Impossibilities, mathematics, and logic

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Abstract

Mathematics is known for its rigor. Owing to its rigor, mathematics is both loved and feared. Proof holds a pivotal position in the whole of mathematical rigor. Proof is required for something to be possible. Interestingly, proof is equally important and required for something to be declared impossible. In this paper, certain beautiful examples of impossibilities are mentioned, which include, among others, the impossibility of the denumerability of real numbers, squaring a circle, and doubling a cube.

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Mathematics is unreasonably effective. This unreasonable effectiveness of mathematics was discussed long back by Eugene Wigner [1], and recently by Ian Stewart [2]. While, on the one hand, mathematics has the power to unravel the unknown and the mysterious, on the other hand, it is capable of bringing to fore the fact that certain things or endeavors are simply impossible. Be that Euler's famous ingenious solution to the Konigsberg bridge problem or the nonexistence of a rational number whose square is two, mathematics has a plethora of eye-openers to offer and astonish us.

One must appreciate that for something to be impossible, it is not merely sufficient to say it or to demonstrate that it is not doable in a specific way or in a particular situation. There is the burden of proof. Mathematics is appreciated not just for its beauty but also for its rigor. In fact, it is this rigor that makes mathematics what it is.

How can one prove or justify the impossibility of something? We require a proof of impossibility. And there are several ways of presenting such proofs. One such proof is proof by contradiction. It is often convenient, if not necessary, to assume the truth of an otherwise false statement just to reach an inescapable unacceptable wrong conclusion thereby demonstrating the flaw in the assumption. This is what a mathematician does when he or she employs the method of contradiction. One famous example is that of the existence of irrational numbers. Assuming that a rational number exists the square of which is two leads to the apparent contradiction that prime factors can be cancelled from the numerator and the denominator a rational number even when the number is in its simplest form.

Let's say there exists a rational number the square of which is 2. In that case,

$$\exists p, q \neq 0 \in Z, \left(\frac{p}{q}\right)^2 = 2$$

$$\Rightarrow \frac{p^2}{q^2} = 2$$

$$\Rightarrow p^2 = 2q^2$$

Since $2q^2$ is an even integer, so must p^2 be. However, the square of an integer can only be an even integer if that integer itself is even. Thus, p must be an even integer. In that case, there must exist some integer, say, r such that $p = 2r$. This leads to $p^2 = 4r^2$. So, we have

$$4r^2 = 2q^2$$

$$\Rightarrow 2r^2 = q^2$$

Thus, q is even. However, that means that $\frac{p}{q}$ is not in its simplest form

Another beautiful example is that of Cantor's diagonal argument wherein assuming that the number of real numbers is the same as that of rational numbers leads to an unacceptable conclusion, a contradiction. Thus, there are more real numbers than rational numbers.

Another proof of impossibility is the proof by descent. This works by assuming, for example, that a smallest solution to a problem must exist (by virtue of the well-ordering principle). We, then, can go on to demonstrate that a solution smaller than the smallest exists thereby exposing the flaw in the assumption, and establishing the impossibility of the thing. The non-solvability of ${}^4 + {}^4 = {}^4$ in non-zero integers can be established by it, for example.

Then there is the method of disproof. Imagine we are required to establish that a statement is not universal. All that is required is to show the existence of a counter-example. Long back, Euler conjectured that at least different th powers are necessary to sum to another th power. This was disproved, in 1966, by the use of powerful mainframe computers, CDC 6600, which came up with the counterexample

$$27^5 + 84^5 + 110^5 + 133^5 = 144^5$$

thereby showing that a proof of Euler's conjecture in the affirmative is impossible[3].

And who could afford to give a wide berth to the impossible constructions sought by the Greeks? They sought a method for trisecting an angle using a straightedge and compasses, a method for doubling a cube, and one for squaring the circle. Pierre Wantzel, in 1837, published a proof of the impossibility of trisecting an arbitrary angle using only a straightedge and compasses. However, the proof was based on field extensions and Galois theory the fruition of which the world witnessed in the early twentieth century. Doubling the cube, aka the Delian problem, that sought the construction of a cube having its volume double the volume of a given cube, was also settled in the negative using field extensions. And similar was the case of the problem of squaring the circle, that sought the construction of a square and a circle with equal area. The fact remains that the theory used to prove these impossibilities seemed very far-fetched and unreal thereby reminding us again of the unreasonable effectiveness of mathematics.

Can a finite formal system be both complete and consistent? The impossibility of any such system to be both is one of the most celebrated and intriguing achievements of the modern logic, thanks to Kurt Gödel.

We, thus, observe that the responsibility of mathematics and mathematicians has not just been to establish existence, but also non-existence. The role of mathematics is not merely to come up with algorithms, but also to show that sometimes none exists.
That is probably why Eric Temple Bell said,

> *"Obvious is the most dangerous word in mathematics".*

References

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