots is equal to zero for the reduced quartics and hence was able to offer a solution by solving a resolvent cubic arising out of the three variables (Nickalls, 2009). Fathi and Sharifan (2013) provided a new method of solving quartic equations by expressing the original root x as a sum of three transformed variables u, v, and w in a manner similar to the solution provided by Cardano. Kulkarni (2006) suggested a unified method for solving polynomial equations which has a more explicit and intuitive form compared to earlier methods.

## 2. Methods

### 2.1. Quadratic Equations

#### 2.1.1. Method A: Complex roots

In the first method for quadratic equations, the root can be derived from the roots of real or complex numbers. The application of De Moivre's theorem is demonstrated in the examples for the calculation of real roots from complex numbers. Let A, B, C, and D be numbers (real or complex) such that:

$$(A+B)^2 = C + D$$

$$(A-B)^2 = C - D$$

Adding and subtracting the above equations in turn gives:

$$A^2 + B^2 = C \text{ and } 2AB = D$$

Eliminating B from the equation containing C gives:

$$A^2 + \left(\frac{D}{2A}\right)^2 - C = 0$$

Rearranging gives:

$$A^4 - CA^2 + \frac{D^2}{4} = 0$$

Let $x = A^2$ so that:

$$X^2 - CX + \frac{D^2}{4} = 0$$

Given the quadratic equation: $X^2 + RX + S$ and equating the constants gives:

$$C = -R \text{ and } D = \sqrt{4S}$$

The value of the root X is determined from the value of A as follows:

$$(A + B) = \pm\sqrt{C + D}$$

$$(A - B) = \pm\sqrt{C - D}$$

As far as X is concerned which is the square of A, the above equations yield two independent values, unlike the four values suggested by the above equations. In other words, it is enough to express A in terms of two independent values, i.e.,

$$A = \frac{1}{2}\left(\sqrt{C + D} \pm \sqrt{C - D}\right)$$

The value of X is then obtained:

$$X = A^2 = \frac{1}{4}\left(\sqrt{C + D} \pm \sqrt{C - D}\right)^2$$

Substituting the values of $C = -R \text{ and } D = \sqrt{4S}$ in the above equation gives the quadratic formula as shown below:

$$X = A^2 = \frac{1}{4}\left(\sqrt{-R + \sqrt{4S}} \pm \sqrt{-R - \sqrt{4S}}\right)^2$$

$$X = A^2 = \frac{1}{4}\left(-R + \sqrt{4S} - R - \sqrt{4S} \pm \sqrt{((-R)^2 - 4S)}\right)$$

$$X = A^2 = \frac{1}{2}\left(-R \pm \sqrt{R^2 - 4S}\right)$$

The examples, in the results section demonstrate the application of the above method. The application of De Moivre's theorem where the roots involve complex numbers follows from the following:

$$X = A^2 = \frac{1}{4}\left(\sqrt{c + di} \pm \sqrt{c - di}\right)^2$$

Let $r = \sqrt{c^2 + d^2}$ and $\theta = Cos^{-1}\left(\frac{c}{r}\right)$

$$\sqrt{c+di} \;=\; r^{1/2}\left(Cos\left(\frac{\theta}{2}\right)+i\,Sin\left(\frac{\theta}{2}\right)\right)$$

$$\sqrt{c-di} \;=\; r^{1/2}\left(Cos\left(\frac{-\theta}{2}\right)+i\,Sin\left(\frac{-\theta}{2}\right)\right)$$

$$X = \;A^2 \;= \frac{1}{4}\left(\left(r^{1/2}\left(Cos\left(\frac{\theta}{2}\right)+i\,Sin\left(\frac{\theta}{2}\right)\right)\right)\pm\left(r^{1/2}\left(Cos\left(\frac{-\theta}{2}\right)+i\,Sin\left(\frac{-\theta}{2}\right)\right)\right)\right)^2$$

$$X = \;A^2 \;=\left\{r\left(Cos\left(\frac{\theta}{2}\right)\right)^2, -r\left(Sin\left(\frac{\theta}{2}\right)\right)^2\right\}$$

### 2.1.2. Method B: Derivation through complex arithmetic

A complex arithmetic is presented here to derive the quadratic formula. Consider the general real quadratic expression:

$$a^2 \;+\; b^2 \;(7)$$

Equation (7) can be factored into two complex factors as follows:

$$a^2 \;+\; b^2 \;=\; (a+bi)\,(a-bi)\;(8)$$

Now it is possible to express the quadratic equation in Equation (1) in the form given by Equation (7). i.e.,

$$x^2+bx\;+c=0=\frac{1}{2}\,(x-p)^2\;+\frac{1}{2}\,(x-q)^2\;(9)$$

Term-by-term comparison of the expressions in Equation (9) gives;

$$p+q=\;-b;\;p^2+\;q^2\;=2c\;(10)$$

Now using the complex factoring given in Equation (8), the solution to Equation (9) can be expressed in the complex form as:

$$\begin{cases} x-p\;+\;(x-q)i\;=0 \\ x-p-\;(x-q)i\;=0 \end{cases}(11)$$

Solving Equation (11) for x one gets:

$$x=\frac{p\;\pm iq}{1\pm i}\;(12)$$

Equation (12) can be reduced by eliminating the complex denominator:

$$x = \frac{p \pm iq}{1 \pm i} * \frac{1 \mp i}{1 \mp i} = \frac{p+q}{2} \pm \frac{1}{2}(p-q)i \ (13)$$

Using the relation in Equation (10), Equation (13) becomes;

$$x = \frac{-b}{2} \pm \frac{1}{2}(p-q)i \ (14)$$

Considering again Equation (10)

$$(p+q)^2 = b^2 = p^2 + 2pq + q^2 = 2c + 2pq$$

From which:

$$2pq = b^2 - 2c \ (15)$$

Considering Equations (14) and (15):

$$[(p-q)i]^2 = 2pq - (p^2+q^2) = b^2 - 2c - 2c = b^2 - 4c \ (16)$$

Taking the square root in (16):

$$(p-q)i = \pm\sqrt{b^2-4c} \ (17)$$

Finally substituting Equation (17) into Equation (14) and knowing that the sign +/- combinations eventually give only two choices gives:

$$x = \frac{-b}{2} \pm \frac{1}{2}(p-q)i = \frac{-b}{2} \pm \frac{1}{2}\sqrt{b^2-4c} = \frac{-b \pm \sqrt{b^2-4c}}{2} \ (18)$$

Equation (18) is the quadratic formula as is known free from the complex expression initially employed to derive it.

## 2.2. Cubic Equations

The solution of the cubic polynomial equation is formulated starting with the cube roots of a complex number whose real and complex part is determined from the equation. Two separate formulations are made, according to the discriminant of the equation. The procedure is described below for the two separate cases

**Case I:**

Given a polynomial equation in the depressed form:

$$x^3 + Rx + S = 0$$

The discriminant $Discr = 4R^3 + 27S^2 \leq 0$, the solution to the equation is obtained through the cubic root of a complex number c + di such that:

$$(a + bi)^3 = c + di$$

The values of c and d are given by:

$$c = \frac{4S}{R};\ d = \pm\frac{4}{3\sqrt{3}\,R}\sqrt{-(4R^3 + 27S^2)}$$

The value of a is computed using De Moivre's Theorem as the real part of the cube root of the complex number c+di as follows:

$$a = r^{1/3}\left[Cos\left(\frac{\theta + 2n\pi}{3}\right)\right]\ n = 0, 1, 2$$

Where the values of r and θ are given by:

$$r = \sqrt{c^2 + d^2};\ \theta = Cos^{-1}\left(\frac{c}{r}\right)$$

The roots of the cubic equation are then given by:

$$x = a^3 - \frac{S}{R}$$

**Case II:**

Given a polynomial equation in the depressed form:

$$x^3 + Rx + S = 0$$

The discriminant $Discr = 4R^3 + 27S^2 \geq 0$, the solution to the cubic equation is obtained through a cubic root of a real number by transforming Case I equation from complex to the real number domain through the substitutions, namely, B = bi and D = di such that:

$$(a - B)^3 = c - D$$

The values of c and D are given by:

$$c = \frac{4S}{R};\ D = \pm\frac{4}{3\sqrt{3}\,R}\sqrt{(4R^3 + 27S^2)}$$

The value of a is given by:

$$a = \frac{1}{2}\sqrt[3]{c + D} + \frac{1}{2}\sqrt[3]{c - D}$$

The roots of the cubic equation are then given by:

$$x = a^3 - \frac{S}{R}$$

### 2.2.1. Proof of Method

The proofs of the formulae given in Case I and II above are provided below separately for each case. First, we start with case I in which the discriminant is negative in which all the roots of the cubic equations are real. It is interesting to note that in the case of Cardan's method, the solution involves the roots of a complex number whereas in this proposed method as demonstrated below it involves the square root of a real number.

**Case I**: $Discr = 4R^3 + 27S^2 \leq 0$

Assuming that the polynomial equation is already converted into the depressed form:

$$x^3 + Rx + S = 0$$

The discriminant $Discr = 4R^3 + 27S^2 \leq 0$, the solution to the equation is obtained through the cubic root of a complex number c + di such that:

$$(a + bi)^3 = c + di$$

Expanding the $(a+bi)^3$ term and equating it to c+di gives:

$$[a^2 - 3ab^2] + [3a^2b - b^3]i = c + di$$

From which it is apparent that:

$$a^2 - 3ab^2 = c;\ 3a^2b - b^3 = d$$

Eliminating the complex coefficient b and expressing the above equation in terms of the real part of a_+bi, i.e., a only gives:

$$a^9 - \frac{3}{4}ca^6 - \frac{[15c^2 + 27d^2]}{64}a^3 - \frac{c^3}{64} = 0$$

Using the substitution $m = a^3$ to convert the above equation into a general cubic equation gives;

$$m^3 - \frac{3}{4}cm^2 - \frac{[15c^2 + 27d^2]}{64}m - \frac{c^3}{64} = 0$$

The above equation is converted into depressed form and equated to the given polynomial equation in x. To do this, the usual variable transformation equation to depressed cubic form is used, i.e.,

$$m = x - \frac{1}{3}\left(\frac{3}{4}c\right) = x + \frac{1}{4}c$$

Using this transformation the cubic equation in m is transformed into x variable as follows:

$$x^3 - \left[\frac{27}{64}(c^2 + d^2)\right]x - \left[\frac{27}{256}(c^3 + d^2c)\right] = 0$$

Equating the terms of the above equation to that of the given equation: $x^3 + Rx + S = 0$,

$$R = -\left[\frac{27}{64}(c^2 + d^2)\right]; \; S = -\left[\frac{27}{256}(c^3 + d^2c)\right]$$

Solving for c and d in terms of R and s will eventually give:

$$c = \frac{4S}{R}; \; d = \pm\frac{4}{3\sqrt{3}\,R}\sqrt{-(4R^3 + 27S^2)}$$

Now working backwards from c and d to the equation in the x variable, since a +bi is the cube root of c +di, the value of a is computed using De Moivre's Theorem as the real part of the cube root of the complex number c +di as follows:

$$a = r^{1/3}\left[Cos\left(\frac{\theta + 2n\pi}{3}\right)\right] \quad n = 0, 1, 2$$

Where the values of r and θ are given by:

$$r = \sqrt{c^2 + d^2}; \; \theta = Cos^{-1}\left(\frac{c}{r}\right)$$

The roots of the cubic equation are given by:

$$x = m - \frac{1}{4}c$$

Using the relation:

$$m = a^3; \; c = \frac{4S}{R}$$

gives:

$$x = a^3 - \frac{S}{R}$$

**Case II:** $Discr = 4R^3 + 27S^2 \geq 0$

In the case where the discriminant of the cubic equation $x^3 + Rx + S = 0$ given as

$$Discr = 4R^3 + 27S^2 \leq 0$$

Represents a cubic equation in which the solution consists of a real number and a complex number with its conjugate. From the relation established in Case I, i.e.,

$$a^2 - 3ab^2 = c; \; 3a^2b - b^3 = d$$

Substituting B = bi and D = di gives after rearrangement;

$$a^2 + 3aB^2 = c; \; 3a^2B + B^3 = di = D$$

Adding and subtracting the above two equations gives:

$$a^2 + 3aB^2 + 3a^2B + B^3 = (a + B)^3 = c + D$$

$$a^2 - 3a^2B + 3aB^2 - B^3 = (a - B)^3 = c - D$$

From the above two relations:

$$a + B = \sqrt[3]{c + D}$$

$$a - B = \sqrt[3]{c - D}$$

Eliminating B from the above two equation yields;

$$a = \frac{1}{2}\sqrt[3]{c + D} + \frac{1}{2}\sqrt[3]{c - D}$$

The values of c and D=di are given as worked out in Case I:

$$c = \frac{4S}{R}; \; D = di = \pm\frac{4}{3\sqrt{3}\,R}\sqrt{(4R^3 + 27S^2)}$$

The roots of the cubic equation are then given by:

$$x = a^3 - \frac{S}{R}$$

It is also interesting to note that opposite to Cardan's method the discriminant is negative where the solution to the cubic equation has one real root only. In the case of the Cardan's Method, the discriminant is positive. It is, however, useful to observe that the solution in the end is formulated using real numbers only in both cases. Case I though requires application of De

Moivre’s theorem to start with whereas case II is all a real number workout. By contrast, Cardan’s method ends up with a solution, which involves complex numbers in the end.

### 2.3. Quartic Equations

Consider a quartic polynomial equation that is in reduced form; f(x) = $x^4 + bx^2 + cx + d = 0$. The solution to this quartic equation is established through an equivalent polynomial P(x) of the form:

$$P(x) = \frac{1}{2}((x^2 + px + q)^2) + \frac{1}{2}((x^2 - px + r)^2) = f(x) = x^4 + bx^2 + cx + d = 0$$

The solution to the quartic polynomial involving complex numbers is:

$$(x^2 + px + q) \pm (x^2 - px + r)i = 0$$

The resulting quadratic equation is of the form:

$$(1 \pm i)x^2 + (1 \mp i)\, px + q \pm ri = 0$$

The root is found using the quadratic formula:

$$x = \frac{-(1 \pm i)p \pm \sqrt{(1 \pm i)^2 p^2 - 4(1 \pm i)(q \pm ri)}}{2(1 \pm i)}$$

To reduce the denominator to a real term the following multiplication is made:

$$x = \frac{-(1 \mp i)p \pm \sqrt{(1 \mp i)^2 p^2 - 4(1 \pm i)(q \pm ri)}}{2(1 \pm i)} \left(\frac{1 \mp i}{1 \mp i}\right)$$

Simplifying further gives:

$$x = \pm\frac{pi}{2} \pm \frac{1}{2}\sqrt{-p^2 + 2(-1 \pm i)(q \pm ri)}$$

The constants p, q, and r are determined by equating P(x) with f(x):

$$P(x) = \frac{1}{2}((x^2 + px + q)^2) + \frac{1}{2}((x^2 - px + r)^2) = f(x) = x^4 + bx^2 + cx + d$$

Expanding P(x) gives:

$$P(x) = x^4 + (p^2 + q + r)x^2 + p(q - r)x + \frac{q^2}{2} + \frac{r^2}{2} = 0$$

Equating the coefficients with the original polynomial f(x):

$$p^2 + q + r = b$$

$$p(q - r) = c$$

$$\frac{q^2}{2} + \frac{r^2}{2} = d$$

Solving for q and r in terms of p using the first two equations:

$$q = \frac{b}{2} - \frac{p^2}{2} + \frac{c}{2p}$$

$$r = \frac{b}{2} - \frac{p^2}{2} - \frac{c}{2p}$$

Inserting the values of q and r in terms of p into the final third equation gives, after simplification:

$$p^6 - 2bp^4 + (b^2 - 4d)p^2 + c^2 = 0$$

Defining a variable $y = p^2$:

$$y^3 - 2by^2 + (b^2 - 4d)y + c^2 = 0$$

The above resolvent cubic equation is solved and the value of p is determined from:

$$p = \sqrt{y}$$

Inserting the expression for q and r in terms of p in the solution for x finally gives:

$$x = \pm\frac{pi}{2} \pm \frac{1}{2}\sqrt{p^2 - 2b \pm \frac{2c}{p}i}$$

The four solutions are:

$$x_1 = +\frac{pi}{2} + \frac{1}{2}\sqrt{p^2 - 2b + \frac{2c}{p}i}$$

$$x_2 = +\frac{pi}{2} - \frac{1}{2}\sqrt{p^2 - 2b + \frac{2c}{p}i}$$

$$x_3 = -\frac{pi}{2} + \frac{1}{2}\sqrt{p^2 - 2b - \frac{2c}{p}i}$$

$$x_4 = -\frac{pi}{2} - \frac{1}{2}\sqrt{p^2 - 2b - \frac{2c}{p}i}$$

# 3. Results and discussion

## 3.1. Quadratic Equation Examples

**Example 1: $x^2$-2x-3 = 0; R = -2 and S = -3**

$$C = -R = 2 \; and \; d = \sqrt{4S} = \sqrt{4 * -3} = \sqrt{12}i$$

$$X = A^2 = \frac{1}{4}\left(\sqrt{C+D} \pm \sqrt{C-D}\right)^2$$

$$X = \frac{1}{4}\left(\sqrt{2+\sqrt{12}i} \pm \sqrt{2-\sqrt{12}i}\right)^2$$

$$= \frac{1}{4}\left(2+\sqrt{12}i + 2 - \sqrt{12}i \pm 2 * \left(\sqrt{2^2 - \left(\sqrt{12}i\right)^2}\right)\right)$$

$$= \frac{1}{4}(4 \pm 8) = \{3, -1\}$$

The application of De Moivre's Theorem is shown below for this example:

Let $r = \sqrt{c^2 + d^2}$ and $\theta = Cos^{-1}\left(\frac{c}{r}\right)$

$$r = \sqrt{2^2 + \sqrt{12}^2} = 4$$

$$\theta = Cos^{-1}\left(\frac{2}{4}\right) = 60^0$$

$$X = A^2 = \left\{r\left(Cos\left(\frac{\theta}{2}\right)\right)^2, -r\left(Sin\left(\frac{\theta}{2}\right)\right)^2\right\}$$

$$X = A^2 = \left\{4\left(Cos(30^0)\right)^2, -4\left(Sin(30^0)\right)^2\right\}$$

$$X = A^2 = \left\{4\left(\frac{\sqrt{3}}{2}\right)^2, -4\left(\frac{1}{2}\right)^2\right\}$$

$$X = A^2 = \{3, -1\}$$

**Example 2: $x^2$+2x+10 = 0; R = 2 and S = 10**

$$C = -R = -2 \; and \; d = \sqrt{4S} = \sqrt{4 * 10} = \sqrt{40}$$

$$X = A^2 = \frac{1}{4}\left(\sqrt{C+D} \pm \sqrt{C-D}\right)^2$$

$$X = \frac{1}{4}\left(\sqrt{-2+\sqrt{40}} \pm \sqrt{-2-\sqrt{40}}\right)^2$$

$$X = \frac{1}{4}\left(-2+\sqrt{40} + -2-\sqrt{40} \pm 2 * \left(\sqrt{(-2)^2 - \left(\sqrt{40}\right)^2}\right)\right)$$

$$X = \frac{1}{4}(-4 \pm 12i) = \{-1+3i, -1-3i\}$$

### 3.2. Cubic equations Examples

The method developed is tested through three cubic equation examples having discriminants negative, zero, and positive respectively. The solutions are worked out for each case as provided below:

**Example 1:** $x^3 - 6x + 4 = 0$

In this equation R = -6 and S = 4. The Discriminant

$$Discr = 4R^3 + 27S^2 = 4(-6)^3 + 27(4)^2 = -432 \leq 0$$

All the solutions of the cubic equations must be real numbers.

The values of c and d are given by:

$$c = \frac{4S}{R} = \frac{4 * 4}{-6} = -\frac{8}{3};$$

$$d = \pm \frac{4}{3\sqrt{3}\,R}\sqrt{-(4R^3 + 27S^2)} = \pm \frac{4}{3\sqrt{3}\,(-6)}\sqrt{-(-432)} = \pm \frac{8}{3}$$

The value of a is computed using De Moivre's Theorem as the real part of the cube root of the complex number c + di as follows:

$$a = r^{1/3}\left[Cos\left(\frac{\theta + 2n\pi}{3}\right)\right] \quad n = 0, 1, 2$$

Where the values of r and θ are given by:

$$r = \sqrt{c^2 + d^2} = \frac{8\sqrt{2}}{3};$$

$$\theta = Cos^{-1}\left(\frac{c}{r}\right) = Cos^{-1}\left(\frac{-1}{\sqrt{2}}\right) = \frac{3\pi}{4}$$

The values of a are worked out as follows:

$$a_1 = r^{1/3}\left[Cos\left(\frac{\theta}{3}\right)\right] = \left(\frac{8\sqrt{2}}{3}\right)^{1/3}\left[Cos\left(\frac{\pi}{4}\right)\right] = \left(\frac{8\sqrt{2}}{3}\right)^{1/3}\left(\frac{1}{\sqrt{2}}\right)$$

$$m_1 = {a_1}^3 = \frac{8\sqrt{2}}{3}\left(\frac{1}{\left(\sqrt{2}\right)^3}\right) = \frac{4}{3}$$

$$a_2 = r^{\frac{1}{3}}\left[Cos\left(\frac{\theta + 2\pi}{3}\right)\right] = \left(\frac{8\sqrt{2}}{3}\right)^{\frac{1}{3}}\left[Cos\left(\frac{11\pi}{12}\right)\right]$$

$$= \left(\frac{8\sqrt{2}}{3}\right)^{1/3}(-0.965925826) = -1.503505501$$

$$m_2 = {a_2}^3 = -3.398717474$$

$$a_3 = r^{\frac{1}{3}}\left[Cos\left(\frac{\theta + 2\pi}{3}\right)\right] = \left(\frac{8\sqrt{2}}{3}\right)^{\frac{1}{3}}\left[Cos\left(\frac{19\pi}{12}\right)\right]$$

$$= \left(\frac{8\sqrt{2}}{3}\right)^{1/3}(0.258819045) = 0.402863084$$

$$m_3 = {a_3}^3 = 0.06538414$$

The root of the cubic equation is then given by:

$$x_1 = {a_1}^3 - \frac{S}{R} = \frac{4}{3} - \left(\frac{4}{-6}\right) = 2$$

$$x_2 = {a_2}^3 - \frac{S}{R} = -3.398717474 - \left(\frac{4}{-6}\right) = -2.732050808$$

$$x_3 = {a_3}^3 - \frac{S}{R} = 0.06538414 - \left(\frac{4}{-6}\right) = 0.732050807$$

**Example 2:** $x^3 - 3x - 2 = 0$

In this equation R = -3 and S = -2. The Discriminant

$$Discr = 4R^3 + 27S^2 = 4(-3)^3 + 27(-2)^2 = 0$$

This cubic equation has repeating roots since Discr = 0.

The values of c and d are given by:

$$c = \frac{4S}{R} = \frac{4 * -2}{-3} = \frac{8}{3};$$

$$d = \pm\frac{4}{3\sqrt{3}\,R}\sqrt{-(4R^3 + 27S^2)} = \pm\frac{4}{3\sqrt{3}\,(-3)}\sqrt{-(0)} = 0$$

The value of a is computed using De Moivre’s Theorem as the real part of the cube root of the complex number c + di as follows:

$$a = r^{1/3}\left[Cos\left(\frac{\theta + 2n\pi}{3}\right)\right] \; n = 0, 1, 2$$

Where the values of r and θ are given by:

$$r = \sqrt{c^2 + d^2} = \frac{8}{3};$$

$$\theta = Cos^{-1}\left(\frac{c}{r}\right) = Cos^{-1}(1) = 0$$

The values of a are worked out as follows:

$$a_1 = r^{1/3}\left[Cos\left(\frac{\theta}{3}\right)\right] = \left(\frac{8}{3}\right)^{1/3}[Cos(0)] = \left(\frac{8}{3}\right)^{1/3}(1)$$

$$m_1 = {a_1}^3 = \frac{8}{3}(1^3) = \frac{8}{3}$$

$$a_2 = r^{\frac{1}{3}}\left[Cos\left(\frac{\theta + 2\pi}{3}\right)\right] = \left(\frac{8}{3}\right)^{\frac{1}{3}}\left[Cos\left(\frac{2\pi}{3}\right)\right]$$

$$= \left(\frac{8}{3}\right)^{1/3}\left(\frac{-1}{2}\right)$$

$$m_2 = {a_2}^3 = \left(\frac{8}{3}\right)\left(\frac{-1}{2}\right)^3 = -\frac{1}{3}$$

$$a_3 = r^{\frac{1}{3}}\left[Cos\left(\frac{\theta + 4\pi}{3}\right)\right] = \left(\frac{8}{3}\right)^{\frac{1}{3}}\left[Cos\left(\frac{4\pi}{3}\right)\right]$$

$$= \left(\frac{8}{3}\right)^{1/3}\left(\frac{-1}{2}\right)$$

$$m_3 = {a_3}^3 = \left(\frac{8}{3}\right)\left(\frac{-1}{2}\right)^3 = -\frac{1}{3}$$

The root of the cubic equation is then given by:

$$x_1 = {a_1}^3 - \frac{S}{R} = \frac{8}{3} - \left(\frac{-2}{-3}\right) = 2$$

$$x_2 = {a_2}^3 - \frac{S}{R} = -\frac{1}{3} - \left(\frac{-2}{-3}\right) = -1$$

$$x_3 = {a_3}^3 - \frac{S}{R} = -\frac{1}{3} - \left(\frac{-2}{-3}\right) = -1$$

The repeating root is x=1 as the solution indicates.

**Example 3:** $x^3 - 2x + 4 = 0$

In this equation R = -2 and S = 4. The Discriminant

$$Discr = 4R^3 + 27S^2 = 4(-2)^3 + 27(4)^2 = 400 > 0$$

This equation has one real root and two complex roots. To get the real root, the formula given in Case 2 of the Methods section is applied.

The values of c and D are given by:

$$c = \frac{4S}{R} = \frac{4 * 4}{-2} = -8;$$

$$D = \pm\frac{4}{3\sqrt{3}\,R}\sqrt{(4R^3 + 27S^2)}$$

$$= \pm\frac{4}{3\sqrt{3}\,(-2)}\sqrt{400} = \mp\frac{40}{3\sqrt{3}} = \mp 7.698003589$$

Both +D and –D give the same result hence choose D = 7.698003589

$$\sqrt[3]{c + D} = \sqrt[3]{-8 + 7.698003589} = -0.670914627$$

$$\sqrt[3]{c - D} = \sqrt[3]{-8 - 7.698003589} = -2.503887477$$

The value of a is given by:

$$a = \frac{1}{2}\sqrt[3]{c + D} + \frac{1}{2}\sqrt[3]{c - D}$$

$$a = \frac{1}{2}(-0.670914627 - 2.503887477) = -1.587401052$$

$$m_3 = {a_3}^3 = (-1.587401052)^3 = -4$$

The real root of the cubic equation is then given by:

$$x = a^3 - \frac{S}{R} = -4 - \left[\frac{4}{-2}\right] = -4 + 2 = -2$$

To obtain the other (complex) roots, synthetic division of the cubic equation by x + 2 gives:

$$\frac{x^3 - 2x + 4}{x + 2} = x^2 - 2x + 2$$

The roots of the quadratic equation using function evaluation:

$$z = -\frac{-2}{2} = 1$$

$$f(z) = 1^2 - 2(1) + 1 = 1$$

$$x = z \pm \sqrt{-f(z)} = 1 \pm i$$

Therefore, the other complex roots of the cubic equation are 1 + i and 1 - i.

### 3.3. Quartic equations Examples

An example is provided below for the application of the formula for solving quartic polynomial equations using complex number arithmetic

**Example:** Given a quartic equation:

$$x^4 - x^3 - 19x^2 - 11x + 30 = 0$$

The solution to the above quartic equation is:

$$\mathrm{x} = \{-3, -2, 1, 5\}$$

The given quartic equation can be reduced using the method of function evaluation (Tiruneh, 2020). This method of function evaluation can also be derived from binomial expansion as follows. Define variables z and t such that:

$$x = z + t$$

The value of z is chosen such that the resulting quartic polynomial equation is in reduced form in terms of the variable t. Using the binomial expansion, the original equation can be expressed in terms of z and t as follows:

$$f(t) = t^4 + \frac{f'''(z)}{3!}t^3 + \frac{f''(z)}{2!}t^2 + f'(z)t + f(z) = 0$$

Since the coefficient of $t^3$ has to be zero in the reduced form, it follows that;

$$f'''(z) = 0$$

$$24z - 6 = 0$$

$$z = \frac{1}{4}$$

Evaluating the other coefficients:

$$f(z) = f\left(\frac{1}{4}\right) = \left(\frac{1}{4}\right)^4 - \left(\frac{1}{4}\right)^3 - 19\left(\frac{1}{4}\right)^2 - 11\left(\frac{1}{4}\right) + 30 = 26.05078125$$

$$f'(z) = f'\left(\frac{1}{4}\right) = 4z^3 - 3z^2 - 38z - 11 = -20.625$$

$$f''(z) = f''\left(\frac{1}{4}\right) = 12z^2 - 6z - 38 = -38.75$$

The reduced quartic then becomes:

$$f(t) = t^4 + \frac{f'''(z)}{3!}t^3 + \frac{f''(z)}{2!}t^2 + f'(z)t + f(z) = 0$$

$$f(t) = t^4 - 19.375\,t^2 - 20.625\,t + 26.05078125 = 0$$

Considering the resolvent cubic equation:

$$y^3 - 2by^2 + (b^2 - 4d)y + c^2 = 0$$

The coefficients corresponding to the given example of a reduced quartic equation are evaluated as follows

$$-2b = -2(-19.375) = 38.75;\; b^2 - 4d = (-19.375)^2 - 4(26.05078123) = 271.1875$$

$$c^2 = (-20.625)^2 = 425.390625$$

The resolvent cubic equation becomes:

$$y^3 + 38.75\,y^2 + 271.1875y + 425.390625 = 0$$

The roots of the above cubic equation are:

$$y = \{-2.25, -6.25, -30.25\}$$

Taking y = -2.25 (the other roots also give the same answer) and evaluating p, q, and r.

$$p = \sqrt{-2.25} = 1.5i$$

The solution in terms of t are:

$$t_1 = +\frac{(1.5i)i}{2} + \frac{1}{2}\sqrt{(1.25i)^2 - 2(-19.375) + \frac{2(-20.625)}{1.5i}i}$$

$$= -0.75 + 0.5 * \sqrt{36.5 - 27.5} = -0.75 + \frac{1}{2}(3) = 0.75$$

$$t_2 = +\frac{(1.5i)i}{2} - \frac{1}{2}\sqrt{(1.25i)^2 - 2(-19.375) + \frac{2(-20.625)}{1.5i}i}$$

$$= -0.75 - 0.5 * \sqrt{36.5 - 27.5} = -0.75 - \frac{1}{2}(3) = -2.25$$

$$t_3 = -\frac{(1.5i)i}{2} + \frac{1}{2}\sqrt{(1.25i)^2 - 2(-19.375) - \frac{2(-20.625)}{1.5i}i}$$

$$= +0.75 + 0.5 * \sqrt{36.5 + 27.5} = +0.75 + \frac{1}{2}(8) = 4.75$$

$$t_4 = -\frac{(1.5i)i}{2} - \frac{1}{2}\sqrt{(1.25i)^2 - 2(-19.375) - \frac{2(-20.625)}{1.5i}i}$$

$$= +0.75 - 0.5 * \sqrt{36.5 + 27.5} = +0.75 - \frac{1}{2}(8) = -3.25$$

Finally, the solution in terms of the original variable x will be:

$$x_1 = t_1 + z = 0.75 + 0.25 = 1$$

$$x_2 = t_2 + z = -2.25 + 0.25 = -2$$

$$x_3 = t_3 + z = 4.75 + 0.25 = 5$$

$$x_4 = t_4 + z = -3.25 + 0.25 = -3$$

The example demonstrates the ability of complex number arithmetic to offer solutions that may be expressed in real terms. Many approaches to solving equations routinely follow the path of real number arithmetic whereas the solution may involve imaginary numbers. This new approach is further proof that complex number arithmetic can be used to eventually solve equations that may be expressed in real or imaginary number forms.

## 4. Conclusion

A method for solving polynomial equations of degrees 2, 3, and 4 is presented that starts with the use of complex number arithmetic in order to arrive at possible real roots of the equations. For cubic equations, it is known that Cardan's method results in solutions involving complex numbers even though all the roots are known to be real numbers. In effect, Cardan's Method starts with real numbers and eventually ends up with complex numbers to contend with in the final solution while the actual roots of the cubic equations are all real. This is revealed by the fact that the Discriminant is negative when all the roots of the cubic equations are real. In the approach demonstrated in this paper, the reverse route is taken whereby the solution starts with complex numbers, and by manipulating the real part of this complex number, the final solution is worked out eventually using real number arithmetic only, without involving complex number arithmetic. On the other hand, the discriminant in this method is a positive number opposite to that of Cardan's method when all the roots are real numbers. As such this method proceeds from complex to real numbers and hence takes a reverse detour to Cardan's Method. The use of complex number arithmetic for solving equations that may be eventually expressed in real number forms is thus demonstrated. This approach is also one further example of the many ways in which polynomial equations can be solved

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