Research Article

Some New Aspects of Quantum Gravity Involving Coupling of Fields to Random Currents with Applications to Astroparticle Physics

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Some basic questions in quantum field theory and cosmology are addressed here. We derive some formulas for the change in the canonical commutation and anticommutation relations at equal time for some of the well known quantum fields in the presence of a background curved space-time metric. We study the problem of quantizing the Klein-Gordon field interacting with the gravitational field of homogeneous and isotropic space time of an expanding universe and also simultaneously interacting with a classical random current field. Formulas for the quantum effective action of the scale factor of the expanding universe are derived by averaging over the Klein-Gordon field, taking into account its interaction with a classical random current field. This gives us information about how the expansion rate of our universe can get affected due to quantum mechanical interaction effects. Finally, we discuss the general problem of symmetry breaking in the quantum effective action of a field when it interacts with a random Gaussian current field source. The symmetry breaking terms are expressed in terms of the correlation field of the random current source. Since the Hessian of the quantum effective action equals the inverse of the propagator kernel, it follows that the former tells us how much mass do particles gain by interacting with a random current field. This fact could provide us with a clue about how particles in our universe acquire masses. We then derive Hawking's temperature formula for a quantum Blackhole using the approximate solution to the Klein-Gordon equation in the vicinity of the Schwarzchild radius. This is a new derivation not present in the literature. Finally, a short discussion is presented about the quantum mechanical meaning of estimating the state of a gravitational wave from continuous noisy measurements on the electromagnetic field with which it interacts on a real time basis using the well known Belavkin quantum filter based on the Hudson-Parthasarathy noisy Schrodinger equation. The appendix provides a simplified analysis of loop quantum gravity based on

Ashtekar's variables and how one constructs the area operator that is crucial in deriving the blackhole entropy.

1. Introduction

In this paper, we present second quantization of the electromagnetic and Dirac fields in the background of classical curved space-times. We discuss how the canonical commutation relations in a curved background space-time lead to deviations in the commutation relations between the position and velocity fields when background curvature is present by taking the Robertson Walker metric of space time with a spatially homogeneous Klein Gordon field also present. We generalize this idea by dropping off the homogeneity condition on the Klein-Gordon metric. This is achieved by expanding the KG field as a linear combination of spatial basis functions with the coefficients in this expansion being functions of time only and serving as the position variables for the KG field. We thus set up the total action for the Robertson-Walker scale factor and the general KG field in this space-time metric. By path integrating over the KG position paths, namely over the paths defined by the linear combination coefficients that appear in the expansion of the KG field in terms of spatial basis functions, we are able to derive the quantum effective action for the scale factor of the universe when gravity interacts with the inhomogeneous KG field. This computation also tells us how to derive the wave function evolution of the scale factor of the universe when the universe consists of scalar KG particles. We then proceed further to calculate the quantum effective action of the scale factor of the universe along with the KG field within it when the KG field interacts with a classical random Gaussian current field. This quantum effective action is obtained by evaluating the classical average of the complex exponential of the total action of the RW gravitational field and the KG field in this metric and in addition, taking into account interactions between the random current field and the KG field. A further path integration of this averaged complex exponential over the KG field then yields us the quantum effective action of the scale factor of the universe alone. This formula can be used to predict how quantum effects with matter and random current fields in the universe can affect the rate of expansion of our universe. We then consider another example of such a situation in which we start with an arbitrary Lagrangian density of a field also depending on the metric of space-time with the metric being a function of a set of random parameter fields. We compute the quantum effective action for such field by assuming that these random parameters have small variances. It should be mentioned that the quantum effective action is computed by forming the Legendre transform of the expected value of the complex exponential of the action w.r.t the mean value of the current in all the cases, and this then yields us the quantum equations of motion satisfied by the quantum effective action. It also gives us symmetry breaking effects induced by the random current field in the sense that the variation of the quantum effective action under the quantum expectation value of the gauge transformation that leaves the classical action invariant is no longer invariant but instead depends on the covariance of the random current field. We illustrate such symmetry breaking using the example of a finite number of KG fields with its Lagrangian having global $O(N)$ symmetry leading to the same symmetry in the quantum effective action but to a breaking of this symmetry when the current with which the field interacts has random Gaussian fluctuations. In the process of discussing the KG field, we analyze its behaviour in the vicinity of the event horizon of a Schwarzchild blackhole and provide an independent derivation of Hawking's formula for the temperature of a blackhole at which it radiates photons. The quantum mechanical problem of estimating the state of a gravitational wave from electromagnetic field measurements using the Belavkin quantum filter is also presented.

2. The quantum electromagnetic field in curved background space-time

The Lagrangian density of the electromagnetic field in the background metric $g_{\mu\nu}(x)$ is given by

$$L(A_\mu, A_{\mu,\nu}) = (-1/4)F_{\mu\nu}F^{\mu\nu}\sqrt{-g} + (a/2)(-g)^{-1/2}(A^\mu\sqrt{-g})_{,\mu})^2 - - - (1)$$

where

$$A^\mu = g^{\mu\nu}A_\nu - - - (2)$$

It is clear that the integral of L over the whole of space-time is invariant w.r.t diffeomorphisms because of the invariance of the 4-volume element $\sqrt{-g}d^4x$ and the fact that

$$(A^\mu\sqrt{-g})_{,\mu} = A^\mu_{:\mu}\sqrt{-g} - - - (3)$$

with $A^\mu_{:\mu}$ is a scalar as also is $F_{\mu\nu}F^{\mu\nu}$. The second term in (1) is to be regarded as a gauge fixing term for the electromagnetic field. It stems from the invariance of the electromagnetic field

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} - - - (4)$$

under the gauge transformation

$$A_\mu \to A_\mu + \phi_{,\mu} - - - (5)$$

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with ϕ a scalar field. The parameter a can be looked upon as a Lagrange multiplier introduced to constrain $(A^{\mu}\sqrt{-g})_{,\mu}$ to vanish, just as we have the Lorentz gauge in flat space-time. It is useful to introduce this gauge fixing term as then the momentum fields corresponding to all the four position fields A_{μ} will be non-zero thereby enabling us to avoid the uncomfortable situation of having to treat this as a Lagrangian/Hamiltonian problem with constraints thereby forcing us to use Dirac brackets in place of Lie/Poisson brackets. Noting that

$$(A^{0}\sqrt{-g})_{,0} = g^{0\mu}\sqrt{-g}A_{\mu,0} + X - - - (6)$$

where X involves non-derivative terms of A_{μ}, we see that the momentum fields P^{μ} corresponding to the position fields A_{μ} are given by

$$P^{\mu}(x) = \partial L/\partial A_{\mu,0} = F^{\mu 0}(x)\sqrt{-g(x)} + a.\,g^{0\mu}(A^{\nu}\sqrt{-g})_{,\nu} - - - (7)$$

The canonical equal time Bosonic commutation relations (CCR) are

$$[A_{\mu}(x), P^{\nu}(y)] = i\delta^{\nu}_{\mu}\delta^{3}(x-y), x^{0} = y^{0} - - - (8)$$

and of course

$$[A_{\mu}(x), A_{\nu}(y)] = [P_{\mu}(x), P_{\nu}(y)] = 0, x^{0} = y^{0} - - - (9)$$

These CCR's imply

$$[A_{\mu}(x), g^{\nu\rho}(y)g^{00}(y)\sqrt{-g(y)}A_{\rho,0}(y) + a.\,ag^{0\nu}(y)g^{0\rho}(y)\sqrt{-g(y)}A_{\rho,0}(y)]$$

$$= i\delta^{\nu}_{\mu}\delta^{3}(x-y), x^{0} = y^{0} - - - (10)$$

or equivalently,

$$(g^{\nu\rho}g^{00} + ag^{\nu 0}g^{\rho 0})(y)\sqrt{-g(y)}[A_{\mu}(z), A_{\rho,0}(y)] = i\delta^{\nu}_{\mu}\delta^{3}(x-y), x^{0} = y^{0} - - - (11)$$

Equivalently, defining the matrix

$$C^{\nu\rho}(y) = (g^{\nu\rho}g^{00} + ag^{\nu 0}g^{\rho 0})(y)\sqrt{-g(y)} - - - (12)$$

and its inverse

$$((K_{\nu\rho}(y))) = K(y) = C(y)^{-1} = ((C^{\nu\rho}(y)))^{-1} - - - (13)$$

we obtain the fundamental CCR for electromagnetics in curved background:

$$C^{\nu\rho}(y)[A_{\mu}(x), A_{\rho,0}(y)] = i\delta^{\nu}_{\mu}\delta^{3}(x-y), x^{0} = y^{0} - - - (14)$$

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or equivalently,

$$[A_\mu(x), A_{\nu,0}(y)] = iK_{\nu\mu}(y)\delta^3(x-y), x^0 = y^0 - - - (15)$$

In flat space-time, we have $g^{\nu\rho}(x) = \eta^{\nu\rho}$ and the CCR simplifies to

$$[A_\mu(x), A_{\nu,0}(y)] = i[(\eta + a.uu^T)^{-1}]_{\nu\mu}\delta^3(x-y), x^0 = y^0 - - - (16)$$

where

$$u = [1, 0, 0, 0]^T - - - (17a)$$

and

$$\eta = diag[1, -1, -1, -1] - - - (17b)$$

is the Minkowskian metric.

Remark: The CCR derived above is not a coordinate and gauge independent formula. It depends on a specific coordinate system chosen as well as the chosen gauge. It follows therefore that since simultaneity of two events depends on the choice of the reference frame, we cannot use the equal time commutation relation derived above in a different system of coordinates. Further, the above formula suggests that the "degree of non-commutativity of the position fields $A_\mu(x)$ and the corresponding velocity fields $A_{\mu,0}(y)$ at equal times $x^0 = y^0$ is metric dependent and also frame and gauge dependent. This suggests that the theoretical variables formulation of quantum mechanics proposed by the second author in a series of papers (see [1] and references there) will lead to degrees of accessibility and inaccessibility of observables being frame and gauge dependent.

3. The Dirac field in a back-ground curved space-time interacting with the electromagnetic field

The covariant derivative is

$$\nabla_\mu = \partial_\mu - igA_\mu + \Gamma_\mu - - - (18)$$

where

$$A_\mu = A_\mu^a T_a, \Gamma_\mu = \omega_\mu^{mn}\gamma_{mn}/4 - - - (19)$$

Note that

$$[T_a, T_b] = -iC(abc)T_c - - - (20)$$

$$[\gamma_{mn}/4, \gamma_{ab}/4] = \eta_{mb}\gamma_{na}/4 + \gamma_{na}\eta_{mb}/4 - \gamma_{ma}\eta_{nb}/3 - \eta_{nb}\gamma_{ma}/4 - - - (20)$$

T_a are Hermitian matrices. Since $\gamma^0, \gamma^0\gamma^n$ are Hermitian, we have

$$(\gamma^0\gamma^m\gamma^n)^* = (\gamma^n)^*\gamma^0\gamma^m = (\gamma^0\gamma^n)^*\gamma^m = \gamma^0\gamma^n\gamma^m - - - (21)$$

and hence,

$$\gamma^0\gamma^{mn} = \gamma^0[\gamma^m, \gamma^n] - - - (22)$$

are skew-Hermitian matrices. Thus,

$$i\gamma^0\Gamma_\mu = \omega_\mu^{mn} i\gamma^0\gamma_{mn}/4 - - - (23)$$

are Hermitian matrices. The Dirac operator is

$$iD - m, D = \gamma^\mu\nabla_\mu, \gamma^\mu(x) = \gamma^a V_a^\mu(x) - - - (24)$$

The Dirac equation is

$$(iD - m)\psi = 0 - - - (25)$$

which is the same as

$$i\gamma^\mu(\partial_\mu - igA_\mu + \Gamma_\mu) - - - (26)$$

The Dirac Lagrangian density from which the Dirac equation is derived is given by

$$L = \psi^*\gamma^0[iD - m]\psi.\sqrt{-g} - - - (27)$$

Hence, the momentum field conjugate to the canonical position field ψ is given by

$$P = \partial/\partial\partial_0\psi = iV_a^0\psi^*\gamma^0\gamma^a\sqrt{-g} = iV_a^0\sqrt{-g}\psi^*\alpha^a = i\sqrt{-g}\psi^*\tilde{\alpha}^0 - - - (28)$$

where

$$\alpha^\mu = \alpha^\mu(x) = \alpha^a V_a^\mu(x) - - - (29)$$

are the non-constant Dirac α-matrices in the gravitational field. Note that the α^a are the constant Dirac α matrices. The canonical equal time anticommutation relations are therefore

$$\{\psi(x), \psi(y)^*\}\tilde{\alpha}^0(y)\sqrt{-g(y)} = \delta^3(x - y), x^0 = y^0 - - - (30)$$

or equivalently,

$$\{\psi(x), \psi(y)^*\} = (-g(x))^{-1/2}(\tilde{\alpha}^0(x))^{-1}\delta^3(x - y), x^0 = y^0 - - - (31)$$

Now, we have the anticommutation relations

$$\{\tilde{\gamma}^\mu(x), \tilde{\gamma}^\nu(x)\} = 2g^{\mu\nu}(x) - - - (32)$$

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and in particular,

$$(\tilde{\alpha}^0(x))^2 = (\gamma^0\tilde{\gamma}^0(x))^2 = (\gamma^0\gamma^a V_a^0(x))^2 = (\alpha^a V_a^0(x))^2 = \{\alpha^a, \alpha^b\} V_a^0 V_b^0/2$$

$$= (1/2)\sum_{a=0}^{3} (V_a^0)^2 = K(x) - - - (33)$$

say. Note that by $K(x)$, we mean $K(x)I$. We have used the fact that the α^a's mutually anticommute and their squares are the identity. It follows therefore that

$$\alpha^0(x)^{-1} = K(x)^{-1}\alpha^0(x) - - - (34)$$

Thus, we obtain the equal time CAR as

$$\{\psi(x), \psi(y)^*\} = (-g(x))^{-1/2} K(x)^{-1}\alpha^0(x)\delta^3(x-y), x^0 = y^0 - - - (35)$$

This is a fundamental equation because it gives us an idea of how much the canonical anticommutation relations of the Dirac field get affected by the presence of a background gravitational field. Specifically, we can evaluate this anticommutator in the background Robertson-Walker metric for an expanding homogeneous and isotropic universe and show that this equal time CAR (Canonical anticommutation relation) becomes proportional to $S(t)^{-3}$.

4. The wave function in cosmological models

The idea of calculating probability amplitudes in cosmology and study the evolution of the scale factor of the universe as it evolves through different histories using the Feynman path integral method is originally due to Hawking (The wave function of the universe). Hawking's idea is to start with the RW model for space time corresponding to a homogensous isotropic universe, ie,

$$d\tau^2 = dt^2 - S(t)^2 f(r)^2 - S(t)^2 r^2 (d\theta^2 + sin^2(\theta) d\phi^2) - - - (36)$$

so that

$$g_{00} = 1, g_{11} + -S(t)^2 f(t)^2, 22 = -S(t)^2 r^2, g_{33} = -S(t)^2 r^2 sin^2(\theta) - - - (37a)$$

where

$$f(r)^2 = 1/(1-kr^2) - - - (37b)$$

with $k = 0, 1, -1$ according as space is flat, spherrical or hyperbolic and then evaluate the Einstein-Hilbert Lagrangian density for this universe

$$L_G(r, \theta, S(t), S'(t)) =$$

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$$g^{\mu\nu}[\Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta} - \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}]\sqrt{-g} - - - (38)$$

Hawking then considers the Lagrangian density for a scalar Klein-Gordon field $\phi(t, r)$ in this background metric:

$$L_{KG}(r, \theta, \chi, \partial_{\mu}\chi | S(t)) = (1/2)g^{\mu\nu}\sqrt{-g}\chi_{,\mu}\chi_{,\nu} - m^2\sqrt{-g}\chi^2/2 - - - (39)$$

He then considers evaluating the Schrodinger wave function of the scale factor of the universe$S$ at time t in a universe filled with such scalar particles using the Feynman path integral

$$\psi(t, S) = C\int exp(iS_G[t, S] + iS_{KG}[t, \chi | S])DS[0, t)D\chi[0, t] - - - (40)$$

where C is a numerical factor and

$$S_G[t, S] = \int_0^t ds \int L_G(r, \theta, S(s), S'(s))drd\theta.\, d\phi$$

$$= \int_0^t L_g(S(s), S'(s))ds - - - (41a)$$

where

$$L_g = \int L_G d^3x - - - (41b)$$

is the Lagrangian of $S(t)$ and

$$S_{KG}[t, \chi | S] = \int_0^t ds \int L_{KG}(r, \theta, \chi(s, r, \theta, \phi), \partial_{\mu}\chi(s, r, \theta, \phi) | S(s))drd\theta.\, d\phi - - - (42)$$

Hawking notes that since the universe is homogeneous and isotropic, we can assume that the KG wave field ψ is a function of only time to a good degree of approximation, so that

$$L_{KG} = (1/2)\chi'(t)^2\sqrt{-g} - m^2\sqrt{-g}\chi(t)^2 - - - (43)$$

where

$$\sqrt{-g} = S^3(t)f(r)r^2 sin(\theta) - - - (44)$$

and so

$$\int \sqrt{-g}d^3x = KS(t)^3, K = 2\pi\int_0^1\int_0^{\pi} f(r)r^2 sin(\theta)drd\theta - - - (45)$$

so that the KG Lagrangian becomes

$$L_{kg}(\chi(t), \chi'(t) | S(t)) = (K/2)S(t)^3[\chi'(t)^2 - m^2\chi(t)^2] - - - (46)$$

which means that the joint wave function of $S(t), \chi(t)$ (ie, the scale factor and the KG field at time t), is given by the formula

$$\psi(t, S, \chi) = C\int exp(i\int_0^t L_g(S(s), S'(s)) + L_{kg}(\chi(s), \chi'(s) | S(s))ds)DS[0, t)D\chi[0, t) - - - (47)$$

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where $S(t) = S, \chi(t) = \chi$ and in particular, the probability density of $S(t)$ is given by

$$p(t,S) = \int |\psi(t,S,\chi)|^2 d\chi - - - (48)$$

Lengthy but elementary calculations show that the Einstein-Hilbert Lagrangian density

$$L_G = g^{\mu\nu} L_{\mu\nu} \sqrt{-g} - - - (49)$$

for the RW metric evaluate as follows:

$$L_G = (g^{00}L_{00} + g^{11}L_{11} + g^{22}L_{22} + g^{33}L_{33})\sqrt{-g} - - - (50)$$

with

$$L_{00} = \Gamma^{\alpha}_{00}\Gamma^{\beta}_{\alpha\beta} - \Gamma^{\alpha}_{0\beta}\Gamma^{\beta}_{0\alpha}$$

$$= -3S^{'2}/S^2 - - - (51)$$

$$L_{11} = \Gamma^{\alpha}_{11}\Gamma^{\beta}_{\alpha\beta} - \Gamma^{\alpha}_{1\beta}\Gamma^{\beta}_{1\alpha}$$

$$= 3S^{'2}f^2(r) - 2/r^2 - - - (52)$$

$$L_{22} = \Gamma^{\alpha}_{22}\Gamma^{\beta}_{\alpha\beta} - \Gamma^{\alpha}_{2\beta}\Gamma^{\beta}_{2\alpha}$$

$$= S^{'2}r^2 - rf^{'}/f^3 - 1/f^2 - cot^2(\theta) - - - (53)$$

$$L_{33} = \Gamma^{\alpha}_{33}\Gamma^{\beta}_{\alpha\beta} - \Gamma^{\alpha}_{3\beta}\Gamma^{\beta}_{3\alpha}$$

$$= (S^{'2}r^2 - rf^{'}/f^3 - 1/f^2)sin^2(\theta) + cos^2(\theta) - - - (54)$$

so that

$$L_G = (4S^{'2}/S^2 + 2/S^2r^2 + 2f^{'}/rS^2f^3 + 2/S^2f^2r^2)S^3fr^2sin(\theta) - - - (55)$$

It is clear then, that the Lagrangian of the scale factor $S(t)$ has the form

$$L_g(S(t), S^{'}(t)) = \int L_G dr d\theta . d\phi = c_1 S^{'2}S + c_2 S - - - (56)$$

where c_1, c_2 are constants and hence that the total Lagrangian of the scale factor $S(t)$ and the KG field $\chi(t)$ in homogeneous and isotropic space-time is given by

$$L(S, S^{'}, \chi, \chi^{'}) = c_1 S^{'2}S + c_2 S + (K/2)S(t)^3[\chi^{'}(t)^2 - m^2\chi(t)^2] - - - (57)$$

This general form can be used in our study of the joint wave function of the scale factor and the KG field and of course, also the classical dynamics of these two quantities. Alternatively, can also quantize this dynamics using the Schrodinger wave mechanics and the Heisenberg matrix mechanics by regarding S, χ as canonical position variables with the corresponding momentum variables given by

$$P_S = \partial L/\partial S^{'} = 2c_1 S S^{'}, P_{\chi} = \partial L/\partial \chi^{'} = KS^3\chi^{'} - - - (58)$$

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The commutation relations (Bosonic) are given by

$$[S, P_S] = i, [\chi, P_\chi] = i, [S, \chi] = [P_S, P_\chi] = 0 - - - (59)$$

so in terms of velocities,

$$[S, S^{'}] = i/2c_1 S, [\chi, \chi^{'}] = i/KS^3 - - - (60)$$

The second equation shows that as the universe keeps expanding, $S(t)$ increases with time, and hence the commutator between χ and $\chi^{'}$ gets smaller and smaller which means that the uncertainty in measuring both the KG field and its rate of change with time gets smaller and smaller. Likewise, the first commutation relations also implies that the uncertainty between the scale factor and its rate of change with time also decreases in the expanding universe. It is interesting to see what this implies from the standpoint of degrees of inaccessibility of theoretical variables.

5. Joint wave function of Klein-Gordon field and the scale factor of the universe from the Feynman path integral

We are now interested in formulating a generalization of Hawking's theory to the case when the scalar particles in our universe are not distributed homogeneously and isotropically, ie, when the KG field is $\chi(t, r, \theta, \phi)$. In order to do so, we choose basis functions $\eta_n(r, \theta, \phi), n = 1, 2, \ldots$ that are functions of only the spatial variables and expand the KG field in terms of them with coefficients being functions of time:

$$\chi(t, r, \theta, \phi) = \sum_n \chi_n(t)\eta_n(\mathbf{r}), \mathbf{r} = (r, \theta, \phi) - - - (61)$$

Then, w.r.t to the RW metric, we have

$$\int g^{\mu\nu}\chi_{,\mu}\chi_{,\nu}\sqrt{-g}d^3x =$$

$$(\chi_{,0}^2 + g^{11}\chi_{,1}^2 + g^{22}\chi_{,2}^2 + g^{33}\chi_{,3}^2)\sqrt{-g}$$

$$= S(t)^3\sum_{n,m} a(n, m)\chi_n^{'}(t)\chi_m^{'}(t) - S(t)\sum_{n,m} b(n, m)\chi_n(t)\chi_m(t) - - - (62)$$

where

$$a(n, m) = \int \eta_n((\mathbf{r})\eta_m(\mathbf{r})f(r)r^2 sin(\theta)drd\theta d\phi - - - (63)$$

$$b(n, m) = \int \eta_{n,1}(\mathbf{r})\eta_{m,1}(\mathbf{r})f(r)^{-1}r^2 sin(\theta)drd\theta d\phi,$$

$$+\int \eta_{n,2}(\mathbf{r})\eta_{m,2}(\mathbf{r})f(r)sin(\theta)drd\theta d\phi +$$

$$\int \eta_{n,3}(\mathbf{r})\eta_{m,3}(\mathbf{r})f(r)sin(\theta)^{-1}drd\theta d\phi - - - (64)$$

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Moreover,

$$m^2\int\chi^2\sqrt{-g}d^3x = S(t)^3\sum_{n,m} c(n,m)\chi_n(t)\chi_m(t) - - - (65)$$

where

$$c(n,m) = m^2\int\eta_n(\mathbf{r})\eta_m(\mathbf{r})f(r)r^2 sin(\theta)d^3x - - - (66)$$

Thus, the KG Lagrangian becomes

$$L_{kg}(\chi_n(t), \chi_n^{'}(t), n \geq 1) =$$

$$(1/2)\int g^{\mu\nu}\chi_{,\mu}\chi_{,\nu}\sqrt{-g}d^3x - (1/2)\int m^2\chi^2\sqrt{-g}d^3x$$

$$= (1/2)S(t)^3\sum_{n,m} a(n,m)\chi_n^{'}(t)\chi_m^{'}(t) - (1/2)S(t)\sum_{n,m} b(n,m)\chi_n(t)\chi_m(t)$$

$$-(1/2)S(t)^3\sum_{n,m} c(n,m)\chi_n(t)\chi_m(t)$$

$$= (1/2)S(t)^3\chi'(t)^T A\chi'(t) - (1/2)S(t)\chi(t)^T B\chi(t) - (1/2)S(t)^3\chi(T)^T C\chi(t) - - - (67)$$

where

$$\chi(t) = ((\chi_n(t)))_{n=1}^{\infty}, A = ((a(n,m))), B = ((b(n,m))), C = ((c(n,m))) - - - (68)$$

Thus the joint wave function of $(S(T), \chi(T))$ is now given by

$$\psi(T, S, \chi) = \psi(T, S, ((\chi_n)))) =$$

$$\int exp(i(c_1 S'(t)^2 S(t) + c_2 S(t) + (i/2)S(t)^3\chi'(t)^T A\chi'(t) - (i/2)S(t)\chi(t)^T B\chi(t)$$

$$-(i/2)S(t)^3\chi(T)^T C\chi(t))DS[0,T)D\chi[0,T) - - - (69)$$

It should be noted that if we were interested only in the wave function of $S(t)$, then we would first evaluate the Gaussian integral w.r.t χ by replacing it by the value χ_0 at which its action is stationary, ie, χ_0 satisfies

$$-S(t)^3 A\chi_0^{''}(t) - S(t)B\chi(t) + S(t)^3 C\chi_0(t) = 0 - - - (70)$$

This is the same as

$$A\chi_0^{''}(t) = -S(t)^{-2}B\chi(t) + C\chi_0(t) - - - (71)$$

This is an infinite dimensional linear second order differential equation with time varying coefficients for the infinite dimensional vector valued function of time

$$\chi_0(t) = ((\chi_{0n}(t))) - - - (72)$$

and its solution will be a function of $S(s), s \leq t$.

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6. Quantum effective action in cosmology in the presence of a random current field interacting with the scalar field

Some remarks on the quantum effective action in quantum cosmology: Consider as above, the joint action of the gravitational field and the scalar KG field but taking into account an interaction between the KG field and a classical random current source $J(t)$. Assuming for simplicity that the KG field depends only on time, this action is given by

$$S_1[S,\chi|J] = \int[c_1 S^{'2}(t)S(t) + c_2 S(t) + (K/2)S(t)^3(\chi^{'}(t)^2 - m^2\chi(t)^2) + J(t)\chi(t)]dt - - - (73)$$

We write this action as

$$\int[L(S(t), S^{'}(t), \chi(t), \chi^{'}(t)) + J(t)S(t)]dt = S_0[S,\chi] + \int J\chi.\,dt - - - (74)$$

Assume for simplicity that $J(t)$ is a Gaussian random current source with mean $M_J(t)$ and covariance

$$Cov(J(t), J(s)) = C_J(t,s) - - - (75)$$

In order to compute the quantum effective action of S,χ after taking into account this interaction, we form the statistical mean of the path integral:

$$Z(M,C) = exp(iW(M,C))$$
$$= \mathrm{E}\int exp(i\int(L(S(t), S^{'}(t), \chi(t), \chi^{'}(t)) + J(t)\chi(t))dt)DS.\,D\chi$$
$$= \int exp(iS_0[S,\chi] + i\int M(t)\chi(t)dt - (1/2)\int C(t,s)\chi(t)\chi(s)dtds)DS.\,D\chi - - - (76)$$

Such a approximation to the path integral is justified when we the scalar field interacts with a cloud of other particles like gravitons distributed all over the cosmos with the the quantum fluctuations in the graviton field being approximated by a classical random field. If we wish to be more accurate, we should take $J(t)$ as a quantum stochastic process in the sense of Hudson and Parthasarathy [2] and calculate quantum expectations of the resulting path integral in a coherent state of the current. We first discuss the classical stochastic approximation and then the quantum stochastic approximation. For given current covariance C, we can define the quantum effective action in the usual way:

$$\Gamma(\chi_0, C) = Ext_M(-i.\,logZ(M,C) - \int M\chi_0 dt) - - - (77)$$

Here, we are assuming that the scale factor process $S(t)$ is a given fixed classical process. Extremizing, we get

$$i\delta logZ(M,C)/\delta M(t) + \chi_0(t) = 0 - - - (78)$$

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Note that this equation implies that the value of the mean current M is that at which the average of $\chi(t)$ equals the classical process $\chi_0(t)$ given the scale factor process $S(.)$ and the classical current covariance C. We now observe that assuming that M satisfies this equation,

$$\delta\Gamma(\chi_0, C)/\delta\chi_0(t) = -M(t) - - - (79)$$

This is the required equation of motion for the classical field $\chi_0(t)$ defined as the classical and quantum average of $\chi(t)$ given the mean current M and the current covariance C. In this context, it is interesting to generalize this equation to the general case when we do not restrict to a homogeneous and isotropic KG scalar field. In that case, proceeding as earlier, the KG action in the presence of a random current field $J(t, \mathbf{r})$ is given by

$$\int L_{kg}(\chi_n(t), \chi_n^{'}(t), n \geq 1)dt + \sum_n \int J_n(t)\chi_n(t)dt$$

$$= \int [(1/2)S(t)^3\chi^{'}(t)^T A\chi^{'}(t) - (1/2)S(t)\chi(t)^T B\chi(t) - (1/2)S(t)^3\chi(T)^T C\chi(t)]dt$$

$$+\int J(t)^T D\chi(t)dt$$

$$= S_0[\chi] + \int J^T D\chi dt - - - (81)$$

where we have expressed the current field in terms of the spatial basis functions $\eta_n(\mathbf{r})$ as

$$J(t, \mathbf{r}) = \sum_n J_n(t)\eta_n(\mathbf{r}) - - - (82)$$

and

$$\chi(t, r, \theta, \phi) = \chi(t, \mathbf{r}) = \sum_n \chi_n(t)\eta_n(\mathbf{r}) - - - (83)$$

so that

$$\int J(t, \mathbf{r})\chi(t, \mathbf{r})d^3r dt = \int J(t)^T D\chi(t)dt - - - (84)$$

where

$$J(t) = ((J_n(t))_n, \chi(t) = ((\chi_n(t)) - - - (85)$$

and

$$D = ((\int \eta_n(t, \mathbf{r})\eta_m(t, \mathbf{r})d^3r))_{n,m} - - - (86)$$

Writing

$$\mathrm{E}J_n(t) = M_n(t), Cov(J_n(t), J_m(s)) = C_{0nm}(t, s) - - - (87)$$

or equivalently,

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$$E(J(t)) = M(t) = ((M_n(t))), Cov(J(t), J(s))$$

$$= E(J(t)J(s)^T) - M(t)M(s)^T = C_0(t,s) = ((C_{0nm}(t,s)))_{n,m} - - - (88)$$

We get

$$Z(M, C_0) = E\int exp(iS_0[\chi] + i\int J(t)^T D\chi(t)dt)D\chi$$

$$= \int exp(iS_0[\chi] + i\int M(t)^T D\chi(t)dt - (1/2)\int \chi(t)^T DC_0(t,s)D\chi(s)dtds)D\chi - - - (89)$$

so that the quantum effective action is given by

$$\Gamma(\chi_0, C_0) = Ext_M(-i.\ log(Z(M, C_0)) - \int M^T D\chi_0 dt)$$

$$= -ilogZ(M_0, C_0) - \int M_0^T D\chi_0 dt - - - (90)$$

where $M_0(t)$ satisfies

$$i\delta(logZ(M_0, C_0))/\delta M(t) + D\chi_0(t) = 0 - - - (91)$$

The corresponding quantum equations of motion are

$$\delta\Gamma(\chi_0, C_0)/\delta\chi_0(t) = -DM_0(t) - - - (92)$$

Noting that

$$Q(\chi, M, C_0) = iS_0[\chi] + i\int M(t)^T D\chi(t)dt - (1/2)\int \chi(t)^T DC_0(t,s)D\chi(s)dtds$$

$$= i\int [(1/2)S(t)^3\chi'(t)^T A\chi'(t) - (1/2)S(t)\chi(t)^T B\chi(t) - (1/2)S(t)^3\chi(t)^T C\chi(t)]dt$$

$$+i\int M(t)^T D\chi(t)dt - (1/2)\int \chi(t)^T DC_0(t,s)D\chi(s)dtds - - - (93)$$

is a linear-quadratic functional of χ, it is easy to evaluate the Gaussian integral

$$Z(M, C_0) = \int exp((-1/2)Q(\chi, M, C_0))D\chi - - - (94)$$

as apart from a multiplicative constant, equal to

$$D_0(S, C_0)^{-1/2}.\ exp((-1/2)Q(\chi_0, M, C_0)) - - - (95)$$

where $D_0(S, C_0)$ is the determinant of the kernel-matrix of the quadratic form Q and χ_0 is the value of χ at which Q becomes stationary, ie, χ_0 satisfies

$$(S(t)^3 A\chi_0'(t))' + S(t)B\chi_0(t) + S(t)^3 C\chi_0(t) - DM(t) + \int DC_0(t,s)D\chi_0(s)ds = 0 - - - (96)$$

Apart from a multiplicative constant, the kernel-matrix of the quadratic form Q is given by

$$K_Q(t,s) = (d/dt)(S(t)^3\delta'(t-s))A + \delta(t-s)(S(t)B + S(t)^3 C) + DC_0(t,s)D - - - (97)$$

and its determinant has to be evaluated.

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7. Implications of classical randomness in the current source on symmetry breaking

It is a well known fact that when the gauge symmetry of a quantum effective action is spontaneously broken by the field acquiring vacuum expectation values, then massless particles are produced, one particle being associated with each gauge degree of freedom that is broken. On the other hand, when the gauge symmetry of the quantum effective action is approximately broken by the presence of small perturbations to it that do not respect gauge symmetry, then massless particles acquire masses, and already massive particles become more massive. So it is important to decide the mechanisms by which the gauge symmetry of a quantum effective action can be broken. We shall consider the problem of symmetry breaking from the standpoint of coupling of the field to a random classical Gaussian current field. To this end, consider the action $I[\phi]$ of a field such that under an infinitesimal gauge transformation

$$\phi \to \phi + \epsilon.\Delta(\phi) - - - (98)$$

the product of the exponentiated action and the path measure remains invariant, ie,,

$$exp(iI[\phi])D\phi = exp(iI[\phi + \epsilon.\Delta(\phi)])D(\phi + \epsilon.\Delta(\phi)) - - - (99)$$

For example, for the complex KG field in a background curved space-time, the action is

$$I[\psi] = \int g^{\mu\nu}\sqrt{-g}\bar{\psi}_{,\mu}\psi_{,\nu}d^4x - m^2\int \psi^* \psi\sqrt{-g}d^4x - - - (100)$$

which is invariant under the infinitesimal $U(1)$ gauge transformation

$$\psi \to exp(i\epsilon)\psi = \psi + i\epsilon\psi, \epsilon \in \mathrm{R}, \epsilon \to 0 - - - (101)$$

More generally, for a vector valued complex KG field $\psi = ((\psi_n))_{n=1}^N$, the action can be taken as

$$I[\psi] = \sum_n \int g^{\mu\nu}\sqrt{-g}\bar{\psi}_{n,\mu}\psi_{n,\nu}d^4x - m^2\sum_n \int \psi_n^* \psi_n\sqrt{-g}d^4x - - - (102)$$

which is invariant under the infinitesimal $U(N)$ gauge transformation

$$\psi \to exp(iX)\psi = \psi + iX.\psi, X \in \mathrm{C}^{N\times N}, X^* = X, \ \| X \| \to 0 - - - (103)$$

Let $T_n, n = 1, 2, \ldots, N^2$ be a basis for the N^2 dimensional real vector space of $N \times N$ Hermitian matrices. Then, the above $U(N)$ symmetry of the complex KG action can also be expressed as

$$\psi \to exp(i\sum_{n=1}^{N} \epsilon(n)T_n)\psi = \psi + i\sum_{n=1}^{N^2} \epsilon(n).T_n\psi, \epsilon(n) \in \mathrm{R}, \epsilon(n) \to 0 - - - (104)$$

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If instead, ψ where a real KG scalar field with action

$$I[\psi] = (1/2)\sum_n \int g^{\mu\nu}\sqrt{-g}\psi_{n,\mu}\psi_{n,\nu}d^4x - (m^2/2)\sum_n \int \psi_n^2 - - - (105)$$

then the symmetry would instead be $O(N, \mathrm{R})$, ie,

$$\psi \to \psi + \epsilon. X. \psi, X \in \mathrm{R}^{N\times N}, X^T = -X - - - (106)$$

this symmetry group now being $N(N-1)/2$-dimensional. Now consider the path integral

$$Z(J) = \int exp(iI[\psi] + i\int J.\psi d^4x)D\psi - - - (107)$$

where J is a non-random current field. It is well known that when $exp(iI[\psi])D\psi$ is invariant under the infinitesimal gauge transformation $\psi \to \psi + \epsilon.\Delta(\psi)$, then the equation

$$Z(J) = \int exp(iI[\psi] + i\int J.\psi)(1 + i\int J.\Delta(\psi))D\psi = 0 - - - (108)$$

gives

$$\int J. < \Delta(\psi) >_J d^4x = 0 - - - (109)$$

which can be expressed in the form of a gauge invariance principle for the quantum effective action

$$\int \frac{\delta\Gamma[\psi_0]}{\delta\psi_0(x)}. < \Delta(\psi) >_J(x)d^4x = 0 - - - (110)$$

where J is that current field for which

$$< \psi > (x)_J = \psi_0(x) - - - (111)$$

In the case when the gauge transformation $\Delta(\psi)$ is linear in ψ as it happens for the three examples considered above, then,

$$< \Delta(\psi) >_J = \Delta(< \psi >_J) = \Delta(\psi_0) - - - (112)$$

and then we get the result that the quantum effective action is also invariant under the same gauge transformation that leaves the classical action invariant, or more precisely as that which leaves the product $exp(iI[\psi])D\psi$ invariant:

$$\int \frac{\delta\Gamma[\psi_0]}{\delta\psi_0(x)}.\Delta(\psi_0)(x)d^4x = 0 - - - (113)$$

Note that the quantum effective action is defined as

$$\Gamma[\psi_0] = Ext_J[-ilogZ(J) - \int J.\psi_0 d^4x] - - - (114)$$

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where Ext denotes extremum w.r.t J. These results make use of the fact that the quantum effective action $\Gamma[\psi_0]$ satisfies its equation of motion

$$\delta\Gamma[\psi_0]/\delta\psi_0(x) = -J(x) - - - (115)$$

and hence

$$\delta\Gamma[\psi_0]/\delta\psi_0(x)\delta\psi_0(y) = -\delta J(x)/\delta\psi_0(y) - - - (116)$$

On the other hand, we have

$$\psi_0(x) = Z(J)^{-1}\int\psi(x).exp(i.I[\psi] + i\int J.\psi)D\psi$$

$$= -i\delta log(Z(J))/\delta J(x) - - - (117)$$

and hence,

$$\delta\psi_0(x)/\delta J(y) = -i\delta^2 Z(J)/\delta J(x)\delta J(y) = -\Delta(x,y) - - - (118)$$

where $\Delta(x,y)$ is the propagator of the field ψ after subtracting out its mean value ψ_0, (ie, the propagator of the quantum fluctuating component of the field around its vacuum expected value). Thus, we get the fundamental formula

$$\delta\Gamma(\psi_0)/\delta\psi_0(x)\delta\psi_0(y) = \Delta^{-1}(x,y) - - - (119)$$

which means that the eigenfunctions of the Hessian matrix of the quantum effective action having zero eigenvalues are precisely the eigenvectors of the propagator having infinite eigenvalues. The eigenvalues of the inverse Bosonic propagator are however the squared masses of the particles in analogy with the fact that the inverse propagator of a free KG particle i s given by $p^2 - m^2$ in the momentum domain, for which if δm^2 is an eigenvalue corresponding to an eigenvector, then this eigenvector is a field perturbation which when added to the vacuum expected field, carries a mass of $m^2 + \delta m^2$. Now let us consider the situation when the current J is a random field with mean M and covariance C. Then, the quantum effective action is computed as

$$\Gamma(\psi_0, C) = Ext_M(-i.logZ(M,C) - \int M.\psi_0) - - - (120)$$

with

$$Z(M,C) = \mathrm{E}exp(iI[\psi] + i\int J.\psi)D\psi$$

$$= \int exp(iI[\psi] + i\int M.\psi - (1/2)\int\psi^T C\psi)D\psi - - - (121)$$

so that

$$\Gamma(\psi_0) = -i.logZ(M_0, C) - \int M_0.\psi_0 - - - (122)$$

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where M_0 satisfies,

$$-i\delta logZ(M_0, C)/\delta M(x) - \psi_0(x) = 0 - - - (123)$$

This gives

$$\delta\Gamma(\psi_0)/\delta\psi_0(x) = -M_0(x) - - - (124)$$

However, now observe that $\Gamma(\psi_0)$ is now no longer gauge invariant since we have on replacing ψ by $\psi + \epsilon.\Delta(\psi)$ in the path integral (a),

$$\int exp(iI[\psi] + i\int M.\psi + i\epsilon\int M.\Delta(\psi) - (1/2)\int \psi^T C\psi - \epsilon\int \psi^T C\Delta(\psi))D\psi = 0 - - - (125)$$

or equivalently,

$$i\int M. < \Delta(\psi) >_M - \int < \psi^T C\Delta(\psi) >_M = 0 - - - (126)$$

or equivalently, making use of the equations of motion

$$\int (\delta\Gamma(\psi_0)/\delta\psi_0(x)). < \Delta(\psi)(x) >_{M_0} = -i\int < \psi(x)C(x, y)\Delta(\psi(y)) > - - - (127)$$

which shows clearly by how much is gauge invariance of the quantum effective action broken when the current to which the field is coupled has random fluctuations.

An example from background general relativity and cosmology: Suppose that $g_{\mu\nu}(x|\theta(x))$ is the metric of space-time where θ is a random parameter field and that the field $\phi(x)$ in this background field is described by the Lagrangian

$$L(\phi(x), \partial_\mu\phi(x), g_{\mu\nu}(x|\theta(x)) - - - (128)$$

Assume that $\theta(x)$ has small random fluctuations around its mean value $\theta_0(x)$, so writing

$$\delta\theta(x) = \theta(x) - \theta_0(x) - - - (129)$$

we can write approximately

$$g_{\mu\nu}(x|\theta(x)) = g_{\mu\nu}(x|\theta_0(x) + g_{\mu\nu,k}(x|\theta_0)\delta\theta_k(x) - - - (130)$$

where

$$g_{\mu\nu,k}(x|\theta_0) = \partial g_{\mu\nu}(x|\theta_0)/\partial\theta_k - - - (131)$$

Another linearization gives approximately

$$L(\phi(x), \partial_\mu\phi(x), g_{\mu\nu}(x|\theta(x)))$$

$$= L(\phi(x), \partial_\mu\phi(x), g_{\mu\nu}(x|\theta_0(x)) +$$

$$(\partial L(\phi(x), \partial_\mu\phi(x), g_{\mu\nu}(x|\theta_0(x))/\partial g_{\mu\nu})g_{\mu\nu,k}(x|\theta_0(x))\delta\theta_k(x) - - - (132)$$

with obvious summation conventions. This expression may be abbreviated as

$$L_0(\phi(x), \partial_\mu \phi(x)) + L_k(\phi(x), \partial_\mu \phi(x))\delta\theta_k(x) - - - (133)$$

and hence the action integral for ϕ has the form

$$S_0(\phi) + \int L_k(\phi)(x)\delta\theta_k(x)d^4x - - - (134)$$

Calling the vector $(\theta_k(x)) = J(x)$ and $L(\phi) = ((L_k(\phi))$, we can express the path integral as

$$Z = \int \mathrm{E}exp(i.\, S_0(\phi) + \int L(\phi)(x)^T J(x)d^4x)D\phi - - - (135)$$

so that when $J(x)$ is a Gaussian field, this evaluates to give

$$Z(M, C) =$$

$$\int exp(iS_0(\phi) + i\int L(\phi)(x)^T M(x)d^4x - (1/2)\int L(\phi)(x)^T C(x, y)L(\phi)(y)d^4xd^4y)D\phi - - - (136)$$

so that the effective action can be defined as (for a given classical field $\phi_0(x)$) as

$$\Gamma(\phi_0, C) = Ext_M(-i.\, logZ(M, C) - \int L(\phi_0)(x)^T M(x)d^4x)$$

$$= -i.\, logZ(M_0, C) - \int L(\phi_0)(x)^T M_0(x)d^4x - - - (137)$$

where M_0 solves

$$i.\, \delta logZ(M_0, C)/\delta M(x) + L(\phi_0)(x) = 0 - - - (138)$$

In other words, M_0 is that current field at which the combined classical and quantum expectation of $L(\phi)(x)$ becomes $L(\phi_0)(x)$. We then get our quantum equations of motion as

$$\delta\Gamma(\phi_0, C)/\delta\phi_0(x) = -\int (\delta L(\phi_0(y))/\delta\phi_0(x))^T M_0(y)d^4y - - - (139)$$

Now, assume that $S_0(\phi)$ has a gauge symmetry

$$S_0(\phi + \epsilon.\, \Delta(\phi)) = S_0(\phi) + o(\epsilon) - - - (140)$$

with the path measure $D\phi$ being invariant under this gauge transformation. More precisely, we assume that the product

$$exp(iS_0(\phi))D\phi - - - (141)$$

is invariant under $\phi \rightarrow \phi + \epsilon.\, \Delta(\phi)$. Then, we get on changing the path integration variable from ϕ to $\phi + \epsilon.\, \Delta(\phi)$ that

$$Z(M, C) =$$

$$\int exp(iS_0(\phi) + i\int L(\phi)(x)^T M(x)d^4x - (1/2)\int L(\phi)(x)^T C(x, y)L(\phi)(y)d^4xd^4y)$$

$$\times (1 + i\epsilon\int (L'(\phi).\, \Delta(\phi))(x)^T M(x)d^4x - \epsilon\int (L'(\phi).\, \Delta(\phi))(x)^T C(x, y)L(\phi)(y)d^4xd^4y)D\phi - - - (142)$$

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(where

$$L'(\phi).\,\Delta(\phi)(x) = \int (\delta L(\phi)(x)/\delta\phi(y)).\,\Delta(\phi)(y)d^4y),$$

which can alternatively be expressed as

$$\int <(L'(\phi).\,\Delta(\phi))(x)>^T_{M,C} M(x)d^4x$$

$$= -i\int Tr(C(x,y) < L(\phi)(y).\,(L'(\phi).\,\Delta(\phi))(x)^T >_{M,C} d^4xd^4y---(143)$$

Substituting for $M(x)$ from the equation of motion rewritten below

$$\delta\Gamma(\phi_0, C)/\delta\phi_0(x) = -\int (\delta L(\phi_0(y))/\delta\phi_0(x))^T M_0(y)d^4y---(144)$$

this equation of broken gauge invariance of the quantum effective action can be expressed in the form

$$\int (\delta\Gamma(\phi_0, C)/\delta\phi_0(x)).\; <\Delta(\phi)(x)>_{M_0,C}$$

$$= -\int(<(L'(\phi) - L'(\phi_0)).\,\Delta(\phi)(x)>^T_{M_0,C} M_0(x)d^4x$$

$$-i\int Tr(C(x,y) < L(\phi)(y).\,(L'(\phi).\,\Delta(\phi))(x)^T >_{M_0,C} d^4xd^4y---(145)$$

This equation tells us the degree to which gauge invariance of the quantum effective action is broken by the mean and covariance of the random current field to which the original field is coupled. If $L(\phi)$ is a linear functional, then only the covariance term will appear. It should be noted if the quantum effective action is gauge invariant, then its Hessian matrix evaluated at its minimum will have zero eigenvalue eigenvectors corresponding to broken gauge symmetries acting on the vacuum state. These are the Goldstone Bosons. However, since the presence of random currents show that it is no longer gauge invariant, these Goldstone vectors will acquire small masses whose values can be easily computed from its Hessian along similar lines as pointed out in Steven Weinberg, "The quantum theory of fields, vol.2".[3]

8. The Klein-Gordon wave equation in the vicinity of the event horizon of a Schwarzchild blackhole and derivation of Hawking's temperature formula

We start with the basic general relativistic diffeomorphic invariant wave equation for a scalar field $\phi(t,r) = \phi(x)$ in a curved space-time having metric $g_{\mu\nu}(x)$:

$$(g^{\mu\nu}\phi_{,\nu}\sqrt{-g})_{,\mu} + \mu^2\sqrt{-g}\phi = 0---(146)$$

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where according to the Einstein energy-momentum relation for a particle of mass m $E^2 = P^2c^2 + m^2c^4$, and the formula for the energy and momentum operators in special relativity, $E = ih\partial_t, P = -ih\nabla$,

$$\mu = mc^2/h - - - (147)$$

In the case of a Schwarzchild metric, $x^0 = t, x^1 = r, x^2 = \theta, x^3 = \phi$,

$$g_{00} = \alpha(r), g_{11} = -\alpha(r)^{-1}/c^2, g_{22} = -r^2/c^2, g_{33} = -r^2 sin^2(\theta)/c^2, \alpha(r) = 1 - 2m/r, m = GM/c^2 - - - (148)$$

$$g^{00} = \alpha(r)^{-1}, g^{11} = -c^2\alpha(r), g^{22} = -c^2/r^2, g^{22} = -c^2/r^2 sin^2(\theta) - - - (149)$$

and of course,

$$g = det((g_{\mu\nu})) = -r^4 sin^2(\theta)/c^6, \sqrt{-g} = r^2 sin(\theta)/c^3 - - - (150)$$

so that assuming radial symmetry of the wave, we get in the case of massless particles like the photon, $\mu = 0$ and

$$\alpha(r)^{-1}r^2\phi_{,tt} - c^2(\alpha(r)r^2\phi_{,r})_{,r} = 0 - - - (151)$$

Assuming $\phi(t, r) = \phi(r)exp(i\omega t)$, ie, we are dealing with KG waves of definite frequency ω, this equation becomes

$$K^2r^3\phi(r) + r(r-2m)^2\phi''(r) + 2(r-m)(r-2m)\phi'(r) = 0 - - - (152)$$

where

$$K = \omega/c - - - (153)$$

To investigate this solution further, we write

$$\phi(r) = exp(iS(r)/h) - - - (154)$$

where $S(r)$ denotes the action function at the fixed frequency ω. We get on substituting this,,

$$K^2r^3 + r(r-2m)^2(iS''(r)/h - S'(r)^2/h^2) + 2(r-m)(r-2m)iS'(r)/h = 0 - - - (155)$$

or

$$K^2h^2r^3 + ihr(r-2m)^2S''(r) - r(r-2m)^2S'(r)^2 + 2h(r-m)(r-2m)iS'(r) = 0 - - - (156)$$

Writing

$$r = 2m + x - - - (157)$$

this equation becomes, with a prime now denoting partial derivative w.r.t x, and $S(x)$ denoting $S(2m+x)$,

$$K^2h^2(2m+x)^3 + hx(2m+x)^2iS''(x) - x^2(2m+x)S'(x)^2 + 2hx(m+x)iS'(x) = 0 - - - (158)$$

For very small $|x|$ (ie, when r is in the vicinity of the event horizon), this equation approximates to

$$8m^3K^2h^2 + 4m^2hxiS''(x) - 2mx^2S'(x)^2 + 2hmxiS'(x) = 0 - - - (159)$$

For very small wavelengths, this equation further approximates to

$$8m^3K^2h^2 - 2mx^2S'(x)^2 = 0 - - - (160)$$

Note that in this approximation, we are assuming that with $r_c = 2m$,

$$|x|\,|S'(x)|/h >> 1 - - - (161)$$

and

$$|x|S'(x)^2/hr_c|S''(x)| >> 1 - - - (162)$$

To see what these two inequalities mean, we observe first that $S'/h = K$ has the interpretation of wave vector, ie $K_0 = 2\pi/\lambda$ where λ is the wavelength and then these two inequalities can be rephrased as

$$K_0|x| >> 1, K_0(x)^2|x|/(r_c|K_0'(x)|) >> 1 - - - (163)$$

Now, $K_0' = -2\pi\lambda'/\lambda^2$ so that $K_0'/K_0^2 = -\lambda'/2\pi$, so these two inequalities are the same as

$$|x| >> \lambda, |\lambda'(x)| << r_c/|x| - - - (164)$$

The first inequality states that the length span in the vicinity of r_c over which the tunneling takes place is much larger that the particle wavelength while the second inequality states that the variations in the wavelength over the length span in the vicinity of r_c are much smaller than r_c, the Schwarzchild radius. If we agree to both of these assumptions, then we get

$$S'(x) = \pm 2m.Kh/x - - - (165)$$

which gives on integration,

$$S(x) = \pm 2m.K.h.log(x) - - - (166)$$

The change in $S(x)$ as x varies from $0-$ to $0+$ or equivalently as r varies from r_c- to r_c+ equals

$$\Delta S = \pm 2mKh.i\pi - - - (167)$$

so that the change in $exp(iS/h)$ over this tunneling zone is given by $exp(\pm 2m.K\pi)$ or equivalently, the change in $|exp(iS/h)|^2$ over this infinitesimal tunneling interval in the vicinity of the event horizon is given by $exp(\pm 4mK\pi) = exp(\pm 4m\pi\omega/c)$. Note that here h stands for Planck's constant divided by 2π. Equating this tunneling probability to $exp(\pm h\omega/kT)$ gives us Hawking's celebrated formula for the equilibrium temperature at which the blackhole radiates massless particles: $T = hc/4\pi mk = hc^3/8\pi GMk$.

9. Conclusions

This paper presents some novel aspects of quantum gravity with applications to the idea of theoretical variables, ie, accessible and inaccessible variables proposed in earlier works by Professor Inge Helland[4]. We demonstrate by taking the example of the gravitational field in a homogeneous and isotropic space-time interacting with a KG field, how the commutation relations of the field vary as the scale factor of the universe expands which means from the measurement standpoint in quantum mechanics, that the degree of uncertainty between two observables or equivalently, the degree of simultaneous non-measurability of two observables varies as the universe expands. At the beginning, we also illustrate this phenomenon using the quantum electromagnetic field interacting with the classical gravitational field. We generalize these results from a spatially homogeneous KG field to a spatially inhomogeneous KG field and then to the anticommutator of the Dirac field in curved space-time. We then explain how to calculate the wave function of one field interacting with another field using the path integral over the second field. Actually, we should instead be talking about TPCP maps corresponding to one field when it interacts with the other. We then explain how to compute the quantum effective action of the scale factor of the universe when it interacts with the quantum KG field present in the form of particles distributed within our universe when in addition, the KG field interacts with a random current field. As an example of such a phenomenon, we consider the metric of space time depending on random classical parameters having small variances, so that the formula for the KG field in a background curved space-time yields an interaction component between the KG field and the classical metric fluctuations with the interaction being quadratic in the KG field and linear in the parameter fluctuations. We then explain how symmetry breaking can occur in the quantum effective action of a field that interacts with a classical random current field, when the classical action has a gauge symmetry. It is a well known result (Steven Weinberg[3], The quantum theory of fields, vol.2) that when the gauge symmetry is linear in the field and the current source is non-random, then the quantum effective action has the same gauge symmetry as the classical action but when the gauge symmetry is nonlinear in the field, the corresponding gauge symmetry of the quantum effective action is not the same as the classical symmetry, rather, it is given by the quantum expectation value of the infinitesimal gauge symmetry of the classical action taken when the current source equals a value at which the quantum field has the same quantum expectation as the classical field. However, when a current source is random, event this gauge symmetry gets broken and we derive a formula for the change in the quantum effective action under the quantum infinitesimal gauge symmetry defined by the the quantum expectation of the classical gauge symmetry in the presence of a non-random current field that yields the quantum expectation value of the field equal to the classical field. This formula for the change in the quantum effective action, namely, the degree by which gauge symmetry is broken is expressed in terms of the statistical correlations of the random current source.

An interesting calculation based on the quantum effective action in the presence of a random current source could provide a clue to the mystery of how particles acquire masses in our universe is to calculate the corrected field propagator to deduce that masses are acquired by particles via symmetry breaking caused by randomly distributed current fields in the form of cosmic microwave background radiation and perhaps also other forms of radiation. One usually treats such constrained problems using Dirac brackets in place of Poisson and Lie brackets. Here, we suggest an approximate method for quantizing such fields based on expressing the constrained position fields in terms of a fewer number of "parameter fields" which form our revised set of position fields and then construct the revised momentum fields using a least squares method for approximately inverting the associated Jacobian matrix of the position field w.r.t the smaller set of parameter fields.

Appendix A1. The Belavkin quantum filter for estimating the state of a gravitational wave from continuous measurements on the electromagnetic field

Let Q, P denote canonical position and momentum fields of the ADM action for the gravitational field. Specifically, Q are the spatial metric tensor coefficient and P the canonical momenta derived from the ADM version of the Einstein-Hilbert action. Let q, p denote the canonical position and momentum fields of the electromagnetic field derived from the action $(-1/4)\int F^{\mu\nu}F_{\mu\nu}\sqrt{-g}d^4x$ with the canonical position fields q being chosen as the components of the vector potential. Here, we work in the Coulomb gauge so that the scalar electric potential vanishes. The electromagnetic action clearly contains the metric coefficients but not their time derivatives and hence the corresponding electromagnetic Hamiltonian can be denoted by $H_2(q, p \mid Q)$. The Hamiltonian of the gravitational field derived from the gravitational action is denoted by $H_1(Q, P)$. In the presence of quantum electromagnetic noise w, we replace q by $q + w$ in H_2 and Taylor expand it upto second degree terms in w. w is expressible as time derivatives of the creation and annihilation processes of the Hudson-Parthasarathy quantum stochastic calculus [2] so that second degree terms in w can be expressed as the conservation process. Substituting this noisy Hamiltonian in

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the Schrodinger equation and adding quantum Ito correction terms shows that the joint unitary evolution of the fields and the noise can be expressed as

$$dU(t) = -i(H_1(Q,P) + H_2(q,p|Q) + P_0)dt + L_1 dA - L_2 dA^* + Sd\Lambda)U(t) - -(168)$$

where L_1, L_2, S are field operators expressible as functions of q, p, Q and A, A^*, Λ are the standard annihilation, creation and conservation quantum stochastic processes of the Hudson-Parthasarathy theory [2]. P_0 is a field operator and is the quantum Ito correction term introduced to ensure unitarity fo the evolution. Non-demolition measurements $Y_o(t) = U(t)^* Y_i(t)U(t)$ can be defined with $Y_i(t) = c(1)A(t) + \bar{c}(1)A(t)^* + c(2)\Lambda(t)$. $Y_i(.)$ form an Abelian family of self-adjoint operator in Boson Fock space $\Gamma_s(L^2(\mathbb{R}_+))$ and these operators commute with the field/system operators X built out of Q, P, q, p and hence using $Y_o(t) = U(T)^* Y_i(t)U(T), T \geq t$, it easily follows that $Y_o(.)$ forms an Abelian family such that $Y_o(t)$ commutes with $j_T(X) = U(T)^* XU(T), T \geq t$. This is the celebrated non-demolition measurement property discovered by V.P.Belavkin and it implies that the conditional expectation $\pi_t(X) = \mathbb{E}(j_t(X)|\eta_o(t))$ can be constructed where $\eta_o(t) = \sigma(Y_o(s) : s \leq t)$ form the measurement algebra upto time t. Here, expectations are taken w.r.t the tensor product of an initial field/system state and a coherent bath state: $\rho(0) = \rho_g(0) \otimes \rho_e(0) \otimes \rho_B(0)$ where $\rho_g(0)$ is the initial state of the gravitational field, $\rho_e(0)$ is the initial state of the electromagnetic field and $\rho_B(0)$ is the bath coherent state. Using the orthogonality principle: $\mathbb{E}(j_t(X)) - \pi_t(X))C(t)) = 0$ in estimation theory where $C(t) \in \eta_o(t)$, one can easily derive the filter differential equations in the form

$$d\pi_t(X) = F_t(X)dt + \sum_{k \geq 1} G_{t,k}(X)dY_o(t)^k - - -(169)$$

where $F_t(X), G_t(X)$ belong to the commutative Von-Neumann algebra $\eta_o(t)$ and can be constructed out of the operators $\pi_t(Z)$ with Z varying over field/system operators. Writing $\pi_t(X) = Tr(\hat{\rho}(t)X)$ where $\hat{\rho}(t)$ is the real time system state estimate, we get by duality a stochastic Schrodinger equation for $\hat{\rho}(t)$:

$$d\hat{\rho}(t) = \theta(\hat{\rho}(t))dt + \sum_{k \geq 1} \theta_{k,t}(\hat{\rho}(t))dY_o(t)^k - - -(170)$$

with θ, θ_k linear maps on the space of system operators and dependent upon the coherent parameter $u(t)$ appearing in $\rho_B(0) = |\phi(u)><\phi(u)|$. Once $\hat{\rho}(T)$ is known, we can determine the state of the gravitational field as the partial trace $\hat{\rho}_g(T) = Tr_e\hat{\rho}(T)$ where the partial trace is over the electromagnetic degrees of freedom. Or, we can choose a $POVMM(a), a \in E$ on the electromagnetic field Hilbert space and by taking measurements get to know the parameters θ on which the initial gravitational state $\rho_g(0) = \rho_g(0|\theta)$ depended. Indeed, if a is the measurement outcome, its probability is $P(a|\theta) = Tr(\hat{\rho}(T)(I \otimes M(a)))$ and a

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maximum likelihood algorithm applied to this probability distribution would yield an estimate of θ. More generally, we can take sequential measurements at times $t_1 < t_2 < \ldots < t_n$ between the state evolutions defined by (a) using this POVM allowing for state collapse after each measurement and from the resultant joint probabilities of the outcomes, estimate the parameter θ. It should be mentioned here that if the gravitational field is described by a weak graviational wave with metric perturbations $h_{\mu\nu}(x)$ around flat space-time, then the Einstein-Hilbert gravitational action is approximately a quadratic form in $h_{\mu\nu,\rho}(x)$ and hence the corresponding gravitational Hamiltonian is a quadratic form in Q, P just like an infinite family of quantum Harmonic oscillators and its interaction with the electromagnetic field generates anharmonic perturbations involving products of a single $h_{\mu\nu}$ and two photon creation-annihilation operators, ie, a cubic interaction and we can study the evolution of the resulting perturbed gravitational state using standard time dependent perturbation theory in the absence of noise. The cubic coupling causes the initial separable state of the gravitons and photons $\rho_g(0|\theta) \otimes \rho_e(0)$ after time T to evolve into an entangled state $U(T)(\rho_g(0|\theta) \otimes \rho_e(0))U(T)^*$ and after applying a POVM $M(a), a \in E$ on the photons, we get probability of a occurring as $p(a|\theta) = Tr(U(T)(\rho_g(0|\theta) \otimes \rho_e(0))U(T)^*(I \otimes M(a)))$ and the maximum likelihood method can be used to estimate the gravitational wave parameter θ.

Appendix A2. A simplified analysis of loop quantum gravity with applications to the derviation of blackhole entropy using the area operator

In order to explain how loop quantum gravity provides a rigorous derivation of the entropy of the blackhole as being proportional to its area horizon, we must first learn to quantize the internal and surface degrees of freedom of a blackhole and then calculate the possible eigenvalues of the surface area of the blackhole.

First, we introduce the flux-holonomy algebra. The flux involves computing the flux of the Ashtekar momentum variable $E_j^a(x)$ (also called the electric field) out of a surface S, ie, $Y_j(S) = \int_S E_j^a(x) n_a(x) dS(x)$. Specifically, if we coordinatize S in terms of $(u_1, \ldots, u_{D-1})$, ie, a $D-1$ surface embedded in D dimensional space, by means of the equations

$$X^a = X^a(u_1, \ldots, u_{D-1}), a = 1, 2, \ldots, D - - - (171)$$

abbreviated as $X = X(u)$, then we can express

$$n_a(X(u))dS(X(u)) = \epsilon(aa_1 \ldots a_{d-1})(\Pi_{k=1}^{D-1} \frac{\partial X^{a_k}}{\partial u_k})du_1 \ldots du_{D-1} - - - (172)$$

The Ashtekar position variable is $A_a^j(x)$. The canonical commutation relations are

$$[A_a^j(x), E_k^b(y)] = \delta^D(x-y) - - - (173)$$

Note that these actually represent equal time commutation relations in $\mathbb{R}^{D+1}$, because, we are assuming that at each time t, we have a D-dimensional space Σ_t embedded inside $\mathbb{R}^{D+1}$ and our variables are all defined in this D-dimensional space.

$h(t)$ is the $SO(D)$ holonomy matrix. Let $A_a^i(x) = \Gamma_a^i(x) + K_a^i(x)$ denote the Ashtekar connection and $E_i^a(x) = \sqrt{q(x)}e_i^a(x)$ the associate Ashtekar canonical momentum field. It should be noted that the spin connection $\Gamma_a^i(x)$ can be expressed as a homogeneous functional of $E_a^i(.)$ of degree zero, ie, $\Gamma_a^i(x)$ is a scale invariant field. From this fact, it can be deduced that

$$[\Gamma_a^i(x), E_j^b(y)] = 0 - - - (174)$$

and hence the CCR

$$[A_a^i(x), E_j^b(y)] = i\delta_j^i\delta_a^b. \delta^D(x-y), [A_a^i(x), A_b^j(y)] = 0, [E_i^a(x), E_j^b(y)] = 0 - - - (174),$$

can be deduced using the CCR

$$[K_a^i(x), E_j^b(y)] = i\delta_j^i\delta_a^b. \delta^D(x-y), [K_a^i(x), K_b^j(y)] = 0, [E_i^a(x), E_j^b(y)] = 0 - - - (175),$$

This latter CCR can be deduced in turn from the CCR derived using the ADM action with the spatial metric coefficients $q_{ab}(x)$ as the position fields and the canonical momenta as $\delta L/\delta\partial_t q_{ab} = P^{(ab)}$ under the condition of the Gauss constraint $G_k(x) = \epsilon(kij)K_{ai}(x)E_j^a(x) = 0$. This constraint has a natural intepretation that $K_{\mu\nu} = q_\mu^{\mu'} q_\nu^{\nu'} q\nabla_{\mu'} n_{\nu'}$ is a symmetric tensor or equivalently, that $K_{ab} = X^\mu_{,a}X^\nu_{,b}\nabla_\mu n_\nu$ is symmetric where Σ_t is a one parameter family of D dimensional space-like surfaces parametrized by time embedded into $\mathbb{R}^{D+1}$ (Usually, D=3, D+1=4) and $X^\mu(x)$ are the coordinates on this surface with n^μ as the unit normal to this surface. Thus, with $x^a, a = 1, 2, 3$ denoting spatial coordinates in Σ_t, we decompose

$$X^\mu_{,t} = N^a X^\mu_{,a} + Nn^\mu - - - (176)$$

into tangential and normal components on Σ_t, ie, $n^\mu X^\nu_{,a} g_{\mu\nu} = 0$ and consequently, one obtains the decomposition of the metric $g^{\mu\nu}$ in $\mathbb{R}^{D+1}$ into spatial tangential components and normal components:

$$g^{\mu\nu} = q^{\mu\nu} + n^\mu n^\nu - - - (177)$$

so that

$$q^{\mu\nu}n_\nu = 0 - - - (178)$$

One then observes that

$$q_{ab} = g_{\mu\nu}X^\mu_{,a}X^\nu_{,b} = q_{\mu\nu}X^\mu_{,a}X^\nu_{,b} - - - (179)$$

are the spatial components of the metric in Σ_t and these serve as the canonical position fields for the ADM action. The Gauss constraint is a natural consequence of the fact that we can write the covariant unit normal to Σ_t as $n_\mu = hf_{,\mu}$ for some scalar functions f, h and then

$$\nabla_\mu n_\nu - \nabla_\nu n_\mu = h_{,\mu}f_{,\nu} - h_{,\nu}f_{,\mu} = (logh)_{,\mu}n_\nu - (\log h)_{,\nu}n_\mu - - - (180)$$

would result in zero if we contract with $q^\mu_\rho q^\nu_\alpha$ since $q^\mu_\rho n_\mu = 0$. One takes a spatial vector field $u_a, a = 1, 2, \ldots, D$, ie, a vector that lives in the D-dimensional space Σ_t and computes its spatial covariant derivative as follows. First define $u_\mu, \mu = 0, 1, 2, \ldots, D$ so that $u_\mu X^\mu_{,a} = u_a$ and then, with ∇_μ denoting the covariant derivative in R^{D+1} and D_μ the projected covariant derivative in R^D, we have

$$D_\mu u_\nu = q^{\mu'}_\mu q^{\nu'}_\nu \nabla_{\mu'}u_{\nu'} - - - (181)$$

Then,

$$D_a u_b = X^\mu_{,a}X^\nu_{,b}D_\mu u_\nu = X^\mu_{,a}X^\nu_{,b}\nabla_\mu u_\nu - - - (182)$$

owing to the obvious formula

$$X^\mu_{,a}q^{\mu'}_\mu = X^{\mu'}_{,a} - - - (183)$$

because

$$q^{\mu'}_\mu + n^{\mu'}n_\mu = g^{\mu'}_\mu = \delta^{\mu'}_\mu, X^\mu_{,a}n_\mu = 0 - - - (184)$$

Likewise, one defines the double spatial covariant derivative of a spatial vector u_a, or u_ν as

$$D_\mu D_\nu u_\rho = q^{\mu'}_\mu q^{\nu'}_\nu q^{\rho'}_\rho \nabla_{\mu'}D_{\nu'}u_{\rho'}$$

$$= q^{\mu'}_\mu q^{\nu'}_\nu q^{\rho'}_\rho \nabla_{\mu'}(q^{\nu''}_{nu'}q^{\rho''}_{\rho'}\nabla_{\nu''}u_{\rho''}) - - - (185)$$

Continuing in this way, we can define the spatial covariant derivatives of any order of a spatial vector. Note our convention: By a spatial vector, we mean a vector that lives in the D-dimensional manifold Σ_t for any fixed time t at which the embedding is carried out and by a space-time vector or simply a vector, we mean a vector that lives in R^{D+1}. The recursion formula for computing the n^{th} order spatial covariant derivative of a spatial vector u_μ or u_a is

$$D_{\mu_n}D_{\mu_{n-1}}\ldots D_{\mu_2}D_{\mu_1}u_\nu =$$

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$$q^{\nu'}_{\nu}(\Pi_{k=1}^{n} q^{\mu'_k}_{\mu_k})\nabla_{\mu_n} D_{\mu'_{n-1}} D_{\mu'_{n-2}} \ldots D_{\mu'_1} u_{\nu'} - - - (186)$$

for $n \geq 1$. Then,

$$D_{a_n} D_{a_{n-1}} \ldots D_{a_2} D_{a_1} u_b = X^{\nu}_{,b}(\Pi_{k=1}^{n} X^{\mu_k}_{,a_k}) D_{\mu_n} D_{\mu_{n-1}} \ldots D_{\mu_2} D_{\mu_1} u_{\nu} - - - (187)$$

Note that the spatial indices are raised and lowered using either q_{ab}, q^{ab} or $q_{\mu\nu}, q^{\mu\nu}$ with

$$q_{ab} = q_{\mu\nu} X^{\mu}_{,a} X^{\nu}_{,b} = g_{\mu\nu} X^{\mu}_{,a} X^{\nu}_{,b} - - - (188)$$

and

$$q^{\mu\nu} = g^{\mu\rho} g^{\nu\sigma} q_{\rho\sigma} - - - (189)$$

$$g^{\mu\nu} = q^{ab} X^{\mu}_{,a} X^{\nu}_{,b} + n^{\mu} n^{\nu} - - - (190)$$

where

$$((q^{ab})) = ((q_{ab}))^{-1}, ((g^{\mu\nu})) = ((g_{\mu\nu}))^{-1} - - - (191)$$

The spatial curvature tensor is defined by

$$R^{D}_{\mu\nu\rho\sigma} u^{\mu} = R^{D\mu}_{\nu\rho\sigma} u_{\mu} = [D_{\sigma}, D_{\rho}] u_{\nu} - - - (192)$$

which can be easily evaluated in terms of $R^{D+1}_{\mu\nu\rho\sigma}$, or more precisely, the projection of $R^{D+1}_{\mu\nu\rho\sigma}$ onto $\Sigma_t = \mathrm{R}^D$ using the spatial projection matrix q^{μ}_{ν} and $K_{\mu\nu}$ using the fact that $u_{\mu} n^{\mu} = 0$ and $K_{\mu\nu} n^{\nu} = 0$.

To check these identities, we require the orthogonality relations

$$(X^{\mu}_{,t} - N^a X^{\mu}_{,a}) X^{\nu}_{,b} g_{\mu\nu} = 0, N^a X^{\mu}_{,a} = N^{\mu} - - - (193)$$

or equivalently,

$$q_{bt} - N^a q_{ab} = 0 - - - (194)$$

so

$$N^a = q^{ab} q_{bt} - - - (195)$$

so that

$$N^2 = g_{\mu\nu}(X^{\mu}_{,t} - N^{\mu})(X^{\nu}_{,t} - N^{\nu}) = g_{\mu\nu}(X^{\mu}_{,t} - N^a X^{\mu}_{,a}) X^{\nu}_{,t} = q_{tt} - q_{at} N^a$$

$$= q_{tt} - q_{at} q^{ab} q_{bt} = 1/h^{tt} - - - (196)$$

and then, with h denoting the D+1-metric w.r.t the coordinates in the D+1-dimensional manifold $\bigcup_t \Sigma_t$ (h_ab=q_ab, h_at=q_at, h_tt=q_tt, but q^{ab} is not the same as h^{ab})

$$g^{\mu\nu} = h^{ab} X^{\mu}_{,a} X^{\nu}_{,b} + h^{at}(X^{\mu}_{,a} X^{\nu}_{,t} + X^{\mu}_{,t} X^{\nu}_{,a}) + h^{tt} X^{\mu}_{,t} X^{\nu}_{,t}$$

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$$= h^{ab}X^{\mu}_{,a}X^{\nu}_{,b} + h^{at}X^{\mu}_{,a}(N^{\nu} + Nn^{\nu}) + X^{\nu}_{,a}(N^{\mu} + Nn^{\mu})) + h^{tt}(N^{\mu} + Nn^{\mu})(N^{\nu} + Nn^{\nu})$$

$$= h^{ab}X^{\mu}_{,a}X^{\nu}_{,b} + (h^{at}X^{\mu}_{,a} + h^{tt}N^{\mu})N^{\nu} + (h^{at}X^{\nu}_{,a} + h^{tt}N^{\nu})N^{\mu} - h^{tt}N^{\mu}N^{\nu} +$$

$$+ h^{tt}N^2 n^{\mu}n^{\nu} + (h^{at}X^{\mu}_{,a} + h^{tt}N^{\mu})Nn^{\nu} + (h^{at}X^{\nu}_{,a} + h^{tt}N^{\nu})Nn^{\mu} - - - (197)$$

Using the above derived orthogonality relations, we easily show that

$$(h^{at}X^{\mu}_{,a} + h^{tt}N^{\mu}) = (h^{at} + h^{tt}N^{a})X^{\mu}_{,a} = 0 - - - (198)$$

In fact, the orthogonality relations are easily shown to be equivalent to

$$h^{at} + h^{tt}N^{a} = 0 - - - (199)$$

or equivalently,

$$N^{a} = -h^{at}/h^{tt} - - - (200)$$

To show this, we must show that

$$-h^{at}/h^{tt} = q^{ab}q_{bt} - - - (201)$$

or equivalently, that

$$q^{ab}q_{bt}h^{tt} + h^{at} = 0 - - - (202)$$

or equivalently, since $((q_{ab}))$ and $((q^{ab}))$ are inverses of each other,

$$q_{at}h^{tt} + q_{ab}h^{bt} = 0 - - - (203)$$

or

$$h_{at}h^{tt} + h_{ab}h^{bt} = 0 - - - (204)$$

But the lhs is $\delta^{t}_{a} = 0, a = 1, 2, 3$ and so we are done with the proof of the claim. Note that the equation

$$\begin{pmatrix} q_{tt} & q^{T}_{1:3,t} \\ q_{1:3,t} & Q \end{pmatrix} \cdot \begin{pmatrix} h^{tt} & h^{1:3,tT} \\ h^{1:3,t} & H \end{pmatrix} = I - - - (205)$$

where $Q = ((q_{ab}))$ and $H = ((h^{ab}))$ shows after some manipulations that

$$h^{tt} = 1/(q_{tt} - q^{T}_{1:3,t}Q^{-1}q_{1:3,t}) = 1/(q_{tt} - q_{at}q^{ab}q_{bt}) - - - (206)$$

Putting all these into (1) finally results in the required decomposition of the contravariant metric into its spatial and normal components with the cross terms cancelling out:

$$g^{\mu\nu} = q^{\mu\nu} + n^{\mu}n^{\nu} - - - (207)$$

where

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$$q^{\mu\nu} = h^{ab}X^{\mu}_{,a}X^{\nu}_{,b} - h^{tt}N^{\mu}N^{\nu} = (h^{ab} + h^{tt}N^{a}N^{b})X^{\mu}_{,a}X^{\nu}_{,b} - - - (208)$$

Now we claim that

$$(h^{ab} - h^{tt}N^{a}N^{b}) = q^{ab} - - - (209)$$

To prove this, its suffices to verify that

$$q_{ab}(h^{bc} - h^{tt}N^{b}N^{c}) = \delta^{c}_{a} - - - (210)$$

or equivalently, since as shown above, $N^{a} = -h^{at}/h^{tt}$, we require to verify that

$$q_{ab}(h^{bc} - h^{bt}h^{ct}/h^{tt}) = \delta^{c}_{a} - - - (211)$$

Now,

$$q_{ab}h^{bc} = q_{ab}h^{bc} + q_{at}h^{tc} - q_{at}h^{tc} = \delta^{c}_{a} - q_{at}h^{tc} - - - (212)$$

so we require to verify that

$$(q_{at}h^{tt} + q_{ab}h^{bt}) = 0 - - - - (213)$$

which is obvious since it is equal to $\delta^{t}_{a} = 0$. Thus, we obtain

$$q^{\mu\nu} = q^{ab}X^{\mu}_{,a}X^{\nu}_{,b} - - - (214)$$

and hence, to summarize,

$$g^{\mu\nu} = q^{\mu\nu} + n^{\mu}n^{\nu}, q^{\mu\nu} = q^{ab}X^{\mu}_{,a}X^{\nu}_{,b} - - - (215),$$

where q_{ab}, $1 \leq a, b \leq 3$ are the spatial components of the metric tensor in Σ_t corresponding to the metric tensor $g_{\mu\nu}$ in R^{D+1} while, $g_{\mu\nu}n^{\mu}n^{\nu}n^{\mu}n_{\mu} = 1$ and $q^{\mu\nu}n_{\nu} = 0$. We then choose an $SO(3)$ triad e^{i}_{a} for q_{ab}, ie, $e^{i}_{a}e^{i}_{b} = q_{ab}$. The expression $R^{D+1}\sqrt{-g}$ for the Einstein-Hilbert Lagrangian can be expressed as the sum of $R^{D}\sqrt{q}N$ and a quadratic form in K_{ab} multiplied by N on making use of identities such as

$$R^{D}_{\mu\nu\rho\sigma}u^{\mu} = [D_{\sigma}, D_{\rho}]u_{\nu} - - - (216)$$

for spatial vectors u_{ν} to obtain

$$R^{D+1}_{\mu\nu\rho\sigma}q^{\nu\rho}q^{\mu\sigma} = R^{D+1}_{abcd}q^{bc}q^{ad} =$$

$$R^{D} + f_1(K_{ab}) - - - (217)$$

where R^{D} is the spatial curvature, ie, curvature in Σ_t and f_1 is a quadratic form in K_{ab} with coefficients given by nonlinear functions of the spatial metric q_{ab}. Observe that the above formula is consistent with

$$R^{D+1}_{abcd} = R^{D+1}_{\mu\nu\rho\sigma}X^{\mu}_{,a}X^{\nu}_{,b}X^{\rho}_{,c}X^{\sigma}_{,d} - - - (218)$$

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because of the spatial relation

$$q^{\mu\nu} = q^{ab} X^{\mu}_{,a} X^{\nu}_{,b} - - - (219)$$

One then relates the curvature scalars R^{D+1} and R^D in R^{D+1} and Σ_t respectively using the relation (obtained using the symmetries and skewsymmetries of the curvature tensor)

$$R^{D+1} = R^{D+1}_{\mu\nu\rho\sigma} g^{\mu\sigma} g^{\nu\rho} =$$

$$R^{D+1}_{\mu\nu\rho\sigma}(q^{\mu\sigma} + n^{\mu}n^{\sigma})(q^{\nu\rho} + n^{\nu}n^{\rho}) =$$

$$= R^{D+1}_{\mu\nu\rho\sigma} q^{\mu\sigma} q^{\nu\rho} + 2R^{D+1}_{\mu\nu\rho\sigma} q^{\nu\rho} n^{\mu} n^{\sigma}$$

$$= R^D + f_1(K_{ab}) + 2R^{D+1}_{\mu\nu\rho\sigma} q^{\nu\rho} n^{\mu} n^{\sigma} - - - (220)$$

with the last term evaluated using

$$R^{D+1}_{\mu\nu\rho\sigma} n^{\mu} = [\nabla_{\sigma}, \nabla_{\rho}] n_{\nu} - - - (221)$$

and expressing the last term evaluated as a quadratic form $f_2(K_{ab})$ in $K_{\mu\nu}$ or equivalently in K_{ab} after neglect of a total covariant divergence which does not contribute to the action integral. In this way, one calculates the Einstein-Hilbert action integral in R^{D+1} in terms of the coordinates (known as the ADM coordinates) in $\bigcup_t \Sigma_t$ in the form

$$\int R^{D+1} \sqrt{-g} d^{D+1} X = \int (R^D + f(K_{ab})) N \sqrt{q} d^D x dt - - - (222)$$

where

$$f(K_{ab}) = f_1(K_{ab}) + f_2(K_{ab}) - - - (223)$$

is a quadratic form in K_{ab} not involving any partial derivative of the latter and with coefficients that are nonlinear functions of q_{ab}. More precisely, f is of the form $a_1 K_{ab} K^{ab} + a_2 K^2$ where a_1, a_2 are numerical constants and $K = q^{ab} K_{ab}$ while $K^{ab} = q^{ac} q^{bd} K_{bd}$.

The next step is to evaluate $K_{ab} = X^{\mu}_{,a} X^{\nu}_{,b} \nabla_{\mu} n_{\nu}$ as an affine linear function of $q_{ab,t}$ (in fact linear in $q_{ab,t}, q_{ab,c}$) and hence derive a formula for the canonical momentum $\partial L^{D+1} / \partial q_{ab,t} = P^{ab}$ corresponding to the position field q_{ab}. Here, L^{D+1} is the Einstein-Hilbert Lagrangian in R^{D+1} and is proportional to $R^{D+1}\sqrt{-g}$. We observe that

$$\sqrt{-g} d^{D+1} X = \sqrt{q} N d^D x dt - - - (224)$$

owing to the fact that the metric in $\bigcup_t \Sigma_t$ is given by

$$((h_{\mu\nu})) = ((h^{\mu\nu}))^{-1} = \begin{pmatrix} q_{tt} & q^T_{1:3,t} \\ q_{1:3,t} & Q \end{pmatrix} - - - (225)$$

and hence the determinant of this matrix is (from a standard formula for the determinant of a block matrix)

$$h = q(q_{tt} - q^T_{1:3,t} Q^{-1} q_{1:3}) = q/h^{tt} = q.N^2 - - - (226)$$

so that

$$\sqrt{-g} d^{D+1} X = \sqrt{h} d^D x dt = N\sqrt{q} d^D x dt - - - (227)$$

Note that $(t, x^a, a = 1, 2, 3)$ are the space-time coordinates in $\bigcup_t \Sigma_t$ and $(X^\mu, \mu = 0, 1, ..., D)$ are the space-time coordinates in R^{D+1}. The formula for P^{ab} is a linear function of K_{ab} and hence the canonical commutation relations between q_{ab} and P^{cd} result in corresponding canonical commutation relations between q_{ab} and K_{cd}. The next step is to define the $SO(D)$ spatial spinor connection Γ^i_{ja} so that the corresponding covariant derivative $\mathcal{D}_a e^i_b = 0$, ie

$$0 = \mathcal{D}_a e^i_b = D_a e^i_b + \Gamma^i_{ja} e^j_b - - - (228)$$

where D_a is the spatial covariant derivative on Σ_t evaluated using the connection derived from the spatial metric q_{ab}:

$$D_a e^i_b = e^i_{b,a} - \Gamma^c_{ab} e^i_c, \Gamma^c_{ab} = q^{cd}\Gamma_{dab}, \Gamma_{dab} = (1/2)(q_{da,b} + q_{db,a} - q_{ab,d}) - - - (229)$$

It should be mentioned that $D_\mu u_\nu$ as a spatial tensor transforms to $D_a u_b$ on using the projection $X^\mu_{,a}$ with D_a being defined as the spatial covariant derivative using the connection derived from the spatial metric q_{ab}. Let us verify this:

$$X^\mu_{,a} X^\nu_{,b} D_\mu u_\nu =$$

$$X^\mu_{,a} X^\nu_{,b} \nabla_\mu u_\nu =$$

$$X^\mu_{,a} X^\nu_{,b} (u_{\nu,\mu} - \Gamma^\rho_{\mu\nu} u_\rho)$$

$$= X^\mu_{,a}((u_\nu X^\nu, b)_{,\mu} - u_\nu X^\nu_{,b\mu}) - \Gamma^\rho_{\mu\nu} u_\rho X^\mu_{,a} X^\nu_{,b}$$

$$= u_{b,a} - u_\nu X^\nu_{,ab} - \Gamma_{\rho\mu\nu} u^c X^\rho_{,c} X^\mu_{,a} X^\nu_{,b} - - - (230)$$

Now

$$\Gamma_{\rho\mu\nu} X^\rho_{,c} X^\mu_{,a} X^\nu_{,b} =$$

$$(1/2)(g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho}) X^\rho_{,c} X^\mu_{,a} X^\nu_{,b} =$$

$$= (1/2)[g_{\rho\mu,\nu} + \ldots]X^\rho_{,c}X^\mu_{,a}X^\nu_{,b} - - - (231)$$

where the three dots refer to similar terms with ρ, μ, ν appropriately permuted. Now this equals

$$(1/2)[(g_{\rho\mu}X^\rho_{,c}X^\mu_{,a})_{,\nu}X^\nu_{,b} - g_{\rho\mu}(X^\rho_{,c}X^\mu_{,a})_{,\nu}X^\nu_{,b}]$$

$$+(1/2)[(g_{\rho\nu}X^\rho_{,c}X^\nu_{,b})_{,\mu}X^\mu_{,a} - g_{\rho\nu}(X^\rho_{,c}X^\nu_{,b})_{,\mu}X^\mu_{,a}]$$

$$-(1/2)[(g_{\mu\nu}X^\mu_{,a}X^\nu_{,b})_{,\rho}X^\rho_{,c} - g_{\mu\nu}(X^\mu_{,a}X^\nu_{,b})_{,\rho}X^\rho_{,c}]$$

$$= (1/2)[q_{ca,b} + q_{cb,a} - q_{ab,c}] -$$

$$(1/2)[g_{\rho\mu}(X^\rho_{,c}X^\mu_{,a})_{,b} + g_{\rho\nu}(X^\rho_{,c}X^\nu_{,b})_{,a} - g_{\mu\nu}(X^\mu_{,a}X^\nu_{,b})_{,c}]$$

$$= (1/2)[q_{ca,b} + q_{cb,a} - q_{ab,c}] -$$

$$(1/2)g_{\mu\nu}[(X^\mu_{,a}X^\nu_{,c})_{,b} + (X^\nu_{,b}X^\mu_{,c})_{,a} - (X^\mu_{,a}X^\nu_{,b})_{,c}]$$

$$= (1/2)[q_{ca,b} + q_{cb,a} - q_{ab,c}]$$

$$-g_{\mu\nu}X^\mu_{,ab}X^\nu_{,c} = \Gamma_{cab} - g_{\mu\nu}X^\mu_{,ab}X^\nu_{,c} - - - (232)$$

Substituting this into (139) finally gives us the required result:

$$X^\mu_{,a}X^\nu_{,b}D_\mu u_\nu =$$

$$= u_{b,a} - u_\nu X^\nu_{,ab} - \Gamma_{\rho\mu\nu}u^c X^\rho_{,c}X^\mu_{,a}X^\nu_{,b}$$

$$= u_{b,a} - u_\mu X^\mu_{,ab} - (\Gamma_{cab} - g_{\mu\nu}X^\mu_{,ab}X^\nu_{,c})u^c$$

$$= u_{b,a} - \Gamma_{cab}u^c = D_a u_b - - - (233)$$

since

$$X^\nu_{,c}u^c = u^\nu, g_{\mu\nu}u^\nu = u_\mu - - - (234)$$

and we have defined the spatial connection

$$\Gamma_{cab} = (1/2)(q_{ca,b} + q_{cb,a} - q_{ab,c}) - - - (235)$$

Now, we address the primary issue required for the canonical quantization of the gravitational field: On passing over from the above ADM action described in terms of the (t, x^a) coordinates in the family $\bigcup_t \Sigma_t$ of D-dimensional spatial manifolds that have been embedded into R^{D+1}, to the corresponding Hamiltonian for the gravitational field described in terms of the canonical position fields q_{ab}, N, N^a and the corresponding momentum fields P^{ab}, Π, Π^a, what are the constrained variables, what is the Hamiltonian constraint and what is the diffeomorphism constraint ? Using the Hamiltonian constraint, how does one arrive at the Wheeler-De-Witt equation, namely the Schrodinger equation for the gravitational field in general relativity ? How does one introduce the Gauss constraint and how does one show by means of the

Ashtekar variables, that the Hamiltonian density for general relativity after multiplying by an appropriate power of $\sqrt{q}$ look exactly like that for a Yang-Mills non-Abelian gauge field with gauge group as $SO(3)$?

In order to answer this question, we first require the fact derived above, namely that the action integral when expressed in the coordinates system adapted to $\bigcup_t \Sigma_t$ has the form

$$S = \int (R^D + f(K))\sqrt{q} N d^D x, d^D x = d^{D-1} x dt - - - (236)$$

with

$$f(K) = a(K_{ab}K^{ab} - K^2), K = q^{ab}K_{ab} - - - (237)$$

The next step is to express K_{ab} in terms of $q_{ab\,,t}$ and $q_{ab\,,c}$: To do so, we first define the Lie derivative w.r.t a vector field of a one form and then of a two form:

$$(L_X\omega)(Y) = X(\omega(Y)) - \omega([X, Y]), \omega(Y) = \omega_\alpha Y^\alpha, X = X^\alpha \partial_\alpha - - - (238)$$

so

$$(L_X\omega)(Y) = X^\rho(\omega_\alpha Y^\alpha)_{,\rho} - \omega_\alpha(X^\beta Y^\alpha_{,\beta} - Y^\beta X^\alpha_{,\beta})$$

$$= (\omega_{\alpha,\rho}X^\rho + \omega_\rho X^\rho_{,\alpha})Y^\alpha - - - (239)$$

so that in terms of the local coordinate system chosen, we have the result that the Lie derivative of the one form ω along the vector field X is given by

$$(L_X\omega)_\alpha = \omega_{\alpha,\rho}X^\rho + \omega_\rho X^\rho_{,\alpha} - - - (240)$$

Next by assuming that the Lie derivative along a vector field when acting on tensors satisfies the derivation property, it immediately follows that the Lie derivative of a covariant two tensor $\omega = \omega_{\alpha\beta}dx^\alpha \otimes dx^\beta$ is given in these local coordinates by

$$(L_X\omega)_{\alpha\beta} = \omega_{\alpha\beta,\rho}X^\rho + \omega_{\rho\beta}X^\rho_{,\alpha} + \omega_{\alpha\rho}X^\rho_{,\beta} - - - (241)$$

Applying this result to the covariant two tensor $q_{\mu\nu}$ with X taken as $T = X^\mu_{,t}\partial_\mu$ gives us

$$(L_Tq)_{\mu\nu} = q_{\mu\nu,\rho}X^\rho_{,t} + q_{\rho\nu}X^\rho_{,t\mu} + q_{\mu\rho}X^\rho_{,t\nu} - - - (242)$$

and finally pulling this Lie derivative from R^{D+1} into $\mathrm{R}^D = \Sigma_t$, gives us

$$(L_Tq)_{ab} = (L_Tq)_{\mu\nu}X^\mu_{,a}X^\nu_{,b} =$$

$$q_{\mu\nu,t}X^\mu_{,a}X^\nu_{,b} + q_{\nu\rho}X^\nu_{,b}X^\rho_{,ta} + q_{\mu\rho}X^\mu_{,a}X^\rho_{,tb}$$

$$= (q_{\mu\nu}X^\mu_{,a}X^\nu_{,b})_{,t} - q_{\mu\nu}(X^\mu_{,a}X^\nu_{,b})_{,t}$$

$$+ q_{\mu\nu}X^\nu_{,b}X^\mu_{,ta} + q_{\mu\nu}X^\mu_{,a}X^\nu_{,tb}$$

$$= q_{ab,t} - - - (243)$$

which is quite a remarkable result. Continuing this thread, we have using

$$X^{\mu}_{,t} = T^{\mu} = N^{\mu} + Nn^{\mu} = N^{a}X^{\mu}_{,a} + Nn^{\mu} - - - (244)$$

and the linearity of the Lie derivative $X \to L_X$ gives

$$(L_T q)_{\mu\nu} = (L_{\mathbf{N}+N\mathbf{n}} q)_{\mu\nu} = (L_{\mathbf{N}} q)_{\mu\nu} + (L_{N\mathbf{n}} q)_{\mu\nu} - - - (245)$$

on the other hand. Now

$$(L_{\mathbf{N}} q)_{\mu\nu} = N^{\rho} q_{\mu\nu,\rho} + q_{\nu\rho} N^{\rho}_{,\mu} + q_{\mu\rho} N^{\rho}_{,\nu} - - - (246)$$

and pulling this back to Σ_t using

$$q_{ab} = q_{\mu\nu} X^{\mu}_{,a} X^{\nu}_{,b}, N^{\rho} = N^{c} X^{\mu}_{,c} - - - (247)$$

gives

$$(L_{\mathbf{N}} q)_{ab} = (L_{\mathbf{N}} q)_{\mu\nu} X^{\mu}_{,a} X^{\nu}_{,b} =$$

$$q_{\mu\nu,\rho} N^{c} X^{\mu}_{,a} X^{\nu}_{,b} X^{\rho}_{,c} + q_{\nu\rho} N^{\rho}_{,\mu} X^{\mu}_{,a} X^{\nu}_{,b} + q_{\mu\rho} N^{\rho}_{,\nu} X^{\mu}_{,a} X^{\nu}_{,b}$$

$$= q_{\mu\nu,c} N^{c} X^{\mu}_{,a} X^{\nu}_{,b} + q_{b\rho} N^{\rho}_{,a} + q_{a\rho} N^{\rho}_{,b}$$

$$= q_{ab,c} N^{c} - N^{c} q_{\mu\nu} (X^{\mu}_{,a} X^{\nu}_{,b})_{,c} + q_{b\rho} (N^{c} X^{\rho}_{,c})_{,a} + q_{a\rho} (N^{c} X^{\rho}_{,c})_{,b}$$

$$= q_{ab,c} N^{c} - N^{c} q_{a\nu} X^{\nu}_{,bc} - N^{c} q_{\mu b} X^{\mu}_{,ac} + q_{bc} N^{c}_{,a} + q_{b\rho} N^{c} X^{\rho}_{,ac} + q_{ac} N^{c}_{,b} + q_{a\rho} N^{c} X^{\rho}_{,bc}$$

$$= q_{ab,c} N^{c} + q_{bc} N^{c}_{,a} + q_{ac} N^{c}_{,b} = (q_{ab,c} - q_{bc,a} - q_{ac,b}) N^{c} + N_{a,b} + N_{b,a} = -2\Gamma_{cab} N^{c} + N_{a,b} + N_{b,a} - - - (248)$$

Finally, we require

$$(L_{N\mathbf{n}} q)_{\mu\nu} = Nn^{\rho} q_{\mu\nu,\rho} + q_{\mu\rho} (Nn^{\rho})_{,\nu} + q_{\nu\rho} (Nn^{\rho})_{,\mu} =$$

$$Nn^{\rho} (q_{\mu\nu,\rho} - q_{\mu\rho,\nu} Nn^{\rho} - q_{\nu\rho,\mu} - - - (249)$$

since

$$q_{\mu\rho} n^{\rho} = 0 - - - (250)$$

This gives on pulling back,

$$(L_{N\mathbf{n}} q)_{ab} = (L_{N\mathbf{n}} q)_{\mu\nu} X^{\mu}_{,a} X^{\nu}_{,b} =$$

$$Nn^{\rho} (q_{\mu\nu,\rho} - q_{\mu\rho,\nu} - q_{\nu\rho,\mu}) X^{\mu}_{,a} X^{\nu}_{,b}$$

$$= Nn^{\rho} [(g_{\mu\nu} - n_{\mu} n_{\nu})_{,\rho} - (g_{\mu\rho} - n_{\mu} n_{\rho})_{,\nu} - (g_{\nu\rho} - n_{\nu} n_{\rho})_{,\mu}) X^{\mu}_{,a} X^{\nu}_{,b}$$

$$= Nn^{\rho} [g_{\mu\nu,\rho} - g_{\mu\rho,\nu} - g_{\nu\rho,\mu} - (n_{\mu} n_{\nu})_{,\rho} + (n_{\mu} n_{\rho})_{,\nu} + (n_{\nu} n_{\rho})_{,\mu}] X^{\mu}_{,a} X^{\nu}_{,b}$$

$$= [-2Nn^{\rho} (\Gamma_{\rho\mu\nu}) + N(n_{\mu,\nu} + n_{\nu,\mu})] X^{\mu}_{,a} X^{\nu}_{,b} - - - (251)$$

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where we used $n_\mu X^\mu_{,a} = 0, n^\nu X^\nu_{,b} = 0$ and $n^\rho n_\rho = 1$. Finally, we can express this as

$$(L_{N\mathbf{n}}q)_{ab} = N(\nabla_\nu n_\mu + \nabla_\mu n_\nu)X^\mu_{,a}X^\nu_{,b} = 2NK_{ab} - - - (252)$$

Finally, on equating the two above derived expressions for $(L_T q)_{ab}$, we arrive at

$$q_{ab,t} = -2\Gamma_{cab}N^c + N_{a,b} + N_{b,a} + 2NK_{ab} = D_b N_a + D_a N_b + 2NK_{ab} - - - (253)$$

or equivalently,

$$K_{ab} = (2N)^{-1}[q_{ab,t} - D_a N_b - D_b N_a] - - - (254)$$

This is a nice expression for $q_{ab,t}$ as a linear form in the time and space derivatives of q_{ab} with coefficients being nonlinear functions of the position fields N, q_{cd}. Now we are in a position to discuss the Hamiltonian constraint, the diffeomorphism constraint and the Gauss constraint. The ADM action functional for general relativity can now be expressed as

$$S = \int [R^D + \alpha(K_{ab}K^{ab} - K^2)]N\sqrt{q}d^Dx, K = q^{ab}K_{ab} - - - (255)$$

and noting that R^D involves only q_ab and its space derivatives $q_{ab,c}$, it follows that with q_{ab} as the canonical position fields, the corresponding canonical momentum fields are

$$P^{ab} = \delta S/\delta q_{ab,t} = \alpha.\sqrt{q}.(K^{ab} - Kq^{ab}) - - - (256)$$

The fact that K_{ab} is symmetric implies that $\epsilon(ijk)K^{ab}e^j_a e^k_b = 0$ and with $D = 3, D+1 = 4$, we define the Ashtekar momentum variable as $E^a_i = \sqrt{q}e^a_i = \epsilon(ijk)\epsilon(abc)e^j_b e^k_c$ so that $\epsilon(ijk)e^j_b e^k_c = \epsilon(abc)E^a_i$. Define $K_{ai} = K_{ab}e^b_i$, so that we get

$$K_{ab}E^b_i = \sqrt{q}.K_{ab}e^b_i = \sqrt{q}.K_{ai} - - - (257)$$

so that the symmetry of K_{ab} can equivalently be expressed in the form

$$G_k = \epsilon(kij)K_{ai}E^a_j = 0 - - - - (258)$$

which is called the Gauss constraint. Now Ashtekar proved that we can define position fields q_{ab} as nonlinear functions of E^a_i and momentum fields P^{ab} as nonlinear functions of E^a_i and K_{ai} that is linear in the latter in such a way that if we introduce commutation relations

$$[E^a_i(x), K_{bj}(y)] = i\delta^a_b\delta_{ij}\delta^3(x-y), x, y \in \Sigma_t - - - (259)$$

then under the Gauss constraint, $G_k = 0$, these commutation relations imply the canonical commutation relations between q_{ab} and P^{cd} (ie, we first compute the corresponding commutation relations between the latter and then impose the constraint) and that if the Gauss constraint holds, these definitions of q_{ab}, P^{cd}

precisely coincide with those defined in the ADM action. Further, Ashtekar observed that if we define spin connection Γ^i_{aj} by the equation

$$\mathcal{D}_a e^i_b = D_a e^i_b + \Gamma^i_{aj} e^j_b = 0 - - - (260)$$

where D_a is the spatial covariant derivative on Σ_t, ie,

$$D_a e^i_b = e^i_{b:a} = e^i_{b,a} - \Gamma^c_{ba} e^i_c - - - (261)$$

which amounts to saying that

$$\Gamma^i_{aj} = -e^b_j e^i_{b:a} - - - (262)$$

and extend the definition of the spatial covariant derivative to contravariant spatial vectors so that

$$\partial_b(E^b_i) = \partial_b(\sqrt{q} e^b_i) = \sqrt{q} e^b_{i:b} = \sqrt{q} D_b e^b_i = \sqrt{q}.\, \Gamma^j_{bi} e^b_j = \Gamma^j_{bi} E^b_j - - - (263)$$

(this amounts to the fact that the spatial covariant derivative of the spatial metric tensor q_{ab} on Σ_t vanishes), then we can express the Gauss constraint in the following way:

$$0 = \partial_a E^a_j - \Gamma^j_{ai} E^a_j = \partial_a E^a_j - \Gamma^j_{ai} E^a_j - G_i =$$

$$\partial_a E^a_j - \Gamma^j_{ai} E^a_j - \epsilon(ikj) K_{ak} E^a_j - - - (264)$$

Note that Γ^j_{ai} is an $SO(3)$ Lie algebra connection because it is skewsymmetric w.r.t interchange of (i, j):

$$\Gamma^i_{aj} = -e^b_j e^i_{b:a} = -(e^b_j e^j_b)_{:a} + e^b_{j:a} e^i_b = e^b_{j:a} e^i_b = -\Gamma^j_{ai} - - - (265)$$

since the indices i, j, k are raised and lowered using the flat space metric δ_{ij} because by definition of triad,

$$q_{ab} = e^i_a e^i_b = \delta_{ij} e^i_a e^j_b - - - (266)$$

Ashtekar's idea was to introduce another connection (Like Γ^i_a, this is also an $SO(3)$ connection)

$$A^i_a = K^i_a + \Gamma^i_a - - - (267)$$

where

$$\Gamma^j_{ai} = \epsilon(ikj) \Gamma^k_a - - - (268)$$

This equation makes sense because Γ^j_{ai} is skew-symmetric w.r.t interchange of (j, i). In fact, we can invert this equation and write

$$\Gamma^k_a = (1/2) \epsilon(ikj) \Gamma^j_{a^i} - - - (269)$$

since

$$\epsilon(imj) \epsilon(ikj) = 2\delta_{km} - - - (270)$$

In terms of this new connection, which is a spinor $SO(3)$ connection, we can express the Gauss constraint as

$$\mathcal{D}^A_a E^a_j = 0 - - - (271)$$

where noe D^A_a is the covariant derivative w.r.t the $SO(3)$ connection A^i_a, ie,

$$D^A_a F^b_i = \partial_a F^b_i + \epsilon(ikj) A^k_a F^b_j - - - (272)$$

This result, namely, that the Gauss constraint can be formulated as the vanishing of the covariant divergence of E^a_j w.r.t the spinor connection A^i_a has given its name, namely coming from Gauss' divergence theorem and Gauss' law of electrostatics in free space, that the divergence of the electric field vanishes. It should be emphasized that Γ^i_a can be expressed as a homogeneous functional of E^a_j of degree zero, ie, Γ^i_a is invariant under constant scale transformations of E^a_i. From this, fact, we deduce the first commutation relation $[A^i_a(x), A^j_b(y)] = 0$ in Ashtekar's theory. The second commutation relation in Ashtekar's theory is derived from the standard one of the ADM action theory, namely, the momentum operators at different spatial points commute:$[K_{ai}(x), K_{bj}(y)] = 0$. The third is $[E^a_i(x), K_{bj}(y)] = \delta_{ij}\delta_{ab}\delta^D(x-y)$. Ashtekar observed that since Γ^i_a can be expressed in terms of the E^a_i, it follows that $[\Gamma^i_a(x), E^b_j(y)] = 0$ and hence that $[A^i_a(x), E^b_j(y)] = [K^i_a(x), E^b_j(y)] = \delta_{ab}\delta_{ij}\delta^D(x-y)$. To prove the first Ashtekar commutation relation, in view of the second one, it suffices to show that $[K^i_a(x), \Gamma^j_b(y)] + [\Gamma^i_a(x), K^j_b(y)] = 0$. This follows directly from the fact that the scale transformations are generated by the scalar product between position E^a_i and momentum K_{bj} and that Γ^j_b is invariant under scale transformations, ie, under scalings of the position fields. The details can be found in .

The above observations suggest to us strongly, that if we use the Ashtekar position and momentum fields $A^i_a(x)$ and $E^a_i(x)$, then the Einstein-Hilbert Hamiltonian should look something like a Yang-Mills non-Abelian gauge field Hamiltonian when the Gauss' constraint is satisfied. It indeed turns out to be so after multiplying by a power of $\sqrt{q}$. Like the Yang-Mills Hamiltonian, it turns out to be a fourth degree functional in the position and momentum variables which enables us to successfully perform the quantization without encountering serious renormalization problems. It should be mentioned here in passing that the formula $G_k = \epsilon(kij)K_{ai}E^a_j$ for the Gauss' constraint is in fact a rotational constraint or more precisely an angular momentum constraint because, it resembles a cross product between the Ashtekar position field E^a_i and the Ashtekar momentum field K_{ai}. In the next section, we shall show that by replacing the position field A^a_i by it holonomy, ie, the $SO(3)$ group elements obtained by parallely translating the $SO(3)$ Lie algebra position field A^i_a along small paths in Σ_t and by replacing the momentum field E_i^a by its flux through a small surface through which the former path pierces, we can effectively get rid of the singularities in the gravitational field quantization process after discretizing space into a graph of edges. Further, this method enables us to obtain an elegant formula for the area operator of a quantum blackhole in terms of $SO(3)$ angular momentum operators and thus derive the Hawking-Beckenstein formula for the blackhole entropy. These matters form the subject of the next section. However, before doing so, we remark that according to the above discussion, the ADM Hamiltonian is given by

$$H = \int (P^{ab}q_{ab,t} - L)N\sqrt{q}d^Dx = \int P^{ab}q_{ab,t}d^Dx - \int LN\sqrt{q}d^Dx$$

$$= \int P^{ab}(D_bN_a + D_aN_b + 2NK_{ab})d^Dx - \int (R^D + \alpha(K_{ab}K^{ab} - K^2))\sqrt{q}.N.d^Dx - - - (273)$$

where

$$P^{ab} = \alpha.\sqrt{q}.(K^{ab} - Kq^{ab}) - - - (274)$$

The second one gives on contraction, using $q_{ab}q^{ab} = 3$, that

$$P = q_{ab}P^{ab} = -2\alpha.\sqrt{q}.K - - - (275)$$

so that

$$K^{ab} = (\alpha\sqrt{q})^{-1}P^{ab} + q^{ab}(-2\alpha\sqrt{q})^{-1}P$$

$$= (\alpha\sqrt{q})^{-1}(P^{ab} - Pq^{ab}/2) - - - (276)$$

This gives (Note that for the real world, $D = 3$)

$$H = \int 2P^{ab}D_aN_bd^Dx - \int [R^Dq^{1/2}N + \alpha^{-1}(q)^{-1/2}N.[(P^{ab} - Pq^{ab}/2)(P_{ab} - Pq_{ab}/2) - P^2/4]d^Dx +$$

$$\int 2N(\alpha\sqrt{q})^{-1}(P^{ab} - Pq^{ab}/2)P_{ab}d^Dx$$

$$= \int 2P^{ab}D_aN_bd^Dx - \int [R^Dq^{1/2}N + \alpha^{-1}(q)^{-1/2}N.[P^{ab}P_{ab} - P^2/2 - P^2/2 + 3P^2/2 - P^2/4]d^Dx$$

$$+\int 2N(\alpha\sqrt{q})^{-1}(P^{ab} - Pq^{ab}/2)P_{ab}d^Dx$$

$$= \int 2P^{ab}D_aN_bd^Dx - \int [\sqrt{q}R^D + \alpha^{-1}q^{-1/2}.[-P^{ab}P_{ab} + 5P^/4]Nd^Dx - - - (277)$$

The first integral on the rhs defines the diffeomorphism constraint. In fact, the equation of motion corresponding to the position field N^a which has zero canonical momentum is given by observing that

$$P^{ab}D_aN_b = P^{ab}(N_{b,a} - \Gamma^c_{ab}N_c) = (P^{ab}N_b)_{,a} - (P^{ab}_{,a} + \Gamma^b_{ac}P^{ac})N_b - - - (278)$$

and using integration by parts:

$$P^{ab}_{,a} + \Gamma^b_{ac}P^{ac} = 0 - - - (279)$$

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The second integral on the rhs defines the Hamiltonian constraint. The equation of motion corresponding to the position field N which has zero canonical momentum is given by

$$[-\sqrt{q}R^D + \alpha^{-1}q^{-1/2}.[P^{ab}P_{ab} - 5P^{/}4] = 0 - - - (280)$$

In quantum mechanics, this equation is to be interpreted as the famous Wheeler-De-Witt equation, namely, the Schrodinger equation for general relativity:

$$H[N]\psi = 0\forall N - - - (281),$$

where

$$H[N] = \int [-\sqrt{q}R^D + \alpha^{-1}q^{-1/2}.[P^{ab}P_{ab} - 5P^{/}4].Nd^Dx - - - (282)$$

Note that R^D is a highly nonlinear function of only the position fields q_{ab} and their spatial derivatives $q_{ab,c}$. While quantization, $q_{ab}(x)$ are interpreted as multiplication operators while $P^{ab}(x) = -ih\delta/\delta q_{ab}(x)$. The wave function ψ is a functional of the position field $q_{ab}(x), x \in \Sigma_t = \mathrm{R}^3$.

Appendix A3. Fundamental idea of loop quantum gravity based on the flux-holonomy algebra

In loop quantum gravity, we introduce the flux-holonomy algebra of operators in place of the Ashtekar position-momentum field operators. This involves partitioning the fabric of space Σ_t into little loops that penetrate little surfaces and we compute the $SO(3)$ group elements obtained by parallely translating the $SO(3)$ Lie algebra element $A_a^i(x)\tau_i$ around these loops parallely according to the same connection where $\tau_i, i = 1,2,3$ form the standard basis for the Lie algebra of $SO(3)$. These group elements are called holonomies and we also simultaneously compute the flux of the Ashtekar momentum field E_i^a through the corresponding little surfaces associated with the different loops. The holonomies are computedby solving the differential equations

$$dh(s)/ds = h(s)\tau_i.A_a^i(x(s))(dx^a(s)/ds), s \in [0,1], h(0) = I_3 - - - (283)$$

where $x(s), s \in [0,1]$ defines a little loop in the space Σ_t. Solving this by the standard series expansion gives the holonomy associated with the path $x(s)$ as

$$h(1) = I + \sum_{n\geq 1}\int_{0<s_1<\ldots<s_n<1} A_{a_1}\left(x\left(s_1\right)\right)A_{a_2}\left(x\left(s_2\right)\right)\ldots A_{a_n}\left(x\left(s_n\right)\right)x^{a_1\prime}\left(s_1\right)\ldots x^{a_n\prime}\left(s_n\right)ds_1\ldots ds_n - - - (284)$$

where

$$A_a(x) = A_a^i(x)\tau_i - - - (285)$$

Further, the flux of $E_i^a(x)$ through a little surface S is computed as

$$Y_i(S) = \int_S E_i^a(x) n_a(x) dS(x) - - - (286)$$

where $n_a(x)$ is the covariant unit normal to the surface S at a point x on it. Suppose we parametrize the surface S as $X^a(u_1, u_2)$. Then, we can write

$$n_a(x) dS(x) = \epsilon(a a_1 a_2) X^{a_1}_{,1}(u) X^{a_2}_{,2}(u) d^2u, x = X(u) - - - (287)$$

We can write

$$Y_i(S) = \int E_i^a(x) \delta^3(x - X(u)) \epsilon(a a_1 a_2) X^{a_1}_{,1}(u) X^{a_2}_{,2}(u) d^2u d^3x - - - (288)$$

We find using the canonical Ashtekar commutation relations that

$$[Y_i(S), h(1)] =$$

$$\sum_{n \geq 1} \int A_{a_1}(x(s_1)) \ldots A_{a_{i-1}}(x(s_{i-1})[Y_i(S), A_{a_i}(x(s_i))] A_{a_{i+1}}(x(s_{i+1})) \ldots A_{a_n}(x(s_n)) ds_1 \ldots ds_n$$

$$= \sum_{n \geq 1} \sum_{k=1}^{n} \int A_{a_1}\left(x\left(s_1\right)\right) \ldots A_{a_{k-1}}\left(x\left(s_{k-1}\right)\left[Y_i(S), A_{a_k}\left(x\left(s_k\right)\right)\right] x^{a_k'}\left(s_k\right) A_{a_{k+1}}\left(x\left(s_{k+1}\right)\right) \ldots A_{a_n}\left(x\left(s_n\right)\right)\right.$$

$$\times \left(\Pi_{j=1, j \neq k}^{n} X^{a_j'}\left(s_j\right)\right) ds_1 \ldots ds_n - - - (289)$$

where

$$[Y_i(S), A_{a_k}(x(s_k))] = -i \int \delta^3(X(u) - x(s_k)) \delta_{a a_k} \delta_{ij} \tau_j n_a(x) dS(X(u))$$

$$= -i \tau_i \int \delta^3(X(u) - x(s_k)) n_{a_k}(X(u)) dS(X(u))$$

$$= -i \tau_i \int \delta^3(X(u) - x(s_k)) \epsilon(a_k a b) X^a_{,1}(u) X^b_{,2}(u) d^2u - - - (290)$$

We now replace the surface S by one having a small thickness δ parametrized by the variable t with the thickness varying along the path $x(s)$ in the vicinity of the point where it intersects the surface S so that the factor $\delta^3(X(u) - x(s_k))$ gets replaced by $\int_{-\delta}^{\delta} dt \delta^3(X(u, t) - x(s_k))$ and we can replace $\delta^3(X(u, t) - x(s_k))$ by $\delta^3((u, t) - (u_k, t_k)) / | \epsilon(cab) X^a_{,1}(u_k) X^b_{,2}(u_k) x^{c'}(t_k) |$ (Using a standard formula for the delta function of a function) with (u_k, t_k) denoting the point at which $X(u_k, t_k) = x(s_k)$. This gives us the result

$$\int_{s_{k-1}}^{s_{k+1}} [Y_i(S), A_{a_k}(x(s_k))] x^{a_k'}(s_k) ds_k = -i \int_{s_{k-1}}^{s_{k+1}}$$

$$ds_k \tau_i \int \delta^2(u - u_k) \delta(t - t_k) sgn(n_{a_k}(X(u_k)). x^{a_k'}(s_k)) dt d^2u$$

$$= -i. \int_{s_{k-1}}^{s_{k+1}} ds_k \tau_i. \epsilon(e, s_k, S) - - - (291)$$

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where $\epsilon(e, s_k, S)$ is $1, -1$ or 0 according as the edge $e = x : [0, 1]\Sigma_t$ intersects S at s_k with a tangent making an acute angle, an obtuse angle or zero angle with the unit normal. It is clear that the range of s_k should in fact be taken as the intersection of $[s_{k-1}, s_{k+1}]$ and the width of the surface introduced by making it have a small thickness and in the limit when this width shrinks to zero, the above integral will become zero except at the end points of the above multiple integral, ie, only when either $k = n$ so that $s_k = s_n, s_{n+1} = 1$ or when $k = 1$ so that $s_0 = 0$. Taking into account all of these facts, we finally end up with the result that

$$[Y_i(S), h(1)] = -i\epsilon(e, S)\tau_i. h(1) - - - (292)$$

or equivalently, in terms of matrix elements,

$$[Y_i(S), h(1)_{AB}] = -i\epsilon(e, S)(\tau_i h(1))_{AB} = -i\epsilon(e, S)(\tau_i)_{AC}(h(1))_{CB} - - - (293)$$

We write $h = h(1) = h([0, 1]) = h(e)$ for simplicity of notation. Here, $\epsilon(e, S)$ is one if the path e intersects S making an acute angle with the unit outward normal to S at its point of intersection -1 if this angle is obtuse and zero if the path falls lies in the surface. By using standard properties of commutators, it then follows that if $f(h)$ is any differentiable function of the path holonomy h, then

$$[Y_i(S), f(h)] = -i\epsilon(e, S)(\tau_i h)_{AB}\frac{\partial f(h)}{\partial h_{AB}} - - - (294)$$

with summation over the repeated index C being implied. Note that h is an $SO(3)$ matrix. Now this formula looks very much like the formula for a right invariant vector field R_j on $SO(3)$ defined by

$$R_j f(h) = \frac{d}{dt}f(exp(-it\tau_j)h)|_{t=0} = -i(\tau_j h)_{AB}\frac{\partial f(h)}{\partial h_{AB}} - - - (295)$$

So we can infer that $Y_i(S)$ is the same as the right invariant vector field $\epsilon(e, S)R_i = E_i(S)$ say. It is clear that R_j is an angular momentum operator obtained by constructing the right regular representation of $SO(3)$ evaluated at the $SO(3)$ generator τ_j. Now consider a general two dimensional surface S embedded into Σ_t parametrized by the equation $u \in \mathrm{R}^2 \to X(u) \in \Sigma_t \subset \mathrm{R}^3$. The area of this surface, from a standard result in differential geometry is given by

$$Area(S) = \int\sqrt{det((q_{ab}X^a_{,i}(u)X^b_{,j}(u)))_{1\le i,j\le 2}}d^2u - - - (296)$$

and we can write

$$q_{ab} = e^k_a e^k_b - - - (297)$$

on the one hand and on the other,

$$E^a_i = \sqrt{q}e^a_i - - - (298)$$

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or equivalently,

$$E^a_i = \epsilon(abc)\epsilon(ijk)e^j_b e^k_c - - - (299)$$

Now,

$$det((q_{ab}X^a_{,i}(u)X^b_{,j}(u)))_{1\leq i,j\leq 2}(d^2u)^2$$

$$= \epsilon(ik)\epsilon(jm)q_{ab}X^a_{,i}X^b_{,j}q_{cd}X^c_{,k}X^d_{,m}(d^2u)^2$$

$$= \epsilon(ik)\epsilon(jm)e^l_a e^l_b e^m_c e^m_d X^a_{,i}X^b_{,j}X^c_{,k}X^d_{,m}(d^2u)^2$$

$$= \epsilon(ik)X^a_{,i}X^c_{,k}d^2ue^l_a e^m_c.\ \epsilon(jm)X^b_{,j}X^d_{,m}.\ d^2ue^l_b e^m_d.\ d^2u$$

$$= \epsilon(acp)n_p dS.\ e^l_a e^m_c.\ \epsilon(bdq)n_q.\ dS.\ e^l_b e^m_d$$

$$= n_p.\ dS.\ \epsilon(lmk)E^p_k.\ n_q.\ dS.\ \epsilon(lmj)E^q_j.\ dS$$

$$= n_p.\ dS.\ E^p_k.\ n_q.\ dS.\ E^q_j\delta(kj) = n_p.\ E^p_k.\ n_q E^q_k(dS)^2 - - - (300)$$

Thus,

$$Area(S) = \int\sqrt{\sum_k(\sum_p n_p E^p_k)^2)}dS = \int\sqrt{\sum_k Y_k(dS)^2} - - - (301)$$

which on discretization of space into infinitesimal paths and surfaces gives

$$Area(S) = \sum\sqrt{\sum_k Y_k(\Delta S)^2} = \sum\sqrt{\sum_k E_k(\Delta S)^2}\,|\,\epsilon(e,\Delta S)\,| - - - (302)$$

This formula has the remarkable interpretation as being equal to the sum of square roots of squared angular momentum operators whose eigenvalues are well known in elementary quantum mechanics ie $j(j+1)$ with j integral. This sum is taken over directed edges that either have their beginning or ends on S and can be used to derive the celebrated Hawking-Beckenstein formula for the quantum Blackhole entropy $Area(S)/4G$ provided that we compute the entropy of the blackhole as the logarithm of the number of eigenvalues of the area operator that have values in a neighbourhood of the classical Blackhole area.

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