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Product of Distributions Applied to Discrete Differential Geometry

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Abstract

We propose a formula for evaluating the product of step discontinuous and delta functions. Using tensor calculus and the above proposed formula, we evaluate of the total curvature of a polyhedron vertex where curvature is infinite and total curvature is finite and therefore the Gaussian curvature can be represented by a Dirac delta function.

From the above calculation we find the well known deficiency angle formula which gives the discrete curvature of a polyhedron vertex and therefore we find an analytic proof of the known results that the Gauss-Bonnet theorem for smooth surfaces and the Descartes deficiency angle theorem for polyhedron, are the same thing.

1 Introduction

Products of distributions are quite common in several fields of both mathematics and physics. Examples arise naturally in quantum field theory, gravitation and in partial differential equations (e.g shock wave solutions in hydrodynamics) see [1]. An important issue, related to product of distributions, is the fact that the product, in the general case, is not well defined in D'. This issue is known as the Schwartz impossibility result (see [1] §1.3). In the Schwartz classical theory, only the product between a smooth function and a distribution is well defined. Historically, products of distributions are addressed by means of algebras of generalised functions developed initially by J. F. Colombeau (see [1] and [2]).

Discrete differential geometry is a rather new field of mathematics which borrows concepts and ideas from both differential geometry and discrete mathematics. Main applications are concerned with the discrete version of several classical concepts of differential geometry such as discrete curvature, minimal surfaces, geodesics coordinates, minimal paths, surfaces of constant curvature, curvature line parametrisation and the discrete version of continuous functionals (see [3]). At the moment, discrete differential geometry uses many tools of discrete mathematics while the classical tools of differential geometry (e.g. tensors and coordinate free exterior calculus) are difficult to be applied. This leads to an ambiguous definition of the various operators (see [4]) which are instead well defined in the continuous counterpart of the theory.

In this paper, we propose a method for evaluating the product of step discontinuous functions and Dirac delta functions, related each other by an integrable function. Moreover, the method is applied to a special class of non differentiable manifolds for which, the classical idea of curvature, together with all tools of differential geometry, needs to be redefined in terms of distribution functions. In particular, the class of manifolds analysed is the one composed of a collection of several Riemannian manifolds glued in such a way the final surface is not differentiable on the resulting edges and vertices. In this case, it is possible to show that vertices and edges carry a concentrated discrete curvature which gives a contribution to the total curvature of the surface, contribution that has to be taken into account in order for the Gauss-Bonnet theorem to work.

For vertices, an important result was already known since the time of Descartes which proved, in the first half of the 17th century, its deficency angle theorem for polyhedra. That idea, using the modern concept of curvature and applied to the class of surfaces defined above, can be stated

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by saying that the discrete total curvature of a vertex is equal to 2π minus the sum of the angles between edges. The concept of discrete curvature can be also easily generalised to edges.

These surfaces with discrete curvature an vertices and edges, characterised by a step discontinuous metric, are typical of problems in many fields, where the usual way to proceed is to brake down the problem along edges and to define boundary conditions (with conserved quantities) in order to keep the whole problem definition consistent (see [5]) or to use methods of discrete mathematics to define the relevant operators (see [4]). The approach proposed in this paper is to use a more direct method derived from the classical differential geometry.

In Paragraphs 2 and 3, we derive a method for evaluating products of step discontinuous and Dirac delta functions).

In Paragraphs 4 and 5, we use the product of step discontinuous and Dirac delta functions, mentioned above, to evaluate the discrete curvature of a polyhedron vertex. In order to do that, we define the step discontinuous metric of polyhedron vertices and we evaluate their Riemann tensors by applying the classical rules of the differential geometry but taking the derivatives in D'. By using this approach, the final result is, as expected, the deficiency angle formula for the total curvature of a polyhedron vertex.

2 Product of steps and delta functions

Proposition 0. *Let $u(x)$ be the Heaviside function, $\delta(x)$ its derivative and $f(x)$ a function which is locally integrable in $A \supseteq [0,1]$. Given the above, it follows that $f(u(x))$ is a step discontinuous function in 0 and:*

$$f(u(x))\delta(x) = \left(\int_0^1 f(x)dx\right)\delta(x) \tag{1}$$

Important Remark: Since product of distribution are not well defined in the Schwartz theory of distributions, which is the most commonly used, in order for the above proposition to make sense we need to clarify what we mean with the product $f(u(x))\delta(x)$. There are two possible approaches:

Approach 1: Given any smooth function g and the relevant sequence of functions $g_n = g(nx)$ with $n \in \mathbb{N}$ such that $\lim_{n\to\infty} g_n(x) = u(x)$ where $u(x)$ is the Heaviside step function, then clearly we have $\lim_{n\to\infty} g_n' = \delta(x)$ and we can define the product $f(u(x))\delta(x)$ to be the following limit:

$$f(u(x))\delta(x) = \lim_{n\to\infty} f(g_n(x))g_n'(x) \tag{2}$$

where all limits above are intended to make sense in the framework of the Schwartz theory of distributions.

Approach 2: A common way to handle products of distributions is the Colombeau Algebras of generalised functions. There are several ways to define these algebras, however the proposition above will work with all of them. Given a Colombeau Algebra of generalised functions $\mathcal{G}$, then $u(x) \in \mathcal{G}$, $\delta(x) \in \mathcal{G}$ and the product $f(u(x))\delta(x)$ is a well defined element of that algebra. In this case the $=$ symbol in Eq. (1) has to be intended as an association relation usually denoted by $\approx$ in the Colombeau theory.

In this paper we will follow approach 1 to make the paper accessible also to the readers which are not familiar with Colombeau Algebras although approach 2 would not be much more difficult.

We give below two non formal proofs of the above proposition to make the reader confident of the fact that it is true in order to jump immediately to the geometric part of the paper. A more formal proof of the proposition is given in the Appendix. Although we said that we will follow the first approach, the first prove below is according to the second approach just because of its extreme simplicity.

Proof 1. Let F a function such that $F'(x) = f(x)$. Then we have that $F(u(x)$ it's a step discontinuous function which has a jump of $F(u(0^+)) - (u(0^-)) = F(1) - F(0)$ in 0 and therefore its derivative is a Dirac delta function of amplitude equal to that jump:

$$\frac{d}{dx}F(u(x)) = f(u(x))\delta(x) = (F(1) - F(0))\delta(x) = \left(\int_0^1 f(x)dx\right)\delta(x) \tag{3}$$

where the derivative above, using the Leibniz Rule, is legal because we are in a Colombeau algebras (i.e. a derivative algebra). □

Note that $F(u(x))$ and $(F(1) - F(0))u(x) - F(0)$ are the same function in D' but they are separate elements of the Colombeau Agebra because they differ by a null function. However, they can be related by an association relation and this is the reason why we need to substitute the symbol $=$ with the symbol $\approx$ in Eq. (1).

Proof 2. Let $h(x)$ be a continuous function which image is contained in A, $\lim_{x\to-\infty} h(x) = 0$ and $\lim_{x\to\infty} h(x) = 1$. It follows that $\lim_{x\to-\infty} h'(x) = 0$, $\lim_{x\to\infty} h'(x) = 0$.

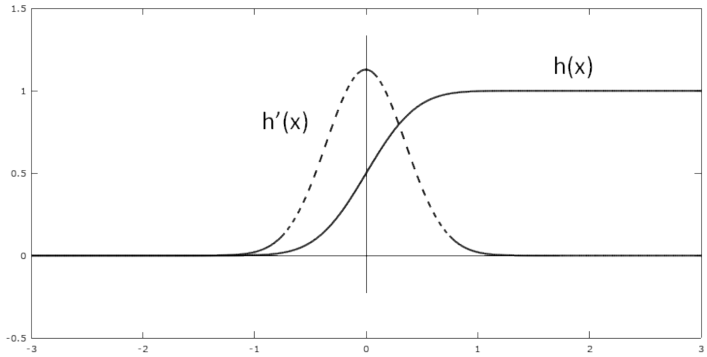

Figure 1: Function h(x)

and:

$$\begin{aligned}\int_{-\infty}^{+\infty} f(h(x))\,h'(x)dx &= \int_{-\infty}^{+\infty} F'(h(x))dx \\ &= F(h(\infty)) - F(h(-\infty)) \\ &= F(1) - F(0) = \int_0^1 f(x)dx\end{aligned} \tag{4}$$

where $F(x)$ is the primitive of $f(x)$ and the value of the integral (4) is independent from $h(x)$. This is the key point of the proof!

If $h(x)$ goes continuously to $u(x)$, then we have that the integrand of the right side of the (4) goes to $f(u(x))\delta(x)$ which is a product of a step and a delta function. This product is a delta function itself since it vanishes everywhere apart from the point $x = 0$ where it is infinite and its integrand has finite value. This product converges therefore to a delta function $\alpha\delta(x)$, where the amplitude α of the delta is given by the the right side of the (4) which, regardless the the shape of h, will be always equal to:

$$\alpha = \int_0^1 f(x)dx \tag{5}$$

□

Proposition 1. *Let $g(x)$ be a function defined as follows:*

$$g(x) = \begin{cases} a & for\, x < 0 \\ b & for\, x > 0 \end{cases} \tag{6}$$

Also let $(b - a)\delta(x)$ be the derivative of $g(x)$. Then:

$$f(g(x))\delta(x) = \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\delta(x) \tag{7}$$

with $a, b \in \mathbb{R}$, and $f(x)$ any locally integrable function in $A \supseteq [a, b]$ (or $[b, a]$ if $b < a$).

Proof. To prove the (7) we can proceed exactly on the same steps of the proof for (1) or we can notice that any function $g(x)$ can be related to the Heaviside function by means an an auxiliary function $t = \chi(x) = (b-a)x+k$ such that $g(x) = \chi(u(x))$. For example the function $sign_(x)$ (which is -1 for $x < 0$ and 1 for $x > 0$) can be written as:

$$sign(x) = 2u(x) - 1 \text{ with } \chi = 2x - 1 \tag{8}$$

We have $t = (b-a)x + k$, $dt = (b-a)dx$ and by a simple change of variable we have:

$$f(\chi(u(x)))\delta(x) = \int_0^1 f(\chi(x))dx = \left(\frac{1}{b-a}\int_a^b f(t)dt\right)\delta(x) \tag{9}$$

which is the (7). □

A more formal prove can be found in Appendix A.2. The above result is already present in the literature (compare with [6]). With respect of the theory developed in [6], in this paper, we have derived the (7) by means of a completely different approach, we require f to be integrable (in [6] f is required to be continuous) and we have generalised the equation to the multidimensional case (see next paragraph).

Note that, even in the case where $f(a) = f(b)$ and therefore there is no step in the discontinuity, proposition 1 is essential to evaluate the product of the discontinuity with a related delta function. For example, is easy to show that $sign^2(x)\delta(x) = \frac{1}{3}\delta(x)$.

We finish this paragraph with a general remark on product of distributions and the way they are addressed in this paper. Every time we define the product in a point x_0, where the distributions are discontinuous, we always want the discontinuities to have each other structure related by a well known law (in this case, one to be the derivatives of the other) so that, if the structure of one distribution in x_0, which is unknown to us, changes, the structure of all other distributions in the same point will change accordingly.

3 The multidimensional case

Proposition 2. *Let $g_1(x)$ and $g_2(y)$ be two functions defined as follows:*

$$g_1(x) = \begin{cases} a & for\ x < 0 \\ b & for\ x > 0 \end{cases} \tag{10}$$

$$g_2(y) = \begin{cases} c & for\ y < 0 \\ d & for\ y > 0 \end{cases} \tag{11}$$

with $a, b, c, d \in \mathbb{R}$ and let $f(x, y)$ be any function locally integrable in $A \supseteq [a, b] \times [c, d]$ (if $b < a$ and/or $d < c$ the definition of A has to be changed accordingly). Also let $(b-a)(d-c)\delta(x, y)$ be the product of the derivatives of $g_1(x)$ and $g_2(y)$. Then:

$$f(g_1(x), g_2(y))\delta(x, y) = \left(\frac{1}{(b-a)(d-c)}\int_c^d dy \int_a^b f(x, y)dx\right)\delta(x, y) \tag{12}$$

This proposition can be proved by following the same steps of Proposition 1 and by using functions h_1 and h_2 having the same property of the function h defined in the paragraph above. For the proof we also need the following identity:

$$\int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} f(h_1(x), h_2(y))h_1'(x)h_2'(y)dx$$

$$= \int_{-\infty}^{+\infty} dy \frac{\partial}{\partial y} \int_{-\infty}^{+\infty} \frac{\partial}{\partial x} F(h_1(x), h_2(y))dx$$

$$= F(1,1) - F(0,1) - F(1,0) + F(0,0) \tag{13}$$

where $F(x,y)$ is a function such that $F_{xy} = F_{yx} = f(x,y)$ and the result is independent from the functions h_1 and h_2. A more formal prove of Proposition 2 is sketched in Appendix A.3.

Obviously, we can interchange the roles of x and y since we may integrate first with respect of y and then with respect of x. Note that the discontinuity $f(g_1(x), g_2(y))$ addressed by this proposition is not the most general step discontinuity we may have in two dimensions.

Note that proposition 2 gives a clear path on the possible way to generalise the idea of products of step discontinuities and delta functions to the case with as many dimensions as we like.

As a final remark, the fact that proposition 1 and 2 are valid for f locally integrable is an important feature. An example, where we use this feature, is given in paragraph 4 and 5 of this paper.

4 Metrics for a polyhedron vertex

The product of step and delta functions, developed in paragraphs 2 and 3, may be applied to a number of fields of both physics and mathematics where the product of step discontinuity and Dirac delta function arise naturally from the theory. Among all, we have decided to focus our attention to applications related to differential geometry and, in particular, to the evaluation of the curvature for those manifolds, described in the introduction, having step discontinuous metric.

As mentioned in the introduction, this kind of variety may have discrete curvature concentrated on edges and vertices. In both cases, Christoffel symbols, Riemann and Ricci tensors, curvature as well as a number of different differential operators, may only be expressed by means of product of step and delta functions. In this case, the relationship between the structures of the step discontinuities and the delta functions codify the geometrical aspects of the non-differentiable point of the surface and proposition 1 (for edges) and proposition 2 (for vertices) turn up to be very useful in finding an expression for the differential quantity of interest

As an example, in this paragraph we will show a convenient and standard way to define a step discontinuous metric for vertices of polyhedra with 3 or 4 concurrent edges, which are very common in many applications, and in paragraph 5 we will show how to use these metrics to evaluate the curvature of that polyhedron in the vertices. Even thought this paragraph is focused on curvatures, the same method can be applied to evaluate any kind of differential parameters and operators (e.g. Laplace-Beltrami operators).

Before we proceed, we need to introduce a definition. For the purpose of this paper, we will call a 2d-step function any function defined as follows:

$$s(x^1, x^2) = \begin{cases} r_1 & for\, x^1 > 0, x^2 > 0 \\ r_2 & for\, x^1 < 0, x^2 > 0 \\ r_3 & for\, x^1 < 0, x^2 < 0 \\ r_4 & for\, x^1 > 0, x^2 < 0 \end{cases} \tag{14}$$

where $r_i \in \mathbb{R}$ and $s(x^1, x^2)$ is not defined on the axis (x^1, x^2). Any function of the kind (14) can always be expressed in the form:

$$s(x^1, x^2) = s_0 + s_1(x^1)s_2(x^2) \tag{15}$$

where $s_0 \in \mathbb{R}$ and s_1, s_2 are defined as follows:

$$s_1(x^1) = \begin{cases} a & for\, x^1 < 0 \\ b & for\, x^1 > 0 \end{cases} \tag{16}$$

$$s_2(x^2) = \begin{cases} c & for\, x^2 < 0 \\ d & for\, x^2 > 0 \end{cases} \tag{17}$$

and where there is always one degree of freedom in the parameters (s_0, a, b, c, d). Conversely any function of the form (15) is always a 2d-step function.

Now, let V be a vertex of a polyhedron whit 4 edges and angles between edges α, β, γ and θ. Let also S be the surface composed of the vertex, the 4 edges and the relevant 4 faces. We can always open S on a (x^1, x^2) plane by stretching each face by a different amount so that each of the 4 edges lies on one of the semi-axes of the plane. By doing so, we basically map each face of S to a specific sector of the plane (x^1, x^2). It is easy to see that the metric of S is:

$$g_{ij} = \begin{pmatrix} 1 & s(x^1,x^2) \\ s(x^1,x^2) & 1 \end{pmatrix} \tag{18}$$

where $s(x^1, x^2)$ is a 2d-step function for which the amplitude, in each sector of the (x^1, x^2) plane, is a function of one of the angles $\alpha, \beta, \gamma, \theta$ and the parameters (s_0, a, b, c, d) are defined as follows:

$$s(x^1,x^2) = \begin{cases} cos(\alpha) = s_0 + bd & for\ x^1 > 0, x^2 > 0 \\ -cos(\beta) = s_0 + ad & for\ x^1 < 0, x^2 > 0 \\ cos(\gamma) = s_0 + ac & for\ x^1 < 0, x^2 < 0 \\ -cos(\theta) = s_0 + bc & for\ x^1 > 0, x^2 < 0 \end{cases} \tag{19}$$

The (19) define at the same time $s(x^1, x^2)$ and the equations to determine its parameters. The minus signs in the (19) is to take into account that we are in a sector with one of the two dx^i negative and therefore the angle to consider in the metrics is the one between dx^1 and dx^2 positive which is equal to π minus the angle of the relevant polyhedron face for that sector. Since $cos(\pi - x) = -cos(x)$ a minus sign is needed.

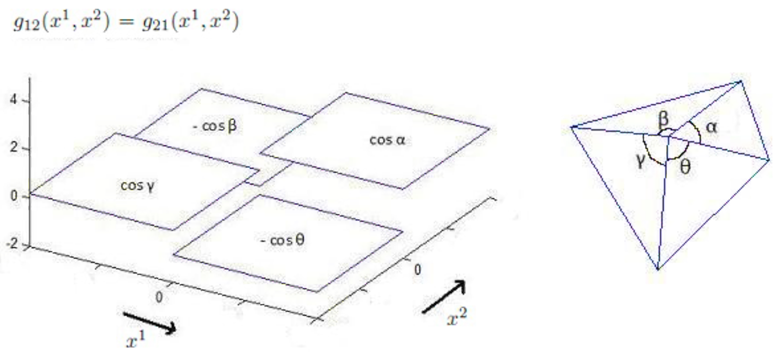

Figure 2: Step discontinuous metric of a polyhedron vertex

As far as vertices with 3 concurrent edges are concerned, we can apply the same procedure by adding a 4th face with angle between edges equal to ϵ and then take the limit for $\epsilon \to 0$. This is equivalent to cut the surface along one of the edges, open the surface on the plane so that each face corresponds to a sector of the axis (x^1, x^2) while the 4th sector remains uncovered and, finally, assign a null metric to that sector (i.e. $s(x^1, x^2) = 1$). This obviously will lead to an infinite inverse metric in the sector. This is not a problem since we are mainly interested in evaluating the curvature in the vertex (i.e. the discontinuity) and not the curvature on faces and edges (which we know to vanish).

An infinite inverse metric will lead to a function $f(x, y)$, of proposition 2 above, which is continuous in $A =]a, b[\times]c, d[$ and that goes to infinity in one of the point of the border of A (the corner related to the null metric). Since proposition 2 works also for this kind of functions, as long as the function is integrable, this is not really an issue.

5 Vertex curvature and deficency angle formula

Given the metric of a vertex defined as for the previous paragraph, we will see now how to evaluate its curvature by means of proposition 2. To do that, we will evaluate all the classical differential parameters, and eventually the curvature, as distributions. First of all we evaluate the $g^{i,j}$. From the (18) we have:

$$g^{ij} = \frac{1}{1-s^2} \begin{pmatrix} 1 & -s \\ -s & 1 \end{pmatrix} \tag{20}$$

The derivatives of the metric are:

$$\Delta_1 = \frac{\partial g_{12}}{\partial x^1} = \frac{\partial g_{21}}{\partial x^1} = (b-a)\delta(x^1)s_2(x^2) \tag{21}$$

$$\Delta_2 = \frac{\partial g_{12}}{\partial x^2} = \frac{\partial g_{21}}{\partial x^2} = (d-c)s_1(x^1)\delta(x^2) \tag{22}$$

all other derivatives vanish. We proceed by evaluating the Christoffel symbol of the first kind. We have (see Eq. (58) in Appendix A.4):

$$\Gamma_{112} = \frac{1}{2}(-0 + \Delta_1 + \Delta_1) = (b-a)\delta(x^1)s_2(x^2) \tag{23}$$

$$\Gamma_{221} = \frac{1}{2}(-0 + \Delta_2 + \Delta_2) = (d-c)s_1(x^1)\delta(x^2) \tag{24}$$

all other coefficients of the Christoffel symbol of the first kind vanish. For our purpose we need to evaluate only one of the coefficients of the Christoffel symbol of the second kind (see Eq. (59) in Appendix A.4):

$$\Gamma^2_{22} = g^{21}\Gamma_{221} + g^{22}\Gamma_{222} = -\frac{(d-c)s}{1-s^2}s_1(x^1)\delta(x^2) \tag{25}$$

We have now all the elements we need to evaluate the Riemann tensor (see Eq. (60) in Appendix A.4):

$$R_{1212} = \frac{(b-a)(d-c)}{1-s^2}(1 - s^2 + s\, s_1 s_2)\delta(x^1, x^2) \tag{26}$$

for surfaces and given the Riemann tensor, a classical formula for evaluating the curvature is the following (see Eq. (61) in Appendix A.4):

$$k = \frac{R_{1212}}{g_{11}g_{22} - g_{12}g_{21}} = \frac{R_{1212}}{1-s^2} \tag{27}$$

as expected the curvature is a Dirac delta function in (0,0). This means that the vertex carries a discrete curvature while the curvature on edges and faces vanishes. The total curvature can be evaluated by integrating the curvature on S:

$$\begin{aligned} k_T &= \iint_S k\sqrt{1-s^2}dx^1dx^2 = \iint_S R_{1212}\frac{\sqrt{1-s^2}}{1-s^2}dx^1dx^2 \\ &= (b-a)(d-c)\iint_S (1 - s^2 + s\, s_1 s_2)(1-s^2)^{-\frac{3}{2}}\delta(x^1, x^2)dx^1dx^2 \end{aligned} \tag{28}$$

since the integrand is impulsive, it is clear that the total curvature is equal to the amplitude of the impulse, which can be evaluated using proposition 2. We have:

$$s_1(x^1) = x;\ s_2(x^2) = y;\ s(x^1, x^2) = s_0 + xy; \tag{29}$$

by using the (29) in the (12) we get the final expression for the total curvature:

$$k_T = \int_a^b dy \int_c^d (1 - s_0^2 - s_0xy)\left[1 - s_0^2 - 2s_0xy - x^2y^2\right]^{-\frac{3}{2}} dx \tag{30}$$

integrating, first with respect of x and then with respect of y, we obtain the primitive $F(x, y)$:

$$F(x, y) = \arctan\left(\frac{s_0 + xy}{\sqrt{1-(s_0+xy)^2}}\right) \tag{31}$$

Let us see how to use the (31) by checking, for example, the value of $F(x, y)$ in (b, d). Given the (19) we have:

$$F(b, d) = \arctan\left(\frac{s_0 + bd}{\sqrt{1-(s_0+bd)^2}}\right) = \arctan\left(\frac{\cos\alpha}{\sin\alpha}\right) = \frac{\pi}{2} - \alpha \tag{32}$$

where we have used the plus sign of the square root. The minus sign corresponds to the case where we swap all the signs in the (19). This is equivalent to choosing a different mapping, between faces and sectors, of the surface on (x^1, x^2). From the (30) we evaluate our final results:

$$k_T = F(b, d) - F(a, d) - F(b, c) + F(a, c) = 2\pi - \alpha - \beta - \gamma - \theta \tag{33}$$

which is, as expected, the deficiency angle formula. It is remarkable that, by means of proposition 2, we have derived the deficiency angle formula, in a non-differentiable point, by using the tools of differential geometry.

Taking the limit for one of the angles going to zero, we get the example, mentioned at the end of the previous paragraph, of a null metric and an infinite inverse metric in a sector. As anticipated above, in this case the function $f(x, y)$ of proposition 2 goes to infinity (compare with the integrand of (30) above) in a point of the integration set. However, the function is still integrable as clearly shown by the (31) where the primitive is finite in the same point.

Appendix

A.1 Relationship between the (7) and Colombeau theory

We show now the relationship between the (7) and the Colombeau theory. What follows cannot be taken as a formal proof of the (7) for two main reasons:

- The relation (34) below is not true with equality in the Colombeau algebra, but only in the sense of association.
- It is not possible to find a well defined notion of convergence for the series (35) below.

For simplicity, we will use $g(x) = u(x)$, the Heaviside function, and $f \in C^\infty$.

Colombeau coefficients are defined as follows (see [1] §3.3):

$$u^n(x)\delta(x) = \frac{1}{n+1}\delta(x) \tag{34}$$

we have:

$$f(u(x))\delta(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} u^n(x)\delta(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!(n+1)}\delta(x) \tag{35}$$

where we have used the (34). With the substitution $k = n + 1$ we have:

$$f(u(x))\delta(x) = \sum_{k=1}^{\infty} \frac{f^{(k-1)}(0)}{k!}\delta(x) = \sum_{k=1}^{\infty} \frac{F^{(k)}(0)}{k!}\delta(x) \tag{36}$$

where F is the primitive of f. We have eventually:

$$f(u(x))\delta(x) = \left[-F(0) + \sum_{k=0}^{\infty} \frac{F^{(k)}(0)}{k!}(1)^k\right]\delta(x) = [F(1) - F(0)]\delta(x) \tag{37}$$

A.2 Formal prove of Proposition 1

Proposition 1. *Let $g(x)$ be a function defined as follows:*

$$g(x) = \begin{cases} a & for\, x < 0 \\ b & for\, x > 0 \end{cases} \tag{38}$$

Also let $(b - a)\delta(x)$ be the derivative of $g(x)$. Then:

$$f(g(x))\delta(x) = \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\delta(x) \tag{39}$$

with $a, b \in \mathbb{R}$, and $f(x)$ any locally integrable function in $A \supseteq [a, b]$ (or $[b, a]$ if $b < a$).

Proof. The proof is given for $a < b$ only, changes to the proof, for the case $b < a$, are trivial.

We start choosing any sequence g_n such that:

$$\begin{array}{l} 1)\ g_n(x) \in C^1\ \ \forall n \in \mathbb{N} \\ 2)\ \lim_{n\to\infty} g_n(x) = g(x) \\ 3)\ \lim_{x\to-\infty} g_n(x) = a\ \ \forall n \in \mathbb{N} \\ 4)\ \lim_{x\to+\infty} g_n(x) = b\ \ \forall n \in \mathbb{N} \\ 5)\ g_n(x)\ \textit{is monotonic}\, \forall n \in \mathbb{N} \end{array} \tag{40}$$

Moreover we want also each $g_n(x)$ such that, if $g_n(x)$ is constant in any $]\alpha, \beta[$ and equal to k, then $f(x)$ is continuous in k.

Given the above sequence of functions, the product in Eq. (39) can to be intended as:

$$f(g(x))\delta(x) = \frac{1}{b-a} \lim_{n\to\infty} f(g_n(x))g_n'(x) \tag{41}$$

We note immediately that, given the (40), the $g_n(x)$ are bounded and converge to $g(x)$. For the dominated convergence theorem, $g_n(x)$ converges in L^1_{loc} and therefore in D'. Also g_n' converges to $(b-a)\delta(x)$ in D'.

First, we prove two useful equations. For any $f \in L^1_{loc}(A)$, for any $g_n(x)$ having the characteristics (40) and given any $\alpha, \beta \in \mathbb{R}$ we have:

$$\int_\alpha^\beta f(g_n(x))g_n'(x)dx = \int_\alpha^\beta \frac{d}{dx}F(g_n(x))dx = F(g_n(\beta)) - F(g_n(\alpha)) \tag{42}$$

where $F(x)$ is the primitive of $f(x)$.

Now, $\lim_{\alpha\to-\infty} g_n(\alpha) = a$ and $\lim_{\beta\to+\infty} g_n(\beta) = b$ and therefore we have:

$$\int_{-\infty}^{+\infty} f(g_n(x))g_n'(x)dx = \int_a^b f(x)dx \tag{43}$$

The (43) does not depend from the function $g_n(x)$ since it depends only on $f(x)$, a and b.

Also, let $[\alpha, \beta]$ be any interval. Given the (40), $g_n' \geq 0$. We write $f(x) = f_+(x) - f_-(x)$ as the sum of its positive and negative part. Note that $f_+(x)$ and $f_-(x)$ are locally integrable on A. We have:

$$\begin{aligned} \int_\alpha^\beta |f(g_n(x))g_n'(x)|dx &= \int_\alpha^\beta f_+(g_n(x))g_n'(x)dx + \int_\alpha^\beta f_-(g_n(x))g_n'(x)dx \\ &= \int_{g_n(\alpha)}^{g_n(\beta)} f_+(x)dx + \int_{g_n(\alpha)}^{g_n(\beta)} f_-(x)dx \\ &= \int_{g_n(\alpha)}^{g_n(\beta)} |f(x)|dx \\ &\leq \int_a^b |f(x)|dx = M > 0 \end{aligned} \tag{44}$$

Now we can prove the proposition. Let $\phi(x)$ be a test function, taking into account the (43) it is possible to write:

$$\begin{aligned} &\left| \int_{-\infty}^{+\infty} f(g_n(x))g_n'(x)\phi(x)dx - \left(\int_a^b f(x)dx \right) \phi(0) \right| \\ &= \left| \int_{-\infty}^{+\infty} f(g_n(x))g_n'(x)[\phi(x) - \phi(0)]dx \right| \\ &\leq I_{m1} + I_{m2} + I_{m3} \end{aligned} \tag{45}$$

where m is any positive integer and:

$$I_{m1} = \int_{-\infty}^{-1/m} |f(g_n(x))g_n'(x)|\, |\phi(x) - \phi(0)|dx \tag{46}$$

$$I_{m2} = \int_{-1/m}^{+1/m} |f(g_n(x))g_n'(x)|\,|\phi(x) - \phi(0)|dx \tag{47}$$

$$I_{m3} = \int_{+1/m}^{+\infty} |f(g_n(x))g_n'(x)|\,|\phi(x) - \phi(0)|dx \tag{48}$$

Since ϕ is a test function, it is continuous at $x = 0$. Given any $\epsilon > 0$, it is possible to find $\delta > 0$ such that, whenever $|x| < \delta$, $|\phi(x) - \phi(0)| < \epsilon$. So, given any $m > \frac{1}{\delta}$, if we choose any $n > m$, we have:

$$\begin{aligned} I_{m2} &= \int_{-1/m}^{+1/m} |f(g_n(x))g_n'(x)|\,|\phi(x) - \phi(0)|dx \\ &\leq \epsilon \int_{-1/m}^{+1/m} |f(g_n(x))g_n'(x)|dx \leq M\epsilon \end{aligned} \tag{49}$$

Where we have used the (44).

Now, ϕ is a continuous function with compact support S and therefore it is bounded. We can find $L > 0$ such that $|\phi(x) - \phi(0)| < L$. We have:

$$\begin{aligned} I_{m1} &= \int_{-\infty}^{-1/m} |f(g_n(x))g_n'(x)|\,|\phi(x) - \phi(0)|dx \\ &\leq \int_S L dx \int_{-\infty}^{-1/m} |f(g_n(x))g_n'(x)|dx \\ &= N \int_a^{g_n(-1/m)} |f(x)|dx \end{aligned} \tag{50}$$

where we have used the (44) and $N > 0$ is the integral of the constant L on S. Since $g_n(-1/m)$ converge to a and given the ϵ above, it is possible to find k such that, whenever $n > k$ then $I_{m1} < N\epsilon$. Applying the same argument to I_{m3} we find that, it is also possible to find k such that, whenever $n > k$ then $I_{m3} < N\epsilon$.

To conclude, given the (45) and given any $\epsilon > 0$, it is possible to find first m and then k such that, whenever we choose $n > k > m$ we have:

$$\left| \int_{-\infty}^{+\infty} f(g_n(x))g_n'(x)\phi(x)dx - \left(\int_a^b f(x)dx \right) \phi(0) \right| \leq (M + 2N)\epsilon \tag{51}$$

This proves that:

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} f(g_n(x))g_n'(x)\phi(x)dx = \left(\int_a^b f(x)dx \right) \phi(0) \tag{52}$$

Now, if we call $(b - a)f(g(x))\delta(x)$ the limit of the sequence of distributions $f(g_n(x))g_n'(x)$, the (52) proves the following:

- the limit exists
- the limit is a Dirac delta function
- the amplitude of the delta function is given by the (39) □

We also note that the constrains (40) used to prove proposition 1 are too stringent and that, in practical calculations, it is possible to relax them (see appendix A.3).

A.3 Formal prove of Proposition 2

Proposition 2. *Let $g_1(x)$ and $g_2(y)$ be two functions defined as follows:*

$$g_1(x) = \begin{cases} a & for\; x < 0 \\ b & for\; x > 0 \end{cases} \tag{53}$$

$$g_2(y) = \begin{cases} c & for\, y < 0 \\ d & for\, y > 0 \end{cases} \tag{54}$$

with $a, b, c, d \in \mathbb{R}$ and let $f(x, y)$ be any function locally integrable in $A \supseteq [a, b] \times [c, d]$ (if $b < a$ and/or $d < c$ the definition of A has to be changed accordingly). Also let $(b - a)(d - c)\delta(x, y)$ be the product of the derivatives of $g_1(x)$ and $g_2(y)$. Then:

$$f(g_1(x), g_2(y))\delta(x, y) = \left(\frac{1}{(b-a)(d-c)} \int_c^d dy \int_a^b f(x, y)dx \right) \delta(x, y) \tag{55}$$

As for proposition 1, in order to prove the above proposition, we first need to prove some useful equations. As an example, we will prove the equivalent of the (43). Let $g_{1n}(x), g_{2n}(y)$ be any two function having characteristics (40) and let $F(x, y)$ be a function such that $F_{xy} = F_{yx} = f(x, y)$. We have:

$$\begin{aligned} &\int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} f(g_{1n}(x), g_{2n}(y)) g'_{1n}(x) g'_{2n}(y) dx \\ &= \int_{-\infty}^{+\infty} dy \frac{\partial}{\partial y} \int_{-\infty}^{+\infty} \frac{\partial}{\partial x} F(g_{1n}(x), g_{2n}(y)) dx \\ &= F(b, d) - F(a, d) - F(b, c) + F(a, c) \end{aligned} \tag{56}$$

where, to prove the (56), we have taken the symbol $\frac{\partial}{\partial y}$ inside the integral (for the linearity of integrals) and applied the definition of $F(x, y)$. The (56) is independent from g_{1n}, g_{2n} and depends only on $f(x, y), a, b, c, d$. Proposition 2 will not be proven in this paper. However, it is possible to prove it by following similar steps to the ones used for proving proposition 1.

A.4 Tensor Formulas

The tensor calculus formulas used in this article, for evaluating Christoffel Symbols and Riemann Tensors, are not the most standard ones but they are consistent and the most convenient for the calculation in place. A reference to those formulas can be found in ([7]).

Given the surface:

$$S = \left(\overline{x}^1(x^1, x^2), \overline{x}^2(x^1, x^2), \overline{x}^3(x^1, x^2)\right) \tag{57}$$

having metric tensor $g_{ij}(x^1, x^2)$, we have:

for Christoffel Symbols (see [7] §6.1 Eq. 6.1a pag. 68):

$$\Gamma_{ijk} = \frac{1}{2}\left(-\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} \right) \tag{58}$$

and (see [7] §6.3 Eq. 6.4 pag. 70):

$$\Gamma^i_{jk} = g^{ir}\Gamma_{jkr} \tag{59}$$

for the Riemann Tensor (see [7] §8.2 Eq. 8.4 pag. 101):

$$R_{ijkl} = \frac{\partial \Gamma_{jli}}{\partial x^k} - \frac{\partial \Gamma_{jki}}{\partial x^l} + \Gamma_{ilr}\Gamma^r_{jk} - \Gamma_{ikr}\Gamma^r_{jl} \tag{60}$$

For a two dimensional manifold the curvature k is equal to (see [7] §8.3 Eq. 8.11 pag. 105):

$$k = \frac{R_{1212}}{g} \tag{61}$$

where $g = det(g_{ij})$.

A.5 Discrete Curvature on a Line

We will give below an example where we evaluate the discrete curvature carried by a line. For simplicity we have chosen an example with a continuous metric which is not differentiable on a line. In this case, product of step and delta function are not present in the various differential parameters. However, the same approach can be used for the more general case where the metric is step discontinuous on a line and therefore the use of proposition 1 is required. Let us consider the following surface:

We want to evaluate the discrete curvature carried by C. We evaluate the discrete curvature of the vertex V. To do that, we cut the cone on one side and we unfold it on a plane so that we can use the deficiency angle formula. We find out easily that the total discrete curvature of the vertex V is:

$$k_T(V) = 2\pi \left(\frac{a-r}{a}\right) \tag{62}$$

We know that the curve C carries a negative discrete curvature, per unit length that compensates exactly the curvature of the vertex:

$$k_T(C) = -k_T(V) = 2\pi r\, k(C) \tag{63}$$

where $k_T(C)$ is the total curvature of C and $k(C)$ is the curvature for unit length on C (constant). We have:

$$k(C) = -\left(\frac{a-r}{ar}\right) \tag{64}$$

which completes our calculation.

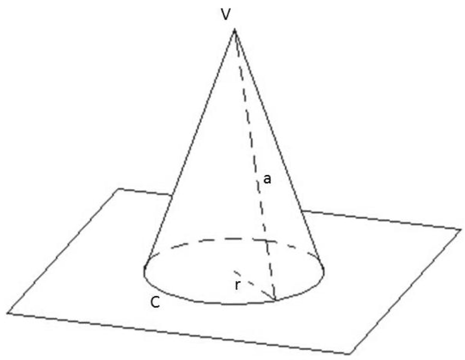

Figure 3: Cone on a Plane

Now, we want to find the same result by using the methods of differential geometry. We define a polar coordinate system (ρ, θ) on the surface of fig. 3 with the centre on the vertex V, where the coordinate $\rho > 0$ is the distance of the point (ρ, θ) from V evaluated on a minimum distance path (i.e. a ray) and θ is a pseudo-angle covering the whole surface for $\theta \in [0, 2\pi]$. In this coordinate system, the metric of the surface is:

$$ds^2 = \begin{cases} d\rho^2 + \left(\frac{r}{a}\rho\right)^2 d\theta^2 & for\; 0 < \rho < a \\ d\rho^2 + (\rho - a + r)^2\, d\theta^2 & for\; \rho > a \end{cases} \tag{65}$$

which is a continuous but not differentiable function and it is not defined for $\rho = 0$.

We define $\nu(x)$ to be a continuous function such that $\nu'(x) = u(x)$, the Heaviside function, and $\nu(0) = 0$. By using the following notation:

$$\begin{aligned} & x^1 = \rho \\ & x^2 = \theta \\ & \alpha_0 = \tfrac{r}{a} \\ & \alpha_1 = 1 - \tfrac{r}{a} \\ & m(x^1) = \left[\alpha_0 x^1 + \alpha_1 \nu(x^1 - a)\right]^2 \end{aligned} \tag{66}$$

we have:

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & m(x^1) \end{pmatrix} \tag{67}$$

$$g^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{m(x^1)} \end{pmatrix} \tag{68}$$

From now on, we proceed as we did in paragraph 5:

$$\Delta = \frac{\partial g_{22}}{\partial x^1} = 2\left[\alpha_0 x^1 + \alpha_1 \nu(x^1 - a)\right]\left[\alpha_0 + \alpha_1 u(x^1 - a)\right] \tag{69}$$

$$\Gamma_{221} = -\frac{1}{2}\Delta \tag{70}$$

$$\Gamma_{212} = \Gamma_{122} = \frac{1}{2}\Delta \tag{71}$$

all other coefficients of the Christoffel symbol of the first kind vanish.

$$\Gamma^2_{21} = g^{21}\Gamma_{211} + g^{22}\Gamma_{212} = \frac{\Delta}{2m} \tag{72}$$

and finally, if we do not write the terms that vanish, we have:

$$\begin{aligned} R_{1212} &= \frac{\partial \Gamma_{221}}{\partial x^1} + \Gamma_{122}\Gamma^2_{21} && (73)\\ &= -\frac{1}{2}\frac{\partial \Delta}{\partial x^1} + \frac{\Delta^2}{4m} && (74)\\ &= -\alpha_0\alpha_1 x^1 \delta(x^1 - a) - (\alpha_1)^2 \nu(x^1 - a)\delta(x^1 - a) && (75) \end{aligned}$$

The curvature of the surface is:

$$k = \frac{R_{1212}}{g_{11}g_{22} - g_{12}g_{21}} = \frac{R_{1212}}{m} \tag{76}$$

Let ϵ be any number such that $0 < \epsilon < a$, the total curvature of the surface can be evaluated as:

$$\begin{aligned} k_T &= \iint_S k\sqrt{m}\, dx^1 dx^2 && (77)\\ &= \int_\epsilon^\infty \int_0^{2\pi} R_{1212} m^{-\frac{1}{2}} dx^1 dx^2 && (78)\\ &= -2\pi\alpha_1 \int_\epsilon^\infty \delta(x^1 - a) dx^1 && (79)\\ &= -2\pi\left(\frac{a - r}{a}\right) = k_T(C) && (80) \end{aligned}$$

as expected. This example is quite trivial. However, the same method can be extended to generic curves where a direct geometrical approach, as the one used at the beginning of this paragraph, cannot be used.

A.6 Examples of Products of Steps and Delta Functions

Example 1:

$$\int_{-\infty}^{+\infty} \frac{d}{dx} u^2(x) dx = \left[u^2(x)\right]_{-\infty}^{+\infty} = 1 = 2\int_{-\infty}^{+\infty} u(x)\delta(x) dx \tag{81}$$

from which we have:

$$u(x)\delta(x) = \frac{1}{2}\delta(x) \tag{82}$$

in agreement with proposition 1.

Example 2:

Given

$$\delta(x) = \frac{1}{2}\frac{d}{dx}sign(x) \tag{83}$$

and

$$1(x) = -1 + \frac{1}{1 - \frac{1}{2}sign^2(x)} \tag{84}$$

by using proposition 1 we have:

$$1(x)\delta(x) = \frac{1}{2}\left[\int_{-1}^{+1}\frac{x^2}{2-x^2}dx\right]\delta(x) = \left[\frac{\sqrt{2}}{2}ln\left(\frac{2+\sqrt{2}}{2-\sqrt{2}}\right) - 1\right]\delta(x) \tag{85}$$

Note that $1(x)$ in D' is the constant function 1 and therefore it has not even a step discontinuity.

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