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Some aspects of quantum fields in curved classical and quantum background space-time using the quantum effective action formalism

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August 6, 2024

abstract

Some basic questions in quantum field theory and cosmology are addressed here. We derive some formulas for the change in the canonical and anticommutation relations at an equal time for some of the well-known quantum fields in the presence of a background curved space-time metric. We then derive formulas for the approximate change in the canonical commutation relations of a field or a set of fields when a small perturbing Lagrangian is added to the unperturbed Lagrangian. We study the problem of quantizing the Klein-Gordon field interacting with the gravitational field of homogeneous and isotropic space-time of an expanding universe and also simultaneously interacting with a classical random current field. Formulas for the quantum effective action of the scale factor of the expanding universe are derived by averaging over the Klein-Gordon field, taking into account its interaction with a classical random current field. This gives us information about how the expansion rate of our universe can be affected due to quantum mechanical interaction effects. Finally, we discuss the general problem of symmetry breaking in the quantum effective action of a field when it interacts with a random Gaussian current field source. The symmetry-breaking terms are expressed in terms of the correlation field of the random current source. We also discuss how canonical commutation relations of a Bosonic field between the position and velocity fields get perturbed approximately when a small perturbing Lagrangian is added.

1 Introduction

In this paper, we present a second quantization of the electromagnetic and Dirac fields in the background of classical curved space-times like the Schwarzchild space-time, the Kerr space-time and the Robertson-Walker space-time. We

https://doi.org/10.32388/QOI6PH

discuss how the canonical commutation relations in a curved background space-time lead to deviations in the commutation relations between the position and velocity fields when background curvature is present by taking the Robertson-Walker metric of space-time with a spatially homogeneous Klein Gordon field also present. We generalize this idea by dropping off the homogeneity condition on the Klein-Gordon metric. This is achieved by expanding the KG field as a linear combination of spatial basis functions with the coefficients in this expansion being functions of time only and serving as the position variables for the KG field. We thus set up the total action for the Robertson-Walker scale factor and the general KG field in this space-time metric. By path integrating over the KG position paths, namely over the paths defined by the linear combination coefficients that appear in the expansion of the KG field in terms of spatial basis functions, we are able to derive the quantum effective action for the scale factor of the universe when gravity interacts with the inhomogeneous KG field. This computation also tells us how to derive the wave function evolution of the scale factor of the universe when the universe consists of scalar KG particles. We then proceed further to calculate the quantum effective action of the scale factor of the universe along with the KG field within it when the KG field interacts with a classical random Gaussian current field. This quantum effective action is obtained by evaluating the classical average of the complex exponential of the total action of the RW gravitational field and the KG field in this metric and in addition, taking into account interactions between the random current field and the KG field. A further path integration of this averaged complex exponential over the KG field then yields us the quantum effective action of the scale factor of the universe alone. This formula can be used to predict how quantum effects with matter and random current fields in the universe can affect the rate of expansion of our universe. We then consider another example of such a situation in which we start with an arbitrary Lagrangian density of a field also depending on the metric of space-time with the metric being a function of a set of random parameter fields. We compute the quantum effective action for such a field by assuming that these random parameters have small variances. It should be mentioned that the quantum effective action is computed by forming the Legendre transform of the expected value of the complex exponential of the action w.r.t the mean value of the current in all the cases, and this then yields us the quantum equations of motion satisfied by the quantum effective action. It also gives us symmetry-breaking effects induced by the random current field in the sense that the variation of the quantum effective action under the quantum expectation value of the gauge transformation that leaves the classical action invariant is no longer invariant but instead depends on the covariance of the random current field. We illustrate such symmetry breaking using the example of a finite number of KG fields with its Lagrangian having global $O(N)$ symmetry leading to the same symmetry in the quantum effective action but to a breaking of this symmetry when the current with which the field interacts has random Gaussian fluctuations.

One of the aims of this paper is to look at the foundations of quantum field theory from the standpoint of accessible and inaccessible variables introduced by Inge Helland when general relativistic effects are considered. We then look at the Wheeler-De-Witt equation also called the Schrodinger equation for general relativity and based on the observation that this is a Lagrangian/Hamiltonian problem with constraints, we derive an intuitive approximate method for quantizing such constrained systems by considering the problem when n position variables are functions of $p < n$ free parameters and we construct p approximate conjugate momenta corresponding to the p canonical positions appearing as the free parameters. This construction is based on a least squares method.

2 The quantum electromagnetic field in curved background space-time

The Lagrangian density of the electromagnetic field in the background metric $g_{\mu\nu}(x)$ is given by

$$L(A_\mu, A_{\mu,\nu}) = (-1/4)F_{\mu\nu}F^{\mu\nu}\sqrt{-g} + (a/2)(-g)^{-1/2}(A^\mu\sqrt{-g})_{,\mu})^2 - - - (1)$$

where

$$A^\mu = g^{\mu\nu}A_\nu$$

It is clear that the integral of L over the whole of space-time is invariant w.r.t diffeomorphisms because of the invariance of the 4-volume element $\sqrt{-g}d^4x$ and the fact that

$$(A^\mu\sqrt{-g})_{,\mu} = A^\mu_{:\mu}\sqrt{-g}$$

with $A^\mu_{:\mu}$ is a scalar as also is $F_{\mu\nu}F^{\mu\nu}$. The second term in (1) is to be regarded as a gauge fixing term for the electromagnetic field. It stems from the invariance of the electromagnetic field

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$$

under the gauge transformation

$$A_\mu \to A_\mu + \phi_{,\mu}$$

with ϕ a scalar field. The parameter a can be looked upon as a Lagrange multiplier introduced to constrain $(A^\mu\sqrt{-g})_{,\mu}$ to vanish, just as we have the Lorentz gauge in flat space-time. It is useful to introduce this gauge fixing term as then the momentum fields corresponding to all the four position fields A_μ will be non-zero thereby enabling us to avoid the uncomfortable situation of having to treat this as a Lagrangian/Hamiltonian problem with constraints thereby forcing us to use Dirac brackets in place of Lie/Poisson brackets. Noting that

$$(A^0\sqrt{-g})_{,0} = g^{0\mu}\sqrt{-g}A_{\mu,0} + X$$

where X involves non-derivative terms of A_μ, we see that the momentum fields P^μ corresponding to the position fields A_μ are given by

$$P^\mu(x) = \partial L/\partial A_{\mu,0} = F^{\mu 0}(x)\sqrt{-g(x)} + a.g^{0\mu}(A^\nu\sqrt{-g})_{,\nu}$$

The canonical equal-time Bosonic commutation relations (CCR) are

$$[A_\mu(x), P^\nu(y)] = i\delta^\nu_\mu \delta^3(x-y), x^0 = y^0$$

and of course

$$[A_\mu(x), A_\nu(y)] = [P_\mu(x), P_\nu(y)] = 0, x^0 = y^0$$

These CCRs imply

$$[A_\mu(x), g^{\nu\rho}(y)g^{00}(y)\sqrt{-g(y)}A_{\rho,0}(y) + a.ag^{0\nu}(y)g^{0\rho}(y)\sqrt{-g(y)}A_{\rho,0}(y)]$$
$$= i\delta^\nu_\mu \delta^3(x-y), x^0 = y^0$$

or equivalently,

$$(g^{\nu\rho}g^{00} + ag^{\nu 0}g^{\rho 0})(y)\sqrt{-g(y)}[A_\mu(z), A_{\rho,0}(y)] = i\delta^\nu_\mu \delta^3(x-y), x^0 = y^0$$

Equivalently, defining the matrix

$$C^{\nu\rho}(y) = (g^{\nu\rho}g^{00} + ag^{\nu 0}g^{\rho 0})(y)\sqrt{-g(y)}$$

and its inverse

$$((K_{\nu\rho}(y))) = K(y) = C(y)^{-1} = ((C^{\nu\rho}(y)))^{-1},$$

we obtain the fundamental CCR for electromagnetics in the curved background:

$$C^{\nu\rho}(y)[A_\mu(x), A_{\rho,0}(y)] = i\delta^\nu_\mu \delta^3(x-y), x^0 = y^0$$

or equivalently,

$$[A_\mu(x), A_{\nu,0}(y)] = iK_{\nu\mu}(y)\delta^3(x-y), x^0 = y^0$$

In flat space-time, we have $g^{\nu\rho}(x) = \eta^{\nu\rho}$ and the CCR simplifies to

$$[A_\mu(x), A_{\nu,0}(y)] = i[(\eta + a.uu^T)^{-1}]_{\nu\mu}\delta^3(x-y), x^0 = y^0$$

where

$$u = [1, 0, 0, 0]^T$$

and

$$\eta = diag[1, -1, -1, -1]$$

is the Minkowskian metric.

Remark: The CCR derived above is not a coordinate and gauge-independent formula. It depends on a specific coordinate system chosen as well as the chosen gauge. It follows therefore that since the simultaneity of two events depends on the choice of the reference frame, we cannot use the equal-time commutation relation derived above in a different system of coordinates. Further, the above formula suggests that the "degree of non-commutativity of the position fields $A_\mu(x)$ and the corresponding velocity fields $A_{\mu,0}(y)$ at equal times $x^0 = y^0$ is metric dependent and also frame and gauge dependent. This suggests that the theoretical variables formulation of quantum mechanics proposed by the second author in a series of papers (see [2] and references there) will lead to degrees of accessibility and inaccessibility of observables being frame- and gauge-dependent.

3 The Dirac field in a background curved space-time interacting with the electromagnetic field

The covariant derivative is

$$\nabla_\mu = \partial_\mu - igA_\mu + \Gamma_\mu$$

where

$$A_\mu = A_\mu^a T_a, \Gamma_\mu = \omega_\mu^{mn}\gamma_{mn}/4$$

Note that

$$[T_a, T_b] = -iC(abc)T_c,$$

$$[\gamma_{mn}/4, \gamma_{ab}/4] = \eta_{mb}\gamma_{na}/4 + \gamma_{na}\eta_{mb}/4 - \gamma_{ma}\eta_{nb}/3 - \eta_{nb}\gamma_{ma}/4$$

T_a are Hermitian matrices. Since $\gamma^0, \gamma^0\gamma^n$ are Hermitian, we have

$$(\gamma^0\gamma^m\gamma^n)^* = (\gamma^n)^*\gamma^0\gamma^m = (\gamma^0\gamma^n)^*\gamma^m = \gamma^0\gamma^n\gamma^m$$

and hence,

$$\gamma^0\gamma^{mn} = \gamma^0[\gamma^m, \gamma^n]$$

are skew-Hermitian matrices. Thus,

$$i\gamma^0\Gamma_\mu = \omega_\mu^{mn} i\gamma^0\gamma_{mn}/4$$

are Hermitian matrices. The Dirac operator is

$$iD - m, D = \gamma^\mu\nabla_\mu, \gamma^\mu(x) = \gamma^a V_a^\mu(x)$$

The Dirac equation is

$$(iD - m)\psi = 0$$

which is the same as

$$i\gamma^\mu(\partial_\mu - igA_\mu + \Gamma_\mu)$$

The Dirac Lagrangian density from which the Dirac equation is derived is given by

$$L = \psi^*\gamma^0[iD - m]\psi.\sqrt{-g}$$

Hence, the momentum field conjugate to the canonical position field ψ is given by

$$P = \partial/\partial\partial_0\psi = iV_a^0\psi^*\gamma^0\gamma^a\sqrt{-g} = iV_a^0\sqrt{-g}\psi^*\alpha^a = i\sqrt{-g}\psi^*\tilde{\alpha}^0$$

where

$$\alpha^\mu = \alpha^\mu(x) = \alpha^a V_a^\mu(x)$$

are the non-constant Dirac α-matrices in the gravitational field. Note that the α^a are the constant Dirac α matrices. The canonical equal-time anticommutation relations are therefore

$$\{\psi(x), \psi(y)^*\}\tilde{\alpha}^0(y)\sqrt{-g(y)} = \delta^3(x-y), x^0 = y^0$$

or equivalently,

$$\{\psi(x),\psi(y)^*\} = (-g(x))^{-1/2}(\tilde{\alpha}^0(x))^{-1}\delta^3(x-y), x^0 = y^0$$

Now, we have the anticommutation relations

$$\{\tilde{\gamma}^\mu(x),\tilde{\gamma}^\nu(x)\} = 2g^{\mu\nu}(x)$$

and in particular,

$$(\tilde{\alpha}^0(x))^2 = (\gamma^0\tilde{\gamma}^0(x))^2 = (\gamma^0\gamma^a V_a^0(x))^2 = (\alpha^a V_a^0(x))^2 = \{\alpha^a,\alpha^b\}V_a^0 V_b^0/2$$

$$= (1/2)\sum_{a=0}^{3}(V_a^0)^2 = K(x)$$

say. Note that by $K(x)$, we mean $K(x)I$. We have used the fact that the α^a's mutually anticommute and their squares are the identity. It follows therefore that

$$\alpha^0(x)^{-1} = K(x)^{-1}\alpha^0(x)$$

Thus, we obtain the equal-time CAR as

$$\{\psi(x),\psi(y)^*\} = (-g(x))^{-1/2}K(x)^{-1}\alpha^0(x)\delta^3(x-y), x^0 = y^0$$

This is a fundamental equation because it gives us an idea of how much the canonical anticommutation relations of the Dirac field get affected by the presence of a background gravitational field. Specifically, we can evaluate this anticommutator in the background Robertson-Walker metric for an expanding homogeneous and isotropic universe and show that this equal-time CAR (Canonical anticommutation relation) becomes proportional to $S(t)^{-3}$.

4 The wave function in cosmological models

The idea of calculating probability amplitudes in cosmology and studying the evolution of the scale factor of the universe as it evolves through different histories using the Feynman path integral method is originally due to Hawking (The wave function of the universe) [reference??]. Hawking's idea is to start with the RW model for space-time corresponding to a homogeneous isotropic universe, i.e.,

$$d\tau^2 = dt^2 - S(t)^2 f(r)^2 - S(t)^2 r^2(d\theta^2 + sin^2(\theta)d\phi^2)$$

so that

$$g_{00} = 1, g_{11} + -S(t)^2 f(t)^2, 22 = -S(t)^2 r^2, g_{33} = -S(t)^2 r^2 sin^2(\theta)$$

where

$$f(r)^2 = 1/(1-kr^2)$$

with $k = 0, 1, -1$ according to whether the space is flat, spherical or hyperbolic and then evaluate the Einstein-Hilbert Lagrangian density for this universe

$$L_G(r, \theta, S(t), S'(t)) =$$

$$g^{\mu\nu}[\Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta} - \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}]\sqrt{-g}$$

Hawking then considers the Lagrangian density for a scalar Klein-Gordon field $\phi(t, r)$ in this background metric:

$$L_{KG}(r, \theta, \chi, \partial_\mu \chi | S(t)) = (1/2)g^{\mu\nu}\sqrt{-g}\chi_{,\mu}\chi_{,\nu} - m^2\sqrt{-g}\chi^2/2$$

He then considers evaluating the Schrodinger wave function of the scale factor of the universe$S$ at time t in a universe filled with such scalar particles using the Feynman path integral

$$\psi(t, S) = C \int exp(iS_G[t, S] + iS_{KG}[t, \chi|S])DS[0, t)D\chi[0, t]$$

where C is a numerical factor and

$$S_G[t, S] = \int_0^t ds \int L_G(r, \theta, S(s), S'(s))drd\theta.d\phi$$

$$= \int_0^t L_g(S(s), S'(s))ds$$

where

$$L_g = \int L_G d^3x$$

is the Lagrangian of $S(t)$ and

$$S_{KG}[t, \chi|S] = \int_0^t ds \int L_{KG}(r, \theta, \chi(s, r, \theta, \phi), \partial_\mu \chi(s, r, \theta, \phi)|S(s))drd\theta.d\phi$$

Hawking notes that since the universe is homogeneous and isotropic, we can assume that the KG wave field ψ is a function of only time to a good degree of approximation, so that

$$L_{KG} = (1/2)\chi'(t)^2\sqrt{-g} - m^2\sqrt{-g}\chi(t)^2$$

where

$$\sqrt{-g} = S^3(t)f(r)r^2 sin(\theta)$$

and so

$$\int \sqrt{-g}d^3x = KS(t)^3, K = 2\pi \int_0^1 \int_0^\pi f(r)r^2 sin(\theta)drd\theta$$

so that the KG Lagrangian becomes

$$L_{kg}(\chi(t),\chi'(t)|S(t)) = (K/2)S(t)^3[\chi'(t)^2 - m^2\chi(t)^2]$$

which means that the joint wave function of $S(t),\chi(t)$ (ie, the scale factor and the KG field at time t), is given by the formula

$$\psi(t,S,\chi) = C\int exp(i\int_0^t L_g(S(s),S'(s)) + L_{kg}(\chi(s),\chi'(s)|S(s))ds)DS[0,t)D\chi[0,t)$$

where $S(t) = S, \chi(t) = \chi$ and in particular, the probability density of $S(t)$ is given by

$$p(t,S) = \int |\psi(t,S,\chi)|^2 d\chi$$

Lengthy but elementary calculations show that the Einstein-Hilbert Lagrangian density

$$L_G = g^{\mu\nu}L_{\mu\nu}\sqrt{-g}$$

for the RW metric evaluate as follows:

$$L_G = (g^{00}L_{00} + g^{11}L_{11} + g^{22}L_{22} + g^{33}L_{33})\sqrt{-g}$$

with

$$\begin{aligned} L_{00} &= \Gamma^{\alpha}_{00}\Gamma^{\beta}_{\alpha\beta} - \Gamma^{\alpha}_{0\beta}\Gamma^{\beta}_{0\alpha} \\ &= -3S^{'2}/S^2, \\ L_{11} &= \Gamma^{\alpha}_{11}\Gamma^{\beta}_{\alpha\beta} - \Gamma^{\alpha}_{1\beta}\Gamma^{\beta}_{1\alpha} \\ &= 3S^{'2}f^2(r) - 2/r^2, \\ L_{22} &= \Gamma^{\alpha}_{22}\Gamma^{\beta}_{\alpha\beta} - \Gamma^{\alpha}_{2\beta}\Gamma^{\beta}_{2\alpha} \\ &= S^{'2}r^2 - rf'/f^3 - 1/f^2 - cot^2(\theta), \\ L_{33} &= \Gamma^{\alpha}_{33}\Gamma^{\beta}_{\alpha\beta} - \Gamma^{\alpha}_{3\beta}\Gamma^{\beta}_{3\alpha} \\ &= (S^{'2}r^2 - rf'/f^3 - 1/f^2)sin^2(\theta) + cos^2(\theta) \end{aligned}$$

so that

$$L_G = (4S^{'2}/S^2 + 2/S^2r^2 + 2f'/rS^2f^3 + 2/S^2f^2r^2)S^3fr^2sin(\theta)$$

It is clear then, that the Lagrangian of the scale factor $S(t)$ has the form

$$L_g(S(t),S'(t)) = \int L_G dr d\theta.d\phi = c_1 S^{'2}S + c_2 S$$

where c_1, c_2 are constants and hence that the total Lagrangian of the scale factor $S(t)$ and the KG field $\chi(t)$ in homogeneous and isotropic space-time is given by

$$L(S,S',\chi,\chi') = c_1 S^{'2}S + c_2 S + (K/2)S(t)^3[\chi'(t)^2 - m^2\chi(t)^2]$$

This general form can be used in our study of the joint wave function of the scale factor and the KG field and of course, also the classical dynamics of these two quantities. Alternatively, can also quantize these dynamics using the Schrodinger wave mechanics and the Heisenberg matrix mechanics by regarding S, χ as canonical position variables with the corresponding momentum variables given by

$$P_S = \partial L/\partial S' = 2c_1 SS', P_\chi = \partial L/\partial \chi' = KS^3\chi'$$

The commutation relations (Bosonic) are given by

$$[S, P_S] = i, [\chi, P_\chi] = i, [S, \chi] = [P_S, P_\chi] = 0$$

so in terms of velocities,

$$[S, S'] = i/2c_1 S, [\chi, \chi'] = i/KS^3$$

The second equation shows that as the universe keeps expanding, $S(t)$ increases with time, and hence the commutator between χ and χ' gets smaller and smaller which means that the uncertainty in measuring both the KG field and its rate of change with time gets smaller and smaller. Likewise, the first commutation relations also imply that the uncertainty between the scale factor and its rate of change with time also decreases in the expanding universe. It is interesting to see what this implies from the standpoint of degrees of inaccessibility of theoretical variables.

More generally, this idea of quantization can be looked upon as a form of restricted quantum gravity or more generally, restricted quantum field theory in the following sense. Suppose

The Wheeler-De Witt equation or the Schrodinger equation of quantum general relativity.

An appropriate quantization method shows that the action for general relativity (more precisely, the ADM action) can be expressed as

$$\int N\sqrt{q}(Q(K) + L(q_{ab}, q_{ab,c}))d^3xdt$$

where q_{ab} is a three dimensional spatial metric and $K = ((K_{ab}))$ is linear in the time derivative $q_{ab,t}$ of the 3-metric with $Q(K)$ being quadratic in K. N is a scalar field that acts as a Lagrange multiplier when we derive the corresponding Hamiltonian having constraints. This action is derived by embedding a one-parameter family of three-dimensional surfaces Σ_t in $\mathbb{R}^4$ with the surfaces being parametrized by time t. The embedding of this surface is defined by the equation

$$X^\mu(t, .) : \Sigma_t \to \mathbb{R}^4$$

with

$$X^\mu_{,t} = T^\mu = B^\mu + Nn^\mu$$

where n^μ is the unit normal to Σ_t while N^μ is tangential to Σ_t, i.e., N^μ is expressible as

$$N^\mu = N^a X^\mu_{,a}$$

Note that the Roman letters a, b, c run over $1, 2, 3$, i.e., the spatial coordinates of points in Σ_t where t is the time coordinate that parametrizes the surface Σ_t. Note that N^μ and n^μ are orthogonal w.r.t the metric $g_{\mu\nu}$ in $\mathbb{R}^4$:

$$g_{\mu\nu} N^\mu n^\nu = 0$$

This decomposition leads us to the following decomposition of the metric in $\mathbb{R}^4$:

$$g^{\mu\nu} = q^{\mu\nu} + n^\mu n^\nu$$

where

$$q^{\mu\nu} = q^{ab} X^\mu_{,a} X^\nu_{,b}$$

and therefore

$$q^{\mu\nu} n_\nu = 0$$

Note that if $\tilde{g}_{\mu\nu}$ is the metric in the $(x^a, t) = (x^\alpha)$ coordinate system where x^a parametrizes the coordinates in Σ_t, then

$$g^{\mu\nu} = \tilde{g}^{\alpha\beta} X^\mu_{,\alpha} X^\nu_{,\beta}$$

$$= \tilde{g}^{ab} X^\mu_{,a} X^\nu_{,b} + 2\tilde{g}^{a0}(X^\mu_{,a} T^\nu + X^\nu_{,a} T^\mu)$$
$$+ \tilde{g}^{00} T^\mu T^\nu$$

Using then the orthogonality of $X^\mu_{,a}$ and n^μ, we get

$$g_{\mu\nu}(T^\mu - N^a X^\mu_{,a}) X^\nu_{,b} = 0$$

or equivalently,

$$\tilde{g}_{0b} - \tilde{g}_{ab} N^a = 0$$

which gives us the above orthogonal decomposition of the metric $g^{\mu\nu}$. The form of the ADM action is a nice one as it decomposes the action into quadratic form in the time derivatives of the spatial metric and a purely spatial component involving only the spatial derivatives of the spatial metric. This form of the ADM action also immediately reveals that it leads to a Hamiltonian problem with constraints. The Hamiltonian has the form

$$H = \int (N^a H_a + N H_0) d^3x$$

where H_0 is quadratic in the canonical momenta $P^{ab} = \partial L / \partial q_{ab,t}$ while H_a is linear in the canonical momenta. The complete set of position fields are N, N^a, q_{ab}, totally ten in number and since the time derivatives of N, N^a do not appear in this Hamiltonian, it follows that N, N^a do not vary with time. H_a is called the diffeomorphism constraint because it is of the form $q_{bc} D^a P^{bc}$

and hence it generates spatial derivatives in view of the canonical commutation relations

$$[P^{ab}(x), cd(y)] = -i\delta^3(x-y)(\delta^a_c\delta^b_d + \delta^a_d\delta^b_c)$$

The functions L, H_0 appearing respectively in the ADM Lagrangian and Hamiltonian are highly nonlinear functions of $q_{ab}, q_{ab,c}$ and are therefore very difficult to deal with in a quantization programme. However, Ashtekar drastically simplified quantization by introducing the so-called Ashtetkar variables in terms of the $SO(3)$ triad e^i_a for the spatial metric q_{ab}, i.e.,

$$q_{ab} = e^i_a e^i_b$$

the summation over $i = 1, 2, 3$ being implicit. In terms of the Ashtekar variables, the canonical Hamiltonian for gravity behaves very much like a Yang-Mills non-Abelian gauge field Hamiltonian in that although this Hamiltonian is highly nonlinear in the Ashtekar variables, the highly nonlinear component is homogeneous of order zero, i.e., it is invariant under scaling and therefore commutes with the generator of the scaling group given by a bilinear combination of the position and momentum fields, also called the Gauss constraint operator. Specifically, if $iP.Q$ is the scaling operator for canonical positions Q and momenta P, since

$$[iP.Q, f(Q)] = Q.f'(Q)$$

and therefore,

$$exp(i\lambda ad(P.Q))(f(Q)) = exp(\lambda Q.\nabla_Q)(f(Q)) = f(exp(\lambda)Q)$$

The Gauss constraint operator is constructed from

$$K_{ab} = X^\mu_{,a} X^\nu_{,b} \nabla_\mu n_\nu$$

(This is the same K that appears in the momentum-containing component of the ADM action), by noting that it is symmetric in (a, b) and therefore

$$K_{[a,b]} = K_{ab} - K_{ba} = 0$$

This means that if e^a_i is the spatial metric triad, then

$$K_{ij} = K_{ab} e^a_i e^b_j$$

is also symmetric. Defining

$$E(c, k) = \epsilon(ijk)\epsilon(abc)e^i_a e^j_b$$

we observe that

$$E(c, k)e^m_c = q^{1/2}\epsilon(ijk)\epsilon(ijm) = q^{1/2}\delta_{jm}$$

and therefore,

$$E(c, k) = q^{1/2} e^c_k$$

Define

$$K_{ai} = K_{ab}e_i^b$$

We then get

$$G_i = \epsilon(ijk)K_{aj}E(a,k) = q^{1/2}\epsilon(ijk)K_{aj}e_k^a$$
$$= q^{1/2}\epsilon(ijk)K_{ab}e_j^b e_k^a = 0$$

because $K_{ab}e_j^b e_k^a$ is symmetric in (j,k). G_i is called the Gauss constraint or the rotational constraint since it can be looked upon as a cross product between the position vector K_{aj} and the momentum vector $E(a,j)$. Under the condition of the Gauss constraint, it is easy to show that the Ashtekar position variables $E(a,k)$ and corresponding Ashtekar momentum variables satisfy the canonical commutation relations.

We are now interested in formulating a generalization of Hawking's theory to the case when the scalar particles in our universe are not distributed homogeneously and isotropically, i.e., when the KG field is $\chi(t,r,\theta,\phi)$. In order to do so, we choose basis functions $\eta_n(r,\theta,\phi), n = 1,2,...$ that are functions of only the spatial variables and expand the KG field in terms of them with coefficients being functions of time:

$$\chi(t,r,\theta,\phi) = \sum_n \chi_n(t)\eta_n(\mathbf{r}), \mathbf{r} = (r,\theta,\phi)$$

Then, w.r.t to the RW metric, we have

$$\int g^{\mu\nu}\chi_{,\mu}\chi_{,\nu}\sqrt{-g}d^3x$$

$$(\chi_{,0}^2 + g^{11}\chi_{,1}^2 + g^{22}\chi_{,2}^2 + g^{33}\chi_{,3}^2)\sqrt{-g}$$
$$= S(t)^3\sum_{n,m} a(n,m)\chi_n'(t)\chi_m'(t) - S(t)\sum_{n,m} b(n,m)\chi_n(t)\chi_m(t)$$

where

$$a(n,m) = \int \eta_n((\mathbf{r})\eta_m(\mathbf{r})f(r)r^2 sin(\theta)drd\theta d\phi,$$

$$b(n,m) = \int \eta_{n,1}(\mathbf{r})\eta_{m,1}(\mathbf{r})f(r)^{-1}r^2 sin(\theta)drd\theta d\phi,$$

$$+\int \eta_{n,2}(\mathbf{r})\eta_{m,2}(\mathbf{r})f(r)sin(\theta)drd\theta d\phi,$$

$$\int \eta_{n,3}(\mathbf{r})\eta_{m,3}(\mathbf{r})f(r)sin(\theta)^{-1}drd\theta d\phi,$$

Moreover,

$$m^2\int \chi^2\sqrt{-g}d^3x = S(t)^3\sum_{n,m} c(n,m)\chi_n(t)\chi_m(t)$$

where

$$c(n,m) = m^2 \int \eta_n(\mathbf{r})\eta_m(\mathbf{r})f(r)r^2 sin(\theta)d^3x$$

Thus, the KG Lagrangian becomes

$$L_{kg}(\chi_n(t), \chi'_n(t), n \geq 1) =$$

$$(1/2)\int g^{\mu\nu}\chi_{,\mu}\chi_{,\nu}\sqrt{-g}d^3x - (1/2)\int m^2\chi^2\sqrt{-g}d^3x$$

$$= (1/2)S(t)^3\sum_{n,m} a(n,m)\chi'_n(t)\chi'_m(t) - (1/2)S(t)\sum_{n,m} b(n,m)\chi_n(t)\chi_m(t)$$

$$-(1/2)S(t)^3\sum_{n,m} c(n,m)\chi_n(t)\chi_m(t)$$

$$= (1/2)S(t)^3\chi'(t)^T A\chi'(t) - (1/2)S(t)\chi(t)^T B\chi(t) - (1/2)S(t)^3\chi(T)^T C\chi(t)$$

where

$$\chi(t) = ((\chi_n(t)))_{n=1}^{\infty}, A = ((a(n,m))), B = ((b(n,m))), C = ((c(n,m)))$$

Thus the joint wave function of $(S(T), \chi(T))$ is now given by

$$\psi(T, S, \chi) = \psi(T, S, ((\chi_n)))) =$$

$$\int exp(i(c_1 S'(t)^2 S(t) + c_2 S(t) + (i/2)S(t)^3\chi'(t)^T A\chi'(t) - (i/2)S(t)\chi(t)^T B\chi(t)$$

$$-(i/2)S(t)^3\chi(T)^T C\chi(t))DS[0,T)D\chi[0,T)$$

It should be noted that if we were interested only in the wave function of $S(t)$, then we would first evaluate the Gaussian integral w.r.t χ by replacing it with the value χ_0 at which its action is stationary, i.e., χ_0 satisfies

$$-S(t)^3 A\chi''_0(t) - S(t)B\chi(t) + S(t)^3 C\chi_0(t) = 0$$

This is the same as

$$A\chi''_0(t) = -S(t)^{-2}B\chi(t) + C\chi_0(t)$$

This is an infinite-dimensional linear second-order differential equation with time-varying coefficients for the infinite-dimensional vector-valued function of time

$$\chi_0(t) = ((\chi_{0n}(t)))$$

and its solution will be a function of $S(s), s \leq t$.

5 Quantum effective action in cosmology in the presence of a random current field interacting with the scalar field

Some remarks on the quantum effective action in quantum cosmology: Consider as above, the joint action of the gravitational field and the scalar KG field but taking into account an interaction between the KG field and a classical random current source $J(t)$. Assuming for simplicity that the KG field depends only on time, this action is given by

$$S_1[S,\chi|J] = \int [c_1 S'^2(t)S(t) + c_2 S(t) + (K/2)S(t)^3(\chi'(t)^2 - m^2\chi(t)^2) + J(t)\chi(t)]dt$$

We write this action as

$$\int [L(S(t), S'(t), \chi(t), \chi'(t)) + J(t)S(t)]dt = S_0[S,\chi] + \int J\chi.dt$$

Assume for simplicity that $J(t)$ is a Gaussian random current source with mean $M_J(t)$ and covariance

$$Cov(J(t), J(s)) = C_J(t,s)$$

In order to compute the quantum effective action of S, χ after taking into account this interaction, we form the statistical mean of the path integral:

$$Z(M,C) = exp(iW(M,C))$$

$$= \mathbb{E}\int exp(i\int (L(S(t), S'(t), \chi(t), \chi'(t)) + J(t)\chi(t))dt)DS.D\chi$$

$$= \int exp(iS_0[S,\chi] + i\int M(t)\chi(t)dt - (1/2)\int C(t,s)\chi(t)\chi(s)dtds)DS.D\chi$$

Such an approximation to the path integral is justified when the scalar field interacts with a cloud of other particles like gravitons distributed all over the cosmos with the quantum fluctuations in the graviton field being approximated by a classical random field. If we wish to be more accurate, we should take $J(t)$ as a quantum stochastic process in the sense of Hudson and Parthasarathy [reference??] and calculate quantum expectations of the resulting path integral in a coherent state of the current. We first discuss the classical stochastic approximation and then the quantum stochastic approximation. For given current covariance C, we can define the quantum effective action in the usual way:

$$\Gamma(\chi_0, C) = Ext_M(-i.logZ(M,C) - \int M\chi_0 dt)$$

Here, we are assuming that the scale factor process $S(t)$ is a given fixed classical process. Extremizing, we get

$$i\delta logZ(M,C)/\delta M(t) + \chi_0(t) = 0$$

Note that this equation implies that the value of the mean current M is that at which the average of $\chi(t)$ equals the classical process $\chi_0(t)$ given the scale factor process $S(.)$ and the classical current covariance C. We now observe that assuming that M satisfies this equation,

$$\delta\Gamma(\chi_0, C)/\delta\chi_0(t) = -M(t)$$

This is the required equation of motion for the classical field $\chi_0(t)$ defined as the classical and quantum average of $\chi(t)$ given the mean current M and the current covariance C. In this context, it is interesting to generalize this equation to the general case when we do not restrict it to a homogeneous and isotropic KG scalar field. In that case, proceeding as earlier, the KG action in the presence of a random current field $J(t, \mathbf{r})$ is given by

$$\int L_{kg}(\chi_n(t), \chi_n'(t), n \geq 1)dt + \sum_n \int J_n(t)\chi_n(t)dt$$

$$= \int [(1/2)S(t)^3\chi'(t)^T A\chi'(t) - (1/2)S(t)\chi(t)^T B\chi(t) - (1/2)S(t)^3\chi(T)^T C\chi(t)]dt$$

$$+ \int J(t)^T D\chi(t)dt$$

$$= S_0[\chi] + \int J^T D\chi dt$$

where we have expressed the current field in terms of the spatial basis functions $\eta_n(\mathbf{r})$ as

$$J(t, \mathbf{r}) = \sum_n J_n(t)\eta_n(\mathbf{r})$$

and

$$\chi(t, r, \theta, \phi) = \chi(t, \mathbf{r}) = \sum_n \chi_n(t)\eta_n(\mathbf{r})$$

so that

$$int J(t, \mathbf{r})\chi(t, \mathbf{r})d^3rdt = \int J(t)^T D\chi(t)dt$$

where

$$J(t) = ((J_n(t))_n, \chi(t) = ((\chi_n(t)),$$

and

$$D = ((\int \eta_n(t, \mathbf{r})\eta_m(t, \mathbf{r})d^3r))_{n,m}$$

Writing

$$\mathbb{E}J_n(t) = M_n(t), Cov(J_n(t), J_m(s)) = C_{0nm}(t, s)$$

or equivalently,

$$\mathbb{E}(J(t)) = M(t) = ((M_n(t))), Cov(J(t), J(s))$$

$$= \mathbb{E}(J(t)J(s)^T) - M(t)M(s)^T = C_0(t,s) = ((C_{0nm}(t,s)))_{n,m}$$

We get

$$Z(M,C_0) = \mathbb{E}\int exp(iS_0[\chi] + i\int J(t)^T D\chi(t)dt)D\chi$$

$$= \int exp(iS_0[\chi] + i\int M(t)^T D\chi(t)dt - (1/2)\int \chi(t)^T DC_0(t,s)D\chi(s)dtds)D\chi$$

so that the quantum effective action is given by

$$\Gamma(\chi_0, C_0) = Ext_M(-i.log(Z(M,C_0)) - \int M^T D\chi_0 dt)$$

$$= -ilogZ(M_0,C_0) - \int M_0^T D\chi_0 dt$$

where $M_0(t)$ satisfies

$$i\delta(logZ(M_0,C_0))/\delta M(t) + D\chi_0(t) = 0$$

The corresponding quantum equations of motion are

$$\delta\Gamma(\chi_0,C_0)/\delta\chi_0(t) = -DM_0(t)$$

Noting that

$$Q(\chi,M,C_0) = iS_0[\chi] + i\int M(t)^T D\chi(t)dt - (1/2)\int \chi(t)^T DC_0(t,s)D\chi(s)dtds$$

$$= i\int [(1/2)S(t)^3\chi'(t)^T A\chi'(t) - (1/2)S(t)\chi(t)^T B\chi(t) - (1/2)S(t)^3\chi(t)^T C\chi(t)]dt$$

$$+i\int M(t)^T D\chi(t)dt - (1/2)\int \chi(t)^T DC_0(t,s)D\chi(s)dtds$$

is a linear-quadratic functional of χ, it is easy to evaluate the Gaussian integral

$$Z(M,C_0) = \int exp((-1/2)Q(\chi,M,C_0))D\chi$$

as apart from a multiplicative constant, equal to

$$D_0(S,C_0)^{-1/2}.exp((-1/2)Q(\chi_0,M,C_0))$$

where $D_0(S,C_0)$ is the determinant of the kernel-matrix of the quadratic form Q and χ_0 is the value of χ at which Q becomes stationary, i.e., χ_0 satisfies

$$(S(t)^3 A\chi_0'(t))' + S(t)B\chi_0(t) + S(t)^3 C\chi_0(t) - DM(t) + \int DC_0(t,s)D\chi_0(s)ds = 0$$

Apart from a multiplicative constant, the kernel-matrix of the quadratic form Q is given by

$$K_Q(t,s) = (d/dt)(S(t)^3\delta'(t-s))A + \delta(t-s)(S(t)B + S(t)^3 C) + DC_0(t,s)D$$

and its determinant has to be evaluated.

6 Implications of classical randomness in the current source on symmetry breaking

It is a well-known fact that when the gauge symmetry of a quantum effective action is spontaneously broken by the field acquiring vacuum expectation values, then massless particles are produced, one particle being associated with each gauge degree of freedom that is broken. On the other hand, when the gauge symmetry of the quantum effective action is approximately broken by the presence of small perturbations that do not respect gauge symmetry, then massless particles acquire masses, and already massive particles become more massive. So, it is important to decide the mechanisms by which the gauge symmetry of a quantum effective action can be broken. We shall consider the problem of symmetry breaking from the standpoint of coupling the field to a random classical Gaussian current field. To this end, consider the action $I[\phi]$ of a field such that under an infinitesimal gauge transformation

$$\phi \to \phi + \epsilon.\Delta(\phi)$$

the product of the exponentiated action and the path measure remains invariant, i.e.,

$$exp(iI[\phi])D\phi = exp(iI[\phi + \epsilon.\Delta(\phi)])D(\phi + \epsilon.\Delta(\phi))$$

For example, for the complex KG field in a background curved space-time, the action is

$$I[\psi] = \int g^{\mu\nu}\sqrt{-g}\bar{\psi}_{,\mu}\psi_{,\nu}d^4x - m^2\int \psi^*\psi\sqrt{-g}d^4x$$

which is invariant under the infinitesimal $U(1)$ gauge transformation

$$\psi \to exp(i\epsilon)\psi = \psi + i\epsilon\psi, \epsilon \in \mathbb{R}, \epsilon \to$$

More generally, for a vector valued complex KG field $\psi = ((\psi_n))_{n=1}^N$, the action can be taken as

$$I[\psi] = \sum_n \int g^{\mu\nu}\sqrt{-g}\bar{\psi}_{n,\mu}\psi_{n,\nu}d^4x - m^2\sum_n \int \psi_n^*\psi_n\sqrt{-g}d^4x$$

which is invariant under the infinitesimal $U(N)$ gauge transformation

$$\psi \to exp(iX)\psi = \psi + iX.\psi, X \in \mathbb{C}^{N\times N}, X^* = X, \parallel X \parallel \to 0$$

Let $T_n, n = 1, 2, ..., N^2$ be a basis for the N^2 dimensional real vector space of $N \times N$ Hermitian matrices. Then, the above $U(N)$ symmetry of the complex KG action can also be expressed as

$$\psi \to exp(i\sum_{n=1}^{N} \epsilon(n)T_n)\psi = \psi + i\sum_{n=1}^{N^2} \epsilon(n).T_n\psi, \epsilon(n) \in \mathbb{R}, \epsilon(n) \to 0$$

If instead, ψ where a real KG scalar field with action

$$I[\psi] = (1/2)\sum_n \int g^{\mu\nu}\sqrt{-g}\psi_{n,\mu}\psi_{n,\nu}d^4x - (m^2/2)\sum_n \int \psi_n^2$$

then the symmetry would instead be $O(N,\mathbb{R})$, i.e.,

$$\psi \to \psi + \epsilon.X.\psi, X \in \mathbb{R}^{N\times N}, X^T = -X$$

this symmetry group now being $N(N-1)/2$-dimensional. Now consider the path integral

$$Z(J) = \int exp(iI[\psi] + i\int J.\psi d^4x)D\psi$$

where J is a non-random current field. It is well known that when $exp(iI[\psi])D\psi$ is invariant under the infinitesimal gauge transformation $\psi \to \psi + \epsilon.\Delta(\psi)$, then the equation

$$Z(J) = \int exp(iI[\psi] + i\int J.\psi)(1 + i\int J.\Delta(\psi))D\psi = 0$$

gives

$$\int J. < \Delta(\psi) >_J d^4x = 0$$

which can be expressed in the form of a gauge invariance principle for the quantum effective action

$$\int \frac{\delta\Gamma[\psi_0]}{\delta\psi_0(x)}. < \Delta(\psi) >_J (x)d^4x = 0$$

where J is the current field for which

$$< \psi > (x)_J = \psi_0(x)$$

In the case when the gauge transformation $\Delta(\psi)$ is linear in ψ as it happens for the three examples considered above, then,

$$< \Delta(\psi) >_J = \Delta(< \psi >_J) = \Delta(\psi_0)$$

and then we get the result that the quantum effective action is also invariant under the same gauge transformation that leaves the classical action invariant, or more precisely as that which leaves the product $exp(iI[\psi])D\psi$ invariant:

$$\int \frac{\delta\Gamma[\psi_0]}{\delta\psi_0(x)}.\Delta(\psi_0)(x)d^4x = 0$$

Note that the quantum effective action is defined as

$$\Gamma[\psi_0] = Ext_J[-ilogZ(J) - \int J.\psi_0 d^4x]$$

where Ext denotes extremum w.r.t J. These results make use of the fact that the quantum effective action $\Gamma[\psi_0]$ satisfies its equation of motion

$$\delta\Gamma[\psi_0]/\delta\psi_0(x) = -J(x)$$

and hence

$$\delta\Gamma[\psi_0]/\delta\psi_0(x)\delta\psi_0(y) = -\delta J(x)/\delta\psi_0(y)$$

On the other hand, we have

$$\psi_0(x) = Z(J)^{-1}\int \psi(x).exp(i.I[\psi] + i\int J.\psi)D\psi$$

$$= -i\delta log(Z(J))/\delta J(x)$$

and hence,

$$\delta\psi_0(x)/\delta J(y) = -i\delta^2 Z(J)/\delta J(x)\delta J(y) = -\Delta(x,y)$$

where $\Delta(x,y)$ is the propagator of the field ψ after subtracting out its mean value ψ_0, (ie, the propagator of the quantum fluctuating component of the field around its vacuum expected value). Thus, we get the fundamental formula

$$\delta\Gamma(\psi_0)/\delta\psi_0(x)\delta\psi_0(y) = \Delta^{-1}(x,y)$$

which means that the eigenfunctions of the Hessian matrix of the quantum effective action having zero eigenvalues are precisely the eigenvectors of the propagator having infinite eigenvalues. The eigenvalues of the inverse Bosonic propagator are however the squared masses of the particles in analogy with the fact that the inverse propagator of a free KG particle is given by $p^2 - m^2$ in the momentum domain, for which if δm^2 is an eigenvalue corresponding to an eigenvector, then this eigenvector is a field perturbation which when added to the vacuum expected field, carries a mass of $m^2 + \delta m^2$. Now let us consider the situation when the current J is a random field with mean M and covariance C. Then, the quantum effective action is computed as

$$\Gamma(\psi_0, C) = Ext_M(-i.logZ(M,C) - \int M.\psi_0)$$

with

$$Z(M,C) = \mathbb{E}exp(iI[\psi] + i\int J.\psi)D\psi$$

$$= \int exp(iI[\psi] + i\int M.\psi - (1/2)\int \psi^T C\psi)D\psi - - - (a)$$

so that

$$\Gamma(\psi_0) = -i.logZ(M_0, C) - \int M_0.\psi_0$$

where M_0 satisfies,

$$-i\delta logZ(M_0, C)/\delta M(x) - \psi_0(x) = 0$$

This gives

$$\delta\Gamma(\psi_0)/\delta\psi_0(x) = -M_0(x) - - - (b)$$

However, now observe that $\Gamma(\psi_0)$ is now no longer gauge invariant since we have to replace ψ by $\psi + \epsilon.\Delta(\psi)$ in the path integral (a),

$$\int exp(iI[\psi]+i\int M.\psi+i\epsilon\int M.\Delta(\psi)-(1/2)\int \psi^T C\psi-\epsilon\int \psi^T C\Delta(\psi))D\psi = 0$$

or equivalently,

$$i\int M. < \Delta(\psi) >_M - \int < \psi^T C\Delta(\psi) >_M = 0$$

or equivalently, making use of the equations of motion

$$\int (\delta\Gamma(\psi_0)/\delta\psi_0(x)). < \Delta(\psi)(x) >_{M_0} = -i\int < \psi(x)C(x,y)\Delta(\psi(y)) >$$

which shows clearly how much is gauge invariance of the quantum effective action is broken when the current to which the field is coupled has random fluctuations.

An example from background general relativity and cosmology: Suppose that $g_{\mu\nu}(x|\theta(x))$ is the metric of space-time where θ is a random parameter field and that the field $\phi(x)$ in this background field is described by the Lagrangian

$$L(\phi(x), \partial_\mu\phi(x), g_{\mu\nu}(x|\theta(x))$$

Assume that $\theta(x)$ has small random fluctuations around its mean value $\theta_0(x)$, so writing

$$\delta\theta(x) = \theta(x) - \theta_0(x),$$

we can write approximately

$$g_{\mu\nu}(x|\theta(x)) = g_{\mu\nu}(x|\theta_0(x) + g_{\mu\nu,k}(x|\theta_0)\delta\theta_k(x)$$

where

$$g_{\mu\nu,k}(x|\theta_0) = \partial g_{\mu\nu}(x|\theta_0)/\partial\theta_k$$

Another linearization gives approximately

$$L(\phi(x), \partial_\mu\phi(x), g_{\mu\nu}(x|\theta(x)))$$

$$= L(\phi(x), \partial_\mu\phi(x), g_{\mu\nu}(x|\theta_0(x))+$$

$$(\partial L(\phi(x), \partial_\mu\phi(x), g_{\mu\nu}(x|\theta_0(x))/\partial g_{\mu\nu})g_{\mu\nu,k}(x|\theta_0(x))\delta\theta_k(x)$$

with obvious summation conventions. This expression may be abbreviated to

$$L_0(\phi(x), \partial_\mu\phi(x)) + L_k(\phi(x), \partial_\mu\phi(x))\delta\theta_k(x)$$

and hence the action integral for ϕ has the form

$$S_0(\phi) + \int L_k(\phi)(x)\delta\theta_k(x)d^4x$$

Calling the vector $(\theta_k(x)) = J(x)$ and $L(\phi) = ((L_k(\phi))$, we can express the path integral as

$$Z = \int \mathbb{E}exp(i.S_0(\phi) + \int L(\phi)(x)^T J(x)d^4x)D\phi$$

so that when $J(x)$ is a Gaussian field, this evaluates to give

$$Z(M,C) =$$

$$\int exp(iS_0(\phi)+i\int L(\phi)(x)^T M(x)d^4x-(1/2)\int L(\phi)(x)^T C(x,y)L(\phi)(y)d^4xd^4y)D\phi$$

so that the effective action can be defined as (for a given classical field $\phi_0(x)$) as

$$\Gamma(\phi_0, C) = Ext_M(-i.logZ(M,C) - \int L(\phi_0)(x)^T M(x)d^4x)$$

$$= -i.logZ(M_0,C) - \int L(\phi_0)(x)^T M_0(x)d^4x$$

where M_0 solves

$$i.\delta logZ(M_0,C)/\delta M(x) + L(\phi_0)(x)^T M_0(x) = 0$$

In other words, M_0 is that current field at which the combined classical and quantum expectation of $L(\phi)(x)$ becomes $L(\phi_0)(x)$. We then get our quantum equations of motion as

$$\delta\Gamma(\phi_0, C)/\delta\phi_0(x) = -\int (\delta L(\phi_0(y))/\delta\phi_0(x))^T M_0(y)d^4y$$

Now, assume that $S_0(\phi)$ has a gauge symmetry

$$S_0(\phi + \epsilon.\Delta(\phi)) = S_0(\phi) + o(\epsilon)$$

with the path measure $D\phi$ being invariant under this gauge transformation. More precisely, we assume that the product

$$exp(iS_0(\phi))D\phi$$

is invariant under $\phi \to \phi+\epsilon.\Delta(\phi)$. Then, we get on changing the path integration variable from ϕ to $\phi + \epsilon.\Delta(\phi)$ that

$$Z(M,C) =$$

$$\int exp(iS_0(\phi)+i\int L(\phi)(x)^T M(x)d^4x-(1/2)\int L(\phi)(x)^T C(x,y)L(\phi)(y)d^4xd^4y)$$

$$\times(1+i\epsilon\int(L'(\phi).\Delta(\phi))(x)^T M(x)d^4x-\epsilon\int(L'(\phi).\Delta(\phi))(x)^T C(x,y)L(\phi)(y)d^4xd^4y)D\phi$$

(where

$$L'(\phi).\Delta(\phi)(x)=\int(\delta L(\phi)(x)/\delta\phi(y)).\Delta(\phi)(y)d^4y),$$

which can alternatively be expressed as

$$\int < (L'(\phi).\Delta(\phi))(x) >^T_{M,C} M(x)d^4x$$

$$= -i\int Tr(C(x,y) < L(\phi)(y).(L'(\phi).\Delta(\phi))(x)^T >_{M,C} d^4xd^4y$$

Substituting for $M(x)$ from the equation of motion rewritten below

$$\delta\Gamma(\phi_0,C)/\delta\phi_0(x)=-\int(\delta L(\phi_0(y))/\delta\phi_0(x))^T M_0(y)d^4y$$

this equation of broken gauge invariance of the quantum effective action can be expressed in the form

$$\int(\delta\Gamma(\phi_0,C)/\delta\phi_0(x)). < \Delta(\phi)(x) >_{M_0,C}$$

$$= -\int(< (L'(\phi)-L'(\phi_0)).\Delta(\phi)(x) >^T_{M_0,C} M_0(x)d^4x$$

$$-i\int Tr(C(x,y) < L(\phi)(y).(L'(\phi).\Delta(\phi))(x)^T >_{M_0,C} d^4xd^4y$$

7 Conclusions

We have in this paper, addressed a few basic questions in quantum cosmology, The first question is related to how much commutation relations between a quantum field and its velocity get altered by the presence of a background gravitational field. The second is related to computing the Feynman path integral in quantum cosmology for the scale factor of our universe in which there are scalar Klein-Gordon particles. We then derive some formulae regarding the joint wave function of the scale factor and the scalar field, first in a situation when the scalar field is homogeneous and isotropic like the gravitational field and then more generally, when the gravitational field is homogeneous and isotropic but the scalar field can be arbitrary functions of space and time. The third question addressed here is related to the computation of the quantum effective action of the scalar field in the presence of a homogeneous and isotropic background gravitational field when the scalar field is coupled to a current source that can have classical randomness. The final question concerns symmetry-breaking terms in an otherwise gauge invariant quantum effective action induced by classical randomness in the current field. We are able to provide a formula for the change in the quantum effective action of the field under a quantum gauge transformation in terms of the covariance matrix field of the random Gaussian current source. This formula can provide us with a clue about how much mass can the field acquire by virtue of random fluctuations in the coupling current field. This result can be applied to cosmology by noting that the KG scalar field action in curved space-time contains the scale factor $S(t)$ of the expanding universe as a parameter and if this parameter undergoes small random fluctuations, then it can be regarded as a current source whose randomness generates extra mass in the KG scalar field particles.

8 References

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9 Appendix 1. Some remarks on the Wheeler-De-Witt equation in cosmology

The ADM Hamiltonian for general relativity (Thiemann [3]) has the form

$$\mathbf{H} = \int (N.H + N^a H_a) d^3x$$

where H_a is the diffeomorpniam constraint and H is the Hamiltonian constraint. $N, N^a, a = 1, 2, 3$ are position fields apart from the six position fields $q_{ab}, 1 \leq a \leq b \leq 3$ that form the spatial components of the metric tensor of the one-parameter family of 3-D surfaces that are embedded into four-dimensional space-time. Note that the constraint equations are $H_a = 0, H = 0$ with the latter being interpreted in the quantum theory as a Schrodinger equation

$$H\psi(q) = 0, q = ((q_{ab}))$$

It is readily verified [Thiemann] that H is the sum of a quadratic form in the momenta P^{ab} with coefficients being nonlinear functions of q plus a nonlinear function of q. This Hamiltonian thus has the form

$$H(Q,P) = P^T F(Q)P + V(Q)$$

where we write P for the six component of $((P^{ab}))$ and Q for the six components of $q = ((q_{ab}))$. Note that the beauty of the ADM action implies that linear terms in P do not appear in the Hamiltonian constraint. The equation

$$H(Q, -i\delta/\delta Q)\psi(Q) = 0$$

is called the Wheeler-De-Witt equation or equivalently, the Schrodinger equation of general relativity. After spatially discretizing the 3-D space on which the position and momentum fields are defined, we can interpret Q, P as $N \times 1$ vector operators where $N = 6M$ with M being the number of spatial pixels. Actually, in loop quantum gravity, the position fields are defined using the Ashtekar connection $A = \Gamma(E) + iK$ with the momentum fields E being the electric flux fields. $\Gamma(E)$ is a homogeneous functional of E of zeroth order. In loop quantum gravity as described in the book by Thiemann, space is discretized by the edges of a graph and since the connection Γ is an $SO(3)$ spin connection, $A = (A_a^i))$ can be viewed as an $SO(3)$-gauge field or equivalently as an $SU(2)$ gauge field, i.e., $A_a = A_a^i \tau_i$ where τ_1, τ_2, τ_3 are the Pauli spin matrices. Rather than using A_a as the position fields, in loop quantum gravity, we parallelly displace the connection A_a along the different edges of the graph giving for each edge e, an $SU(2)$-group matrix $g(e)$, i.e., $g(e) = exp(i\int_e A_a^i(x(s))\tau_i dx^a(s)) \in SU(2)$ with the edge being parametrized as $(x^a(s)), 0 \leq s \leq 1$. The wave function of the universe is then regarded as a function of all the $g(e)'s$ and can be represented using a basis comprising of the products over all the edges e of matrix elements of the irreducible representations of $SU(2)$ evaluated at the $g(e)'s$. Such a basis is called a basis of spin network functions.

In the case of restricted quantum gravity, our position variables are very few in number, call these θ with the position vector Q being a function of these, i.e., $Q(\theta)$. Assume that Q has N components while θ has $p < N$ components. Then, our wave function ψ must be regarded as a function of the θ. The problem then becomes how to approximate the N momentum components P as linear combinations of $-i\partial/\partial\theta$. To do so, we define the matrix

$$B(\theta) = Q'(\theta) = ((\partial Q_i/\partial\theta_j)) \in \mathbb{R}^{N\times p}$$

We write $P_\theta = -i\partial/\partial\theta$ so that by the chain rule

$$\partial/\partial\theta_i = (\partial Q_j/\partial\theta_i)\partial/\partial Q_j$$

summation over j being implied. In matrix-vector notation, this is the same as

$$P_\theta = B(\theta)^T P$$

We must now find an $N \times p$ matrix $C(\theta)$ so that $C(\theta)B(\theta)^T \approx I_N$, i.e., C must be chosen so that

$$\| CB^T - I_N \|^2$$

is a minimum where the matrix norm used is the Frobenius norm.

10 Appendix 2. Some techniques of constructing operators (Heisenberg observables) in quantum mechanics from the Feynman path integral

Let $L(q(t), q'(t), t)$ be the Lagrangian of a particle moving in one dimension. The Schrodinger evolution kernel in position space between times 0 and T is given by

$$K_T(q(T)|q(0)) = C. \int exp(i \int_0^T L(q(t), q'(t), t)dt) Dq(0, T)$$

where

$$Dq(0, T) = \Pi_{0<t<T} dq(t)$$

ie, the path integral excludes $q(0)$ and $q(T)$. If $\psi_0(q)$ is the initial wave function, then at time T, the wave function is

$$\psi_T(q(T)) = \int K_T(q(T)|q(0))\psi_0(q(0))dq(0)$$

We denote by $U(t_2|t_1)$ the Schrodinger unitary evolution between times t_1 and t_2. Thus,

$$|\psi_T >= U(T|0)|\psi_0 >$$

in position space, reads

$$\psi_T(q) =< q|\psi_T >=< q|U(T|0)|\psi_0 >= \int K_T(q|q(0))\psi_0(q(0))dq(0)$$

The momentum conjugate to the canonical position $q(t)$ is

$$p(t) = \partial L(q(t), q'(t), t)/\partial q'(t)$$

if O_t is a Heisenberg observable at time t, then in most situations, it can be expressed as a function of $(q(t), p(t))$, i.e. $O(q(t), p(t))$ or in view of the above definition of the momentum, as $O_t = O(q(t), q'(t))$. The average value of O_t is given by

$$< \psi_0|O_t|\psi_0 >$$

or equivalently, in terms of the unitary evolution operator $U_t = U(t|0)$, as

$$< \psi_0|O(q(t), q'(t))|\psi_0 >=< \psi_0|U_t^* O(q(0), q'(0))U_t|\psi_0 >$$

$$=< \psi_t|O(q(0), q'(0))|\psi_t >$$

which expresses precisely the duality between Heisenberg's matrix mechanics in which the observables evolve but states are fixed and Schrodinger's wave mechanics in which observables are fixed in time but states evolve. (Note that the Heisenberg evolution of observables is given by $q(t) = U_t^* q(0)U_t$ and $q'(t) = U_t^* q'(0)U_t$, so by unitarity of $U_t, O(q(t), q'(t)) = U_t^* O(q(0), q'(0))U_t$). Equivalently, in terms of the path integral,

$$< \psi_0|O(q(t), q'(t))|\psi_0 >=< \psi_0|U_t^* O(q(0), q'(0))U_t|\psi_0 >$$

$$=< \psi_0|U_T^* U_T U_t^* O(q(0), q'(0))U_t|\psi_0 >$$

$$=< \psi_T|U(T|t)O(q(0), q'(0))U(t|0)|\psi_0 >$$

$$= \int \bar{\psi}_T(q(T))dq(T)$$

$$\int exp(i \int_t^T L(q(s), q'(s), s)ds).O(q(t), q'(t))$$

$$.exp(i \int_0^t L(q(s), q'(s), s)ds)Dq(0, T)\psi_0(q(0))dq(0)$$

$$= \int \bar{\psi}_T(q(T)).\psi_0(q(0)).O(q(t), q'(t)).exp(i \int_0^T L(q(t), q'(t), t)dt)Dq[0, T]$$

The action of the operator O_t on the wave function $\psi_0(q)$ on the other hand can be represented by a kernel $O_t(q'', q')$:

$$O_t\psi_0(q'') = \int O_t(q'', q')\psi_0(q')dq'$$

If ϕ_0 is another initial wave function which evolves to ϕ_T in time T, then the matrix element of O_t w.r.t the two states ψ_0, ϕ_0 is along the same lines, given by

$$< \phi_0|O_t|\psi_0 >$$

$$= \int \bar{\phi}_T(q(T)).\psi_0(q(0))O(q(t), q'(t))exp(iS_T(q))Dq[0, T]$$

This immediately gives us the formula for the kernel of O_t in terms of the path integral as

$$\int \phi_0(q'')dq'' O_t(q'', q(0)) = \int \phi_T(q(T))O(q(t), q'(t))exp(iS_T(q))Dq(0, T]$$

$$= \int \phi_0(q'')dq'' K_T(q(T)|q'')O(q(t), q'(t))exp(iS_T(q))Dq(0, T]$$

or equivalently,

$$O_t(q'',q(0)) = \int K_T(q(T)|q'')O(q(t),q'(t))exp(iS_T(q))Dq(0,T], t \in [0,T]$$

Note that we have used the notation

$$S_T(q) = \int_0^T L(q(t),q'(t),t)dt$$

Using the evolution composition property

$$U(t_2|t_1)U(t_1|t_0) = U(t_2|t_0), t_2 > t_1 > t_0$$

in the kernel form

$$\int K(T,q(T)|t,q(t))dq(t).K(t,q(t)|0,q(0)) = K(T,q(T)|0,q(0))$$

as can also be derived readily from the basic property of the path integral, we can equivalently write

$$O_t(q'',q(0))$$

$$= \int K(T,q(T)|0,q'')O(q(t),q'(t))exp(iS_{0,t-0}(q))exp(iS_{t+0,T}(q))Dq(0,T]$$

$$= \int K(T,q(T),|0,q'')O(q(t),q'(t))K(t-0,q(t-0)|0,q(0))K(T,q(T)|t+0,q(t+0))$$

$$Dq[t-0,t+0]dq(T)$$

$$= \int K(t+0,q(t+0)|0,q'').O(q(t),q'(t)).K(t-0,q(t-0)|0,q(0))Dq[t-0,t+0]$$

This is the path integral version of the Heisenberg matrix mechanics formula for the evolution of observables:

$$O_t = U(t|0)^* O_0 U(t|0)$$

Note that $q'(t)$ is a function of $(q(t-0),q(t),q(t+0))$ which is why we required a partition of the above form. Note that the above path integral formula for the kernel of a Heisenberg observable, more precisely, can be expressed as

$$O_t(q'',q')$$

$$= lim_{\delta\to 0} \int K(t+\delta,q(t+\delta)|0,q').O(q(t),q'(t)).K(t-\delta,q(t-\delta)|0,q')Dq[t-\delta,t+\delta]$$

11 Appendix 3, A remark on how equal-time commutation relations of a Bosonic quantum field get affected when it interacts with another quantum field

Let $L_0(\phi(x), \partial_\mu \phi(x))$ denote the unperturbed Lagrangian density of a quantum field $\phi(x)$. Taking $\phi(x)$ as our canonical position field, the corresponding conjugate momentum field is given by

$$P_0(x) = \partial L_0 / \partial \partial_0 \phi(x)$$

The equal-time commutation relations are then given by

$$[\phi(t,r), P_0(t,r')] = i\delta^3(r - r')$$

Now suppose that the field ϕ interacts with another quantum field $\chi(x)$ in accordance with the interaction Lagrangian density $L_1(\phi(x), \chi(x), \partial_\mu \phi(x), \partial_\mu \chi(x))$. Here, we are also assuming that the self Lagrangian of $\chi(x)$ is also present as a component in L_1. Then, the canonical momentum conjugate to the position field $\phi(x)$ is given by

$$P_\phi(x) = \partial L_0 / \partial \partial_0 \phi(x) + \partial L_1 / \partial \partial_0 \phi(x)$$

$$= P_0(x) + \delta P(x)$$

say where P_0 is the first term and δP is the second term on the rhs. We thus obtain the revised equal-time commutation relationship

$$[\phi(t,r), P_\phi(t,r')] = [\phi(t,r), P_0(t,r') + \delta P(t,r')] = i\delta^3(r - r')$$

or equivalently,

$$[\phi(t,r), P_0(t,r')] = i\delta^3(r - r') - [\phi(t,r), \delta P(t,r')]$$

In order to evaluate the last commutator, we note that in the expression

$$\delta P(t,r) = \partial L_1 / \partial \partial_0 \phi(t,r)$$

the rhs is a function of $\phi, \chi, \nabla\phi, \nabla\chi, \partial_0\phi and \partial_0\chi$, all evaluated at (t,r). We also note that the spatial gradients $\nabla\phi, \nabla\chi$ are functions of the position field at the same time and hence their equal-time commutation relations with the position fields vanish. In order to evaluate $[, \phi(t,r), \delta P(t,r')]$, we therefore have to express the canonical velocity fields $\partial_0\phi, \partial_0\chi$ in terms of the canonical position and momentum fields, all at the same time and then make use of the canonical commutation relations between the position and momentum fields at the same time. We proceed to do this analysis below:

Remark: A typical example of this situation is present in the main body of this paper where $\phi(x)$ is represented by $S(t)$ the scale factor of the universe and $\chi(x)$ is the KG field.

Now we observe that $\chi(x)$ is the second position field and as such we have the canonical equal-time commutation relation

$$[\phi(t,r),\chi(t,r')]=0$$

and of course, also

$$[\phi(t,r),\phi(t,r')]=0$$

and

$$[\phi(t,r),P_\chi(t,r)]=0,[P_\phi(t,r),P_\chi(t,r)]=0,$$
$$[\chi(t,r),P_\chi(t,r')]=i\delta^3(r-r')$$

Suppose that we are able to solve the defining relations for P_ϕ, P_χ to express $\partial_0\phi(t,r), \partial_0\chi(t,r)$ in terms of $P_\phi(t,r), P_\chi(t,r), \nabla\phi(t,r), \nabla\chi(t,r)$. We express these solutions as

$$\partial_0\phi(t,r)=F_1(P_\phi(t,r),P_\chi(t,r),\nabla\phi(t,r),\nabla\chi(t,r)),$$

$$\partial_0\chi(t,r)=F_2(P_\phi(t,r),P_\chi(t,r),\nabla\phi(t,r),\nabla\chi(t,r)),$$

Then, we easily find using standard properties of the commutator the following equal-time commutation relations between the canonical position and canonical velocity fields:

$$[\phi(t,r),\partial_0\phi(t,r')]=i\frac{\partial F_1(P_\phi(t,r'),P_\chi(t,r'),\nabla\phi(t,r'),\nabla\chi(t,r'))}{\partial P_\phi(t,r')}\delta^3(r-r')$$

and likewise for the field χ. This equation is fundamental because it tells us how much the interaction of a quantum field with other quantum fields will affect the equal-time commutation relations between the field and its velocity. We can go a step further by assuming that the interaction Lagrangian L_1 is weak. To this end, for the sake of illustration, we shall assume that χ is a background classical field and hence, we can express the perturbed Lagrangian of ϕ as

$$L_0(\phi(x),\phi_{,\mu}(x))+L_1(x,\phi(x),\phi_{,\mu}(x))$$

The explicit dependence on x of the second term here arises owing to the presence of the classical field $\chi(x)$. Assume that in the absence of the perturbing term L_1, the position field is $\phi^0(x)$ and the corresponding momentum field is $P_0(x)$. Thus,

$$P_0(x)=\partial L_0(\phi^0(x),\phi^0(x)_{,\mu})/\partial\phi_{,0}(x)$$

Solving this gives

$$\phi^0_{,0}(x)=F_0(P_0(x),\phi^0(x),\phi^0_{,r}(x))$$

where by $\phi^0_{,r}$ we mean the components of the spatial gradient of ϕ^0 or in other words, that r varies over the spatial indices $1,2,3$ in contrast to μ which varies over the space-time indices $0, 1, 2, 3$. Then the perturbed momentum is given up to the first order of smallness by

$$P_0 + \delta P = \partial L_0(\phi, \phi_{,\mu})/\partial \phi_{,0}$$
$$+\partial L_1(\phi^0, \phi^0_{,\mu})/\partial \phi_{,0}$$

solving which, we get

$$\phi^0_{,0} + \delta\phi_{,0} = \phi_{,0} = F_0(P_0 + \delta P - F_1(\phi^0, \phi^0_{,\mu}), \phi, \phi_{,\mu})$$
$$= F_0(P_0 + \delta P - F_1(\phi^0, \phi^0_{,\mu}), \phi^0 + \delta\phi, \phi^0_{,\mu} + \delta\phi_{,\mu})$$

where

$$F_1(\phi^0, \phi^0_{,\mu}) = \partial L_1(\phi^0, \phi^0_{,\mu})/\partial \phi_{,0}$$

Equivalently, up to first-order terms,

$$\delta\phi_{,0} = (\partial F_0(P_0, \phi^0, \phi^0_{,\mu})/\partial P)(\delta P - F_1(\phi^0, \phi^0_{,\mu}))$$
$$+(\partial F_0(P_0, \phi^0, \phi^0_{,\mu})/\partial \phi)\delta\phi$$
$$+(\partial F_0(P_0, \phi^0, \phi^0_{,\mu})/\partial \phi_{,\mu}))\delta\phi_{,\mu}$$

Our aim is to calculate the equal-time commutator ($x^0 = y^0$)

$$[\phi(x), \phi_{,0}(y)] = [\phi^0(x) + \delta\phi(x), \phi^0_{,0}(y) + \delta\phi_{,0}(y)]$$
$$= [\phi^0(x), \phi^0_{,0}(y)] + [\delta\phi(x), \phi^0_{,0}(y)] + [\phi^0(x), \delta\phi_{,0}(y)]$$

up to the first order of smallness. This is equivalent to computing the perturbation in the equal-time commutator of the position and velocity fields:

$$\delta[\phi(x), \phi_{,0}(y)] =$$
$$[\phi(x), \phi_{,0}(y)] - [\phi^0(x), \phi^0_{,0}(y)] =$$
$$[\delta\phi(x), \phi^0_{,0}(y)] + [\phi^0(x), \delta\phi_{,0}(y)]$$

Note that we already have available with us the corresponding unperturbed commutator between position and velocity fields at equal times:

$$[\phi^0(x), \phi^0_{,0}(y] =) = [\phi^0(x), F_0(P_0(x), \phi^0(x), \phi^0_{,r}(x))]$$
$$= i\delta^3(x - y)\partial F_0(P_0(y), \phi^0(y), \phi^0_{,r}(y))/\partial P_0(y)$$
$$= i\delta^3(x - y)\partial F_0(P_0(x), \phi^0(x), \phi^0_{,r}(x))/\partial P_0(x)$$

Actually, this result should be dependent upon the orders in which the various operators appear in the function F_0. We therefore agree to the convention that in F_0, $P_0(x)$ appears to the left of the commuting fields $\phi^0, \phi^0_{,r}$. If not, we can always use the commutation relation

$$[\phi^0(x), P_0(y)] = i\delta^3(x - y)$$

to arrange these matters after discarding infinite constants, or equivalently, replacing infinite constants by large numbers. This problem will not be so severe if we discretize space into pixels of finite size, and replace Dirac δ-functions with Kronecker *delta* functions divided by the volume of each spatial pixel.

12 Conclusions

This paper presents some novel aspects of quantum gravity with applications to the idea of theoretical variables, i.e., accessible and inaccessible variables proposed in earlier works by Professor Inge Helland. We demonstrate by taking the example of the gravitational field in a homogeneous and isotropic space-time interacting with a KG field, how the commutation relations of the field vary as the scale factor of the universe expands which means from the measurement standpoint in quantum mechanics, that the degree of uncertainty between two observables or equivalently, the degree of simultaneous non-measurability of two observables varies as the universe expands. In the beginning, we also illustrate this phenomenon using the quantum electromagnetic field interacting with the classical gravitational field. We generalize these results from a spatially homogeneous KG field to a spatially inhomogeneous KG field and then to the anticommutator of the Dirac field in curved space-time. We then explain how to calculate the wave function of one field interacting with another field using the path integral over the second field. Actually, we should instead be talking about TPCP maps corresponding to one field when it interacts with the other. We then explain how to compute the quantum effective action of the scale factor of the universe when it interacts with the quantum KG field present in the form of particles distributed within our universe when in addition, the KG field interacts with a random current field. As an example of such a phenomenon, we consider the metric of space-time depending on random classical parameters having small variances, so that the formula for the KG field in a background curved space-time yields an interaction component between the KG field and the classical metric fluctuations with the interaction being quadratic in the KG field and linear in the parameter fluctuations. We then explain how symmetry breaking can occur in the quantum effective action of a field that interacts with a classical random current field, when the classical action has a gauge symmetry. It is a well-known result (Steven Weinberg, The quantum theory of fields, vol.2)that when the gauge symmetry is linear in the field and the current source is non-random, then the quantum effective action has the same gauge symmetry as the classical action but when the gauge symmetry is nonlinear in the field, the corresponding gauge symmetry of the quantum effective action is not the same as the classical symmetry, rather, it is given by the quantum expectation value of the infinitesimal gauge symmetry of the classical action taken when the current source equals a value at which the quantum field has the same quantum expectation as the classical field. However, when a current source is random, event this gauge symmetry gets broken and we derive a formula for the change in the quantum effective action under the quantum infinitesimal gauge symmetry defined by the quantum expectation of the classical gauge symmetry in the presence of a non-random current field that yields the quantum expectation value of the field equal to the classical field. This formula for the change in the quantum effective action, namely, the degree by which gauge symmetry is broken is expressed in terms of the statistical correlations of the random current source. In the last section of this paper, we derive some formulas for the change in the field propagator of a field in the presence of a random current source and from this change in the propagator, it is possible to infer how much mass the field gains by its interaction with both another quantum field as well as with a classical random current source field. This calculation could provide a clue to the mystery of how particles acquire masses in our universe by saying that masses are acquired by particles via symmetry breaking caused by randomly distributed current fields in the form of cosmic microwave background radiation and perhaps also other forms of radiation. We then consider the Wheeler-De-Witt equation for the wave function of the metric field in general relativity and observe that the associated Hamiltonian is constrained. One usually treats such constrained problems using Dirac brackets in place of Poisson and Lie brackets. Here, we suggest an approximate method for quantizing such fields based on expressing the constrained position fields in terms of a smaller number of "parameter fields" which form our revised set of position fields and then construct the revised momentum fields using a least squares method for approximately inverting the associated Jacobian matrix of the position field w.r.t the smaller set of parameter fields.

Acknowledgements: I wish to thank Professor Inge Helland for introducing me to his theory of accessible and inaccessible variables as a new way of looking at the foundations of quantum theory.