New adaptative numerical algorithm for solving partial integro-differential equations

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Abstract

The paper introduces an acurrate numerical appraoch based on orthonormal Bernoulli polynomials for solving parabolic partial integro- differential equations (PIDEs). This type of equations arises in physics and engineering. Some operational matrix are given for these polynomials and are also used to obtain the numerical solution. By this approach, the problem is transformed into a nonlinear algebraic system. Convergence analysis is given and some experiment tests are studied to examine the good accuracy of the numerical algorithm, the proposed technique is compared with some other well known methods.

Keywords: Shifted orthonormal Bernoulli polynomials, parabolic integro- differential equation, collocation method, numerical solution.
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1. Introduction

Various approximations by orthonormal family of functions have been investigated in physical sciences, engineering,etc. This type of numerical approximations can be also used in optimal control problems and in general to approximate solutions of dynamical systems. Integral equations arise in many physical problems, diffusion problems, concrete problem of physics and mechanics and some others problems of engineering, different applications of potential theory, synthesis problem, mathematical modelling of economics, population, geophysics, antenna, genetics, communication theory, radiation problems, concerning transport of particles, etc [1, 17, 20, 22, 24, 27, 29]. There are various problems such as differential, integral and partial integro-differential equations which uses polynomial series and orthogonal functions to approximate their numerical solutions [2, 8, 11, 28, 30, 31, 33]. Spectral methods are methods based on polynomial ap-

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proximation, we can see that the convergence of the approximations is exponentiel when the functions to be approximated are analytic that's means that the order of convergence is limited by the choice of the regularity of the exact solution. In many science fields, systems are described by partial differential equations (PDE). In [12] Fakha et al used Legendre collocation technique for solving parabolic PDE. Radial basis functions are also used for the approximate solutions of nonlinear parabolic type Volterra partial integro-differential equations [33]. In [19], Brunner et al studied the numerical solution of parabolic Volterra integro-differential equations on unbounded spatial domains. For more applications, we can see [4, 7, 16]. A method based on Hermite-Taylor matrix to solve partial integro-differential equations is given in [8]. A matrix method for solving two- dimensional time-dependent diffusion equations is given by Zogheib et al. [34]. In [30], the author presented a new technique by using Bernoulli operational matrix to solve SDE. In this article, we use pseudo-spectral method based on orthonormal Bernoulli polynomials to approximate the following PIDEs

$$u_t(x,t) + \lambda_1 u_{xx}(x,t) = \lambda_2 \int_0^t K_1(x,t,s,u(x,s))ds + \lambda_3 \int_0^T K_2(x,t,s,u(x,s))ds + g(x,t), \qquad (1)$$

with initial and boundary conditions

$$u(x,0) = h_0(x), \ u(0,t) = h_1(t), u(b,t) = h_2(t), \qquad (2)$$

where $x \in [0,b], t \in [0,T], \lambda_1, \lambda_2$ and λ_3 are constants and the functions $g(x,t), k_1(x,t,s,u)$ and $k_2(x,t,s,u)$ are supposed to be sufficiently smooth on $\Gamma := [0,b] \times [0,T]$ and D where $D := \{(x,t,s) : x \in [0,b], s,t \in [0,T]\}$. The existence and uniqueness of equation (1) are given in [18].

To approximate the integrals oppeared in (1), we use the Gauss Legendre quadrature on the interval $[-1,1]$ given by

$$\int_{-1}^1 f(x)dx = \sum_{j=0}^r \omega_j f(x_j), \qquad (3)$$

where x_j are the roots of $L_{r+1}(x)$ and $\omega_j = \dfrac{2}{(1 - x_j^2)[L'_{r+1}(x_j)]^2}$, $j = 0, 1, \ldots, r$. For Gauss Legendre quadrature on $[a,b]$, we have

$$\int_a^b f(x)dx = \frac{b-a}{2} \sum_{j=0}^r \omega_j f\left(\frac{b-a}{2}x_j + \frac{b+a}{2}\right). \qquad (4)$$

2. Othonormal Bernoulli polynomials and approximation

The Bernoulli polynomials $B_n(x)$, are given in [21] and satisfied the following relation

$$\sum_{k=0}^n \binom{n+1}{k} B_k(x) = (n+1)x^n, \quad n = 0, 1, \ldots.$$

2

The Bernoulli polynomials form a complete basis over the interval $[0, 1]$ [9]. In this paper, we use the shifted OBPs $P_{i,T}(t)$ and $P_{i,b}(x)$ over $[0, T]$ and $[0, b]$ as follows

$$P_{i,b}(x) = \sqrt{\frac{2i+1}{b}} \sum_{k=0}^{i} (-1)^{i-k} \binom{i}{i-k}\binom{i+k}{k}\left(\frac{x}{b}\right)^k, \; i = 0, 1, \ldots \tag{5}$$

$$\Phi_{N,b}(.) = (P_{0,b}(.), P_{1,b}(.), \ldots, P_{N,b}(.))^T, \tag{6}$$

contains first $N + 1$ orthonormal Bernoulli polynomials. Any function $u \in L^2[0, b]$ has a best approximation $\hat{u} \in span\{\Phi_{N,b}(.)\}$ such that

$$\forall v \in span\{\Phi_{N,b}(.)\}, \; \|u - \hat{u}\| \leq \|u - v\|,$$

and

$$u(x) \simeq \hat{u} = \sum_{i=0}^{N} u_i P_{i,b}(x) = U^T \Phi_{b,N}(x), \tag{7}$$

where $U = (u_0, u_1, u_2, \ldots, u_N)^T$, and u_i can be computed by the formula

$$u_i = \int_0^b u(x) P_{i,b}(x) dx, \; i = 0, 1, \ldots, N. \tag{8}$$

Any function $u(x, t)$ defined over $[0, b] \times [0, T]$ can be approximated by shifted OBPs as follows:

$$u_{N,M}(x, t) = \sum_{i=0}^{N} \sum_{j=0}^{M} u_{ij} P_{i,b}(x) P_{j,T}(t) = \Phi_{b,N}(x)^T U \Phi_{T,M}(t), \tag{9}$$

where $U = [u_{ij}]$ is a matrix of order $(N + 1) \times (M + 1)$ with

$$u_{ij} = \int_0^b \int_0^T u(x) P_{i-1,b}(x) P_{j-1,T}(x) dt dx, \; i = 1, 2, \ldots N + 1, \; j = 1, 2, \ldots, M + 1. \tag{10}$$

3. Pseudo-spectral method for Solving PIDE

In this section, we describe our numerical technique t to solve PIDEs (1). Let the solution of (1) be approximated by the polynomial $u_{N,M}(x, t)$ such that

$$u_{N,M}(x, t) = \sum_{i=0}^{N} \sum_{j=0}^{M} u_{ij} P_{i,b}(x) P_{j,T}(t) = \Phi_{b,N}(x)^T U \Phi_{T,M}(t). \tag{11}$$

In this work, we take $N = M$. It is easy to approximate the derivatives of approximate solution of (1). In view of (11), one can write

$$\frac{\partial}{\partial t}[u_{N,M}(x, t)] = \sum_{i=0}^{N} \sum_{j=0}^{M} u_{ij} P_{i,b}(x) \frac{\partial}{\partial t}[P_{j,T}(t)] = \Phi_{b,N}(x)^T U \frac{\partial}{\partial t}[\Phi_{T,M}(t)], \tag{12}$$

3

and

$$\frac{\partial^2}{\partial x^2}[u_{N,M}(x,t)] = \sum_{i=0}^{N}\sum_{j=0}^{M} u_{ij}\frac{\partial^2}{\partial x^2}[P_{i,b}(x)][P_{j,T}(t)] = \frac{\partial^2}{\partial x^2}[\Phi_{b,N}(x)]^T U[\Phi_{T,M}(t)], \tag{13}$$

Now we give some relations for the derivatives of the shifted OBPs, we use technique used by various researchers for solving different kind of integral equations [12, 23, 25]. Let given the vector defined in (6), we can write

$$\Phi_{b,N}(x) = T_{b,N}\Psi_N(x), \tag{14}$$

where

$$\Psi_N(x) = [1, x, x^2, \ldots, x^{N-1}, x^N]^T, \tag{15}$$

with $T_{b,N}$ is a lower triangular square matrix of order $N + 1$ with entries

$$[T_{b,N}]_{ij} = \begin{cases} (-1)^{i-j}\sqrt{2i-1}\dfrac{1}{b^{j-1/2}}\dbinom{i-1}{i-j}\dbinom{i+j-2}{j-1} & i \geq j \\ \\ 0 & i < j \end{cases}$$

The matrix $T_{b,N}$ is invertible. From equation (14), we have

$$\frac{\partial}{\partial x}\Phi_{b,N}(x) = T_{b,N}\frac{\partial}{\partial x}\Psi_N(x) = T_{b,N}A_N\Psi_N(x) = T_{b,N}A_N T_{b,N}^{-1}\Phi_{b,N}(x), \tag{16}$$

where A_N is a square matrix of order $N + 1$ given by

$$A_N = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & \ldots & 0 & 0 & 0 & 0 \\ . & . & . & \ldots & 0 & 0 & 0 & 0 \\ . & . & . & \ldots & N-2 & 0 & 0 & 0 \\ . & . & . & \ldots & 0 & N-1 & 0 & 0 \\ . & . & . & \ldots & 0 & 0 & N & 0 \end{pmatrix},$$

and

$$\frac{\partial^2}{\partial x^2}\Phi_{b,N}(x) = T_{b,N}\frac{\partial^2}{\partial x^2}\Psi_N(x) = T_{b,N}A_N^1\Psi_N(x) = T_{b,N}A_N^1 T_{b,N}^{-1}\Phi_{b,N}(x), \tag{17}$$

4

where A_N^1 is a square matrix of order $N + 1$ given by

$$A_N^1 = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ (3-1)(3-2) & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & (4-1)(4-2) & 0 & \ldots & 0 & 0 & 0 & 0 \\ . & . & . & \ldots & 0 & 0 & 0 & 0 \\ . & . & . & \ldots & 0 & 0 & 0 & 0 \\ . & . & . & \ldots & (N-1)(N-2) & 0 & 0 & 0 \\ . & . & . & \ldots & 0 & N(N-1) & 0 & 0 \end{pmatrix}.$$

Then, we get

$$\frac{\partial}{\partial t}[u_{N,M}(x,t)] = \sum_{i=0}^{N} \sum_{j=0}^{M} u_{ij} P_{i,b}(x) \frac{\partial}{\partial t}[P_{j,T}(t)] = \Phi_{b,N}(x)^T U \frac{\partial}{\partial t}[\Phi_{T,M}(t)] = \Phi_{b,N}(x)^T U T_{T,M} A_M T_{T,M}^{-1} \Phi_{T,M}(t),$$

(18)

and

$$\frac{\partial^2}{\partial x^2}[u_{N,M}(x,t)] = \sum_{i=0}^{N} \sum_{j=0}^{M} u_{ij} \frac{\partial^2}{\partial x^2}[P_{i,b}(x)][P_{j,T}(t)] = \frac{\partial^2}{\partial x^2}[\Phi_{b,N}(x)]^T U[\Phi_{T,M}(t)] = [T_{b,N} A_N^1 T_{b,N}^{-1} \Phi_{b,N}(x)]^T U[\Phi_{T,M}(t)].$$

(19)

Thus, by substituting (18) and (19) into equation (1) we get

$$\Phi_{b,N}(x)^T U T_{T,M} A_M T_{T,M}^{-1} \Phi_{T,M}(t) + \lambda_1 \left[[T_{b,N} A_N^1 T_{b,N}^{-1} \Phi_{b,N}(x)]^T U[\Phi_{T,M}(t)] \right] = \lambda_2 \int_0^t k_1(x,t,s,u(x,s))ds \qquad (20)$$
$$+ \lambda_2 \int_0^T k_2(x,t,s,u(x,s))ds + g(x,t),$$

with $x \in [0,b]$ and $t \in [0,T]$, We collocate equation (20) at points t_j, $j = 0,1,\ldots,r_1$ which are the Gauss-Legendre noeuds on the interval $[0,T]$ defined by (3), then we obtain

$$\Phi_{b,N}(x)^T U T_{T,M} A_M T_{T,M}^{-1} \Phi_{T,M}(t_j) + \lambda_1 \left[[T_{b,N} A_N^1 T_{b,N}^{-1} \Phi_{b,N}(x)]^T U[\Phi_{T,M}(t_j)] \right] = \lambda_2 \int_0^{t_j} K_1(x,t_j,s,u(x,s))ds \qquad (21)$$
$$+ \lambda_3 \int_0^T K_2(x,t_j,s,u(x,s))ds + g(x,t_j).$$

The integrals oppeared in equation (21) are approximated by using Gauss-Legendre quadrature as follows

$$\int_0^{t_j} K_1(x,t_j,s,u(x,s))ds = \sum_{k=0}^{r_1} \beta_k K_1(x,t_j,l_k,u(x,l_k)), \qquad (22)$$

5

where $l_k = \dfrac{t_j + t_j x_k}{2}$ and $\beta_k = \dfrac{t_j}{(1 - x_k)^2 [L'_{r_1+1}(x_k)]^2}$, $k = 0, 1, 2, \ldots, r_1$. are Gauss Legendre nodes and weight on $[0, t_j]$.

$$\int_0^T K_2(x, t_j, s, u(x, s))ds = \sum_{k=0}^{r_2} \omega_k K_2(x, t_j, s_k, u(x, s_k)), \tag{23}$$

where $s_k = \dfrac{T + T x_k}{2}$ and $\omega_k = \dfrac{T}{(1 - x_k)^2 [L'_{r_2+1}(x_k)]^2}$, $k = 0, 1, 2, \ldots, r_2$ are nodes and weight of Gauss Legendre quadrature on $[0, T]$. Substituting equations (22) and (23) in equation (21) we get

$$\Phi_{b,N}(x)^T U T_{T,M} A_M T_{T,M}^{-1} \Phi_{T,M}(t_j) + \lambda_1 \Big[[T_{b,N} A_N^1 T_{b,N}^{-1} \Phi_{b,N}(x)]^T U[\Phi_{T,M}(t_j)] \Big] = \sum_{k=0}^{r_1} \beta_k K_1(x, t_j, l_k, u(x, l_k))$$

$$+ \sum_{k=0}^{r_2} \omega_k K_2(x, t_j, s_k, u(x, s_k)) + g(x, t_j).$$

Let $RES(x, t_j)$ the residual function given by

$$RES(x, t_j) = \Phi_{b,N}(x)^T U T_{T,M} A_M T_{T,M}^{-1} \Phi_{T,M}(t_j) + \lambda_1 \Big[[T_{b,N} A_N^1 T_{b,N}^{-1} \Phi_{b,N}(x)]^T U[\Phi_{T,M}(t_j)] \Big] \tag{24}$$

$$- \lambda_2 \sum_{k=0}^{r_1} \beta_k K_1(x, t_j, l_k, u(x, l_k)) - \lambda_3 \sum_{k=0}^{r_2} \omega_k K_2(x, t_j, s_k, u(x, s_k)) - g(x, t_j) = 0.$$

Using relation (9), initial and boundary conditions given in (6), we get

$$\begin{cases} \Phi_{b,N}(x)^T U \Phi_{T,M}(0) - h_0(x) = 0, \\ \Phi_{b,N}(0)^T U \Phi_{T,M}(t) - h_1(t) = 0, \\ \Phi_{b,N}(b)^T U \Phi_{T,M}(t) - h_2(t) = 0. \end{cases}$$

Now, we extract the below $(N + 1) \times (M + 1)$ algebraic system

$$\begin{cases} RES(s_i, t_j) = 0, \ 2 \le i \le N, \ 2 \le j \le M + 1, \\ \Phi_{b,N}(x)^T U \Phi_{T,M}(0) - h_0(s_i) = 0, \ 1 \le i \le N + 1, \\ \Phi_{b,N}(0)^T U \Phi_{T,M}(t_j) - h_1(t_j) = 0, \ 2 \le j \le M + 1 \\ \Phi_{b,N}(b)^T U \Phi_{T,M}(t_j) - h_2(t_j) = 0, \ 2 \le j \le M + 1, \end{cases}$$

where $s_i = \dfrac{b + b x_i}{2}$ and $t_j = \dfrac{T + T x_j}{2}$ are respectively the nodes of Gaus-Legendre quadrature on $[0, b]$ and $[0, T]$, with x_i are the corresponding nodes on $[-1, 1]$.

4. Error bound of the present method

In [5], the error estimates for some orthogonal systems are given in the norms of the Sobolev spaces $H^\mu(\Omega)$, with $\Omega = I^d \subset \mathbb{R}^d$ and I is a bounded open interval of $\mathbb{R}$. In this section, we consider orthogonal

approximations in mutiple dimensions. Let $k = (k_1, k_2; \ldots, k_d)$, $|k| = \sum_{i=1}^{d} k_i$, k_i being any non-negative integers, and $\partial_x^k \Phi = \dfrac{\partial^{|k|} u}{\partial_{x_1}^{k_1} \ldots \partial_{x_d}^{k_d}}$. For $\mu \geq 0$, we define the Sobolev space

$$H^\mu(\Omega) = \left\{ \Phi, \, \partial_x^k \Phi \in L^2(\Omega), \, 0 \leq |k| \leq \mu \right\}, \tag{25}$$

with the norm

$$\|\Phi\|_\mu^2 = \sum_{k \in \mathbb{N}^d, \, k_1 + k_2 + \ldots k_d < \mu} \int_\Omega \left| \left(\prod_{j=1}^{d} D_j^{k_j} \right) \Phi \right|^2 dx, \tag{26}$$

where $D_j = \dfrac{\partial}{\partial x^{(j)}}$. If $\{\Phi_k\}_{k=0}^{\infty}$ is the system of orthonormal in $L^2(I)$ with $deg\Phi_k = k$ then the system $\{\Phi_k\}_{k \in \mathbb{N}^d}$, where $\Phi_k(x) = \prod_{j=1}^{d} \Phi_{k_j}(x^{(j)}$ is complete and orthonormal in $L^2(\Omega)$ and any $u \in L^2(\Omega)$ is as follows

$$u = \sum_{k \in \mathbb{N}^d} u_k \Phi_k, \quad u_k = \langle u, \Phi_k \rangle, \tag{27}$$

with $\|u\|_0^2 = \sum_{k \in \mathbb{N}^d} |u_k|^2$. Setting,

$$S_N = S_N(\Omega) = span\left\{ \Phi_k : k \in \mathbb{N}^d, |k|_\infty < N \right\}, \tag{28}$$

S_N is the set of all polynomials of degree at most N in each variable $x^{(j)}$, $j = 1, \ldots, d$. Let $P_N : L^2(\Omega) \longrightarrow S_N$ the orthogonal projection on S_N in $L^2(\Omega)$.

Theorem 1. *[5] For any real $\mu > 0$, there exists a constant C such that*

$$\|u - P_N u\|_0 \leq CN^{-\mu} \|u\|_\mu, \quad \forall u \in H^\mu(\Omega). \tag{29}$$

For the error estimation of $u - P_N$ in Sobolev space, we need the following lemmas.

Lemma 1. *[5] For any real μ and r such that $0 < r < \mu$, there exists a constant C such that*

$$\|u\|_\mu \leq CN^{2(r-\mu)} \|u\|_r, \quad \forall u \in S_N. \tag{30}$$

Lemma 2. *For the two real r and μ with $0 \leq r \leq \mu - 1$ there exists a constant C so that for $j = 1, \ldots, d$*

$$\|(P_N D_j - D_j P_N) u\|_r \leq CN^{(2r-\mu+3/2)} \|u\|_\mu, \quad \forall u \in H^\mu(\Omega). \tag{31}$$

The following theorems gives the error estimation of the approximation of u

Theorem 2. *For the two real μ and r with $0 \le r \le \mu$, we can get a constant C so that*

$$\|u - P_N u\|_r \le C N^{e(r,\mu)} \|u\|_\mu, \quad \forall u \in H^\mu(\Omega), \tag{32}$$

where

$$e(r,\mu) = \begin{cases} 2r - \mu - 1/2, & r \ge 1 \\ 3r/2 - \mu, & 0 \le r \le 1 \end{cases}$$

Proof 1. *For $r = 0$, 32 reduces to 29. Now suppose that 32 holds for any integer $r \le m - 1$ by inductive hypothesis. Then*

$$
\begin{aligned}
\|u - P_N u\|_m &\le \sum_{j=1}^{d} \|D_j u - D_j P_N u\|_{m-1} \\
&\le \sum_{j=1}^{d} \|D_j u - P_N D_j u\|_{m-1} + \sum_{j=1}^{d} \|P_N D_j u - D_j P_N u\|_{m-1},
\end{aligned}
$$

by taking the inductive hypothesis for $D_j \in H^{\mu-1}$ and by using 31, we get

$$\|u - P_N u\|_m \le C' N^{e(m-1,\mu-1)} \sum_{j=1}^{d} \|D_j u\|_{\mu-1} + c'' N^{e(m,\mu)} \|u\|_\mu,$$

we have $e(m - 1, \mu - 1) < e(m, \mu)$, then we get the results 32.

Theorem 3. *Suppose that $u(x,t) \in H^\mu(\Omega)$ and $\bar{u}(x,t) \in H^\mu(\Omega)$ with $\mu \ge 0$ be the exact and the numerical solution of equation (1), respectively. Also, suppose K_1 and K_2 satisfy the following uniform Lipschitz conditions*

$$|K_1(x,t,s,u_1) - K_1(x,t,s,u_2)| \le l_1|u_1 - u_2|, \quad |K_2(x,t,s,u_1) - K_2(x,t,s,u_2)| \le l_2|u_1 - u_2|. \tag{33}$$

Then, for any real μ and r such that $0 \le r < \mu$ the error bound E_N of the present method is given by

$$\|E_N\|_{H^r(\Omega)} \le \left[\left(C + \lambda_1 C\right) N^{(2r-\mu+3/2)} + \left(l_1\lambda_2 + l_2\lambda_3\right) c N^{e(r,\mu)}\right] \|u\|_{H^\mu(\Omega)}, \tag{34}$$

where

$$e(r,\mu) = \begin{cases} 2r - \mu - 1/2, & r \ge 1 \\ 3r/2 - \mu, & 0 \le r \le 1. \end{cases}$$

8

Proof 2. *Using equation 1, we get*

$$\|E_N\|_{H^r(\Omega)} = \|u_t(x,t) + \lambda_1 u_{xx}(x,t) - \lambda_2 \int_0^t K_1(x,t,s,u(x,s))ds - \lambda_3 \int_0^T K_2(x,t,s,u(x,s))ds \tag{35}$$

$$- \left(\bar{u}_t(x,t) + \lambda_1 \bar{u}_{xx}(x,t) - \lambda_2 \int_0^t K_1(x,t,s,\bar{u}(x,s))ds - \lambda_3 \int_0^T K_2(x,t,s,\bar{u}(x,s))ds\right)\|_{H^r(\Omega)}$$

$$\leq \|u_t(x,t) - \bar{u}_t(x,t)\|_{H^r(\Omega)} + \lambda_1 \|u_{xx}(x,t) - \bar{u}_{xx}\|_{H^r(\Omega)}$$

$$+ \lambda_2 \| \int_0^t \left(K_1(x,t,s,u(x,s)) - K_1(x,t,s,\bar{u}(x,s))\right)ds\|_{H^r(\Omega)} + \lambda_3 \| \int_0^T \left(K_2(x,t,s,u(x,s)) - K_2(x,t,s,\bar{u}(x,s))\right)ds\|_{H^r(\Omega)}$$

$$\leq \|u_t(x,t) - \bar{u}_t(x,t)\|_{H^r(\Omega)} + \lambda_1 \|u_{xx}(x,t) - \bar{u}_{xx}\|_{H^r(\Omega)}$$

$$+ \lambda_2 \int_0^t \|K_1(x,t,s,u(x,s)) - K_1(x,t,s,\bar{u}(x,s))\|_{H^r(\Omega)}ds + \lambda_3 \int_0^T \|K_2(x,t,s,u(x,s)) - K_2(x,t,s,\bar{u}(x,s))\|_{H^r(\Omega)}ds,$$

since K_1 and K_2 satisfied Lipschitz conditions, then we have

$$\|E_N\|_{H^r(\Omega)} \leq \|u_t(x,t) - \bar{u}_t(x,t)\|_{H^r(\Omega)} + \lambda_1 \|u_{xx}(x,t) - \bar{u}_{xx}\|_{H^r(\Omega)} \tag{36}$$

$$+ l_1\lambda_2 \int_0^t \|u(x,s)) - \bar{u}(x,s))\|_{H^r(\Omega)}ds + l_2\lambda_3 \int_0^T \|u(x,s)) - \bar{u}(x,s))\|_{H^r(\Omega)}ds,$$

by using Lemma31 and theorem 2, we get

$$\|E_N\|_{H^r(\Omega)} \leq CN^{(2r-\mu+3/2)}\|u\|_{H^\mu(\Omega)} + \lambda_1 CN^{(2r-\mu+3/2)}\|u\|_{H^\mu(\Omega)} \tag{37}$$

$$+ l_1\lambda_2 \int_0^t cN^{e(r,\mu)}\|u\|_{H^\mu(\Omega)}ds + l_2\lambda_3 \int_0^T cN^{e(r,\mu)}\|u\|_{H^\mu(\Omega)}ds$$

$$\leq \left[\left(C + \lambda_1 C\right)N^{(2r-\mu+3/2)} + \left(l_1\lambda_2 + l_2\lambda_3\right)cN^{e(r,\mu)}\right]\|u\|_{H^\mu(\Omega)},$$

where

$$e(r,\mu) = \begin{cases} 2r - \mu - 1/2, & r \geq 1 \\ 3r/2 - \mu, & 0 \leq r \leq 1. \end{cases}$$

Then if u is is infinitely smooth, then $\|E_N\|_{H^r(\Omega)} \longrightarrow 0$ as $N \longrightarrow \infty$.

5. Numerical implementation of the proposed algorithm

In this section, some numerical test equations are considered to shown the accuracy of the presented algorithm, where we have calculated the maximum absolute errors at different time. In these examples, the linear and nonlinear algebraic systems are solved by Newton iterative method and using MATLAB software.

Example 1. *Consider the PIDEs*

$$u_t(x, t) - u_{xx}(x, t) = g(x, t) - \int_0^t e^{s-t} u(x, s) ds, \tag{38}$$

with initial and boundary conditions $u(x, 0) = x$, $u(0, t) = 0$, $x \in [0, 1]$, $u(1, t) = e^{-t}$, $t \in [0, 1]$, *and* $g(x, t) = (2t - x^2 - t^2 x) \exp(-xt) + \dfrac{x(\exp(-t) - \exp(-xt))}{x - 1}$. *The analytical solution for this example is* $u(x, t) = xe^{-xt}$. *The numerical experiments are given in table* (1).

Table 1: Errors using OBP method for test (1)

t	$N = 3$	$N = 4$	$N = 5$	$N = 6$
0.0625	2.4918 E-4	1.6010 E-5	7.4614 E-7	2.5947 E-8
0.1250	3.1382 E-4	1.4463 E-5	3.7456 E-7	6.8578 E-9
0.1875	2.6376 E-4	6.1076 E-6	1.2642 E-7	1.3100 E-8
0.2500	1.5539 E-4	2.3487 E-6	3.5723 E-7	9.0718 E-9
0.3125	6.4499 E-5	7.4193 E-6	2.7735 E-7	2.9712 E-9
0.3750	1.4299 E-5	8.0728 E-6	1.1176 E-7	1.0793E-8
0.4375	2.5590E-4	6.6364 E-5	2.1644 E-7	9.4010E-9
0.500	3.5960 E-4	1.1409 E-5	3.4232 E-7	8.3644 E-9
0.5625	4.6127 E-4	1.8037 E-5	6.0631 E-7	1.6629 E-8
0.6250	6.2660 E-4	2.6913 E-5	1.0077 E-6	3.0685 E-8
0.6875	8.2174 E-4	3.8470 E-5	1.5912 E-6	5.3212E-8
0.7500	1.0486 E-3	5.3175 E-5	2.4089 E-6	8.7621 E-8
0.8125	1.3090 E-3	7.1535 E-5	3.5395E-6	2.1009 E-7
0.8750	1.6050 E-3	9.4094 E-5	5.0403 E-6	3.0975 E-7
0.9375	1.9385 E-3	1.2143 E-4	6.9854 E-6	4.4474 E-7

Example 2. *Let given the nonlinear equation*

$$u_t(x, t) = u_{xx}(x, t) + g(x, t) - \int_0^1 u^2(x, s) ds, \tag{39}$$

with conditions $u(x, 0) = x^2$, $u(0, t) = t^2$, $x \in [0, 1]$, $u(1, t) = t^2 + 1$, $t \in [0, 1]$. *The function* $g(x, t)$ *is obtained from the analytical solution* $u(x, t) = t^2 + x^2$. *The numerical results are presented in figure 1.*

10

Figure 1: Exact (left) and approximate (right) solutions for example 2 for ($N = M = 3$).
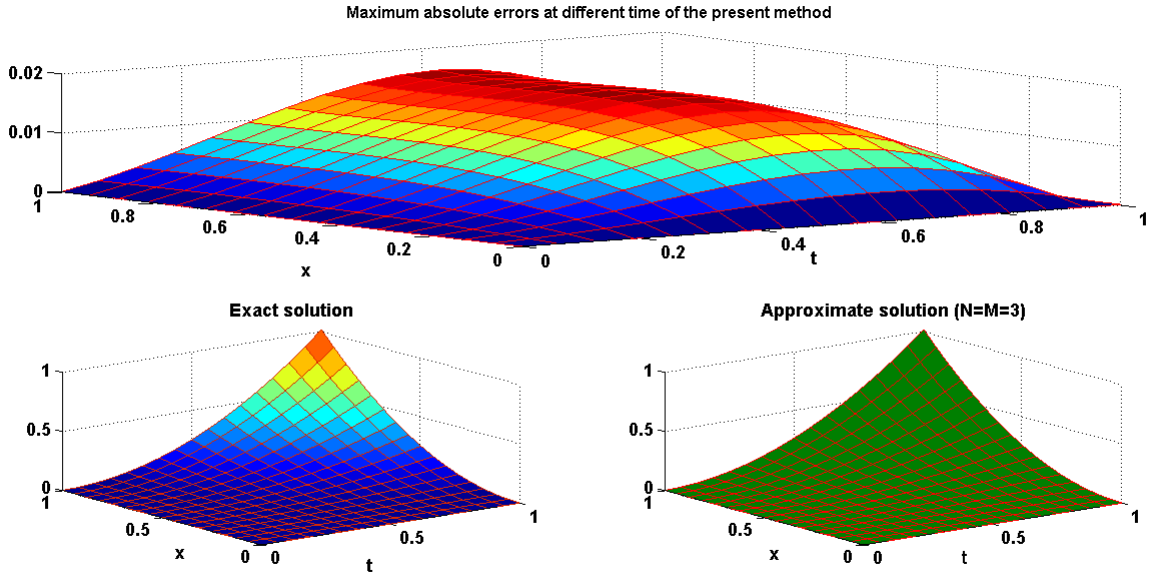
Example 3. *[3] Let consider the following partial integro-differential equation*

$$u_t(x,t) - u_{xx}(x,t) = g(x,t) - \int_0^t e^{x(t-s)}u(x,s)ds, \tag{40}$$

with conditions

$$u(x,0) = 0, \quad x \in [0,1], u(0,t) = \sin(t), \quad u(1,t) = 0, \quad t \in [0,1],$$

with $g(x,t) = (1-x^2)\cos(t) + 2\sin(t) + \dfrac{(x^2-1)\cos(t) + x\sin(t) - e^{xt}}{x^2+1}$. *The exact solution is given by* $u(x,t) = (1-x^2)\sin(t)$. *The numerical results of example 3 are summarized in table 2 and figure 2. Table 3 gives a comparison between the proposed method in [3] and Cardinal Chebyshev functions [13]. better accuracy than the others methods.*

Example 4. *In this example [12], we take a diffusion problem as*

$$u_t(x,t) = u_{xx}(x,t) + g(x,t) - \int_0^t \frac{t-s+1}{x+1}u(x,s)ds, \quad x,t \in [0,1], \tag{41}$$

where $g(x,t)$ *is determined such that the solution is* $u(x,t) = \dfrac{1-x^2}{1+t^2}$. *The numerical results for this example are summariezd in figure 3. Our numerical tests are better than that given by Legendre multi-wavelets collocation method [3].*

11

Table 2: Errors of the present method using OBP method for test (3)

t	$N = 3$	$N = 4$	$N = 5$	$N = 6$
0.0625	2.2589 E-4	4.2169 E-5	1.4692 E-6	1.5646 E-7
0.1250	3.2716 E-4	4.9342 E-5	1.4308 E-6	1.2011 E-7
0.1875	3.3538 E-4	3.9105 E-5	8.5875 E-7	5.3754 E-8
0.2500	2.8047 E-4	2.2820 E-5	3.0315 E-7	1.2997 E-8
0.3125	1.9333 E-4	1.0381 E-5	1.9737 E-7	4.1951 E-9
0.3750	9.6781 E-5	1.1357 E-5	1.4119 E-7	1.5299 E-8
0.4375	1.0161 E-4	8.9784 E-6	1.6239 E-7	1.4767 E-8
0.500	1.2012 E-4	6.8816 E-6	2.3785 E-7	1.0269 E-8
0.5625	1.1537 E-4	6.8278 E-6	2.0673 E-7	1.2452 E-8
0.6250	1.0561 E-4	1.0505 E-6	1.3352 E-7	1.4616 E-8
0.6875	7.0710 E-5	9.1550 E-6	2.0449 E-7	1.2901 E-8
0.7500	9.5995 E-5	6.0418 E-6	2.5585 E-7	1.1463 E-8
0.8125	1.4199 E-4	6.3528 E-6	2.2374 E-7	1.5682 E-8
0.8750	1.1942 E-4	1.2261 E-5	2.2730 E-7	5.8400 E-9
0.9375	7.4179 E-5	8.711 E-6	2.8581 E-7	2.0667 E-8

Table 3: Errors of the present method using OBBP method for test (3)

	LMW Collocation Method [3]			Chebyshev CF [13]	Present Method
t	$N = 8$	$N = 16$	$N = 32$	$N = 8$	$N = 8$
0.0625	7.4383 E-5	4.6240 E-6	1.2106 E-5	2.2070 E-8	2.5266 E-10
0.1250					1.2593 E-10
0.1875	7.5155 E- 5	1.2275 E-5	2.4685E-5	1.1514 E- 9	4.6585 E-11
0.2500					3.1922 E-11
0.3125	1.4643 E-4	2.5696 E-5	3.5745 E- 5	4.8570 E-8	2.8864 E-11
0.3750					1.3091 E-11
0.4375	7.5929 E-5	4.2169 E-5	4.5563 E-5	1.4616 E- 9	1.2449 E-11
0.500					1.0695 E-11
0.5625	1.2180 E-4	6.0743 E-5	5.3926 E- 5	1.7855 E-9	1.2261 E-11
0.6250					9.3796 E-12
0.6875	1.0567 E-4	8.1933 E-5	6.0499 E- 5	1.0870 E-7	6.6454 E-12
0.7500					1.3368 E-11
0.8125	4.7215 E- 5	1.0738 E- 4	6.4915 E-5	5.3619 E- 9	1.1561 E-11
0.8750					1.5320 E-11
0.9375	2.1869 E-4	1.3833 E-4	6.6396 E-5	3.8717 E-7	1.2016 E-11

Figure 2: Exact (left) and approximate (right) solutions for example 3 for ($N = M = 6$).
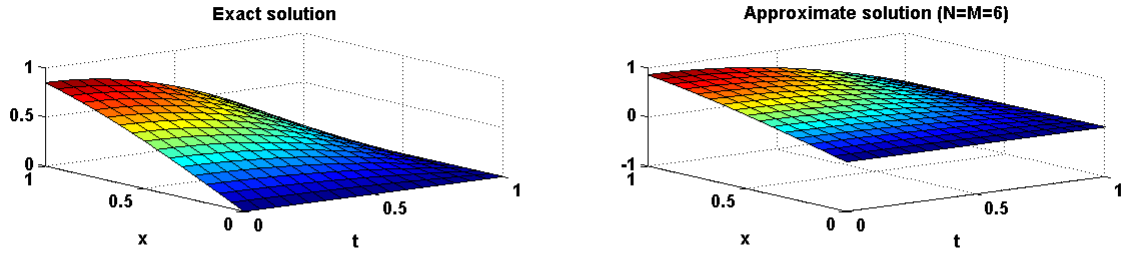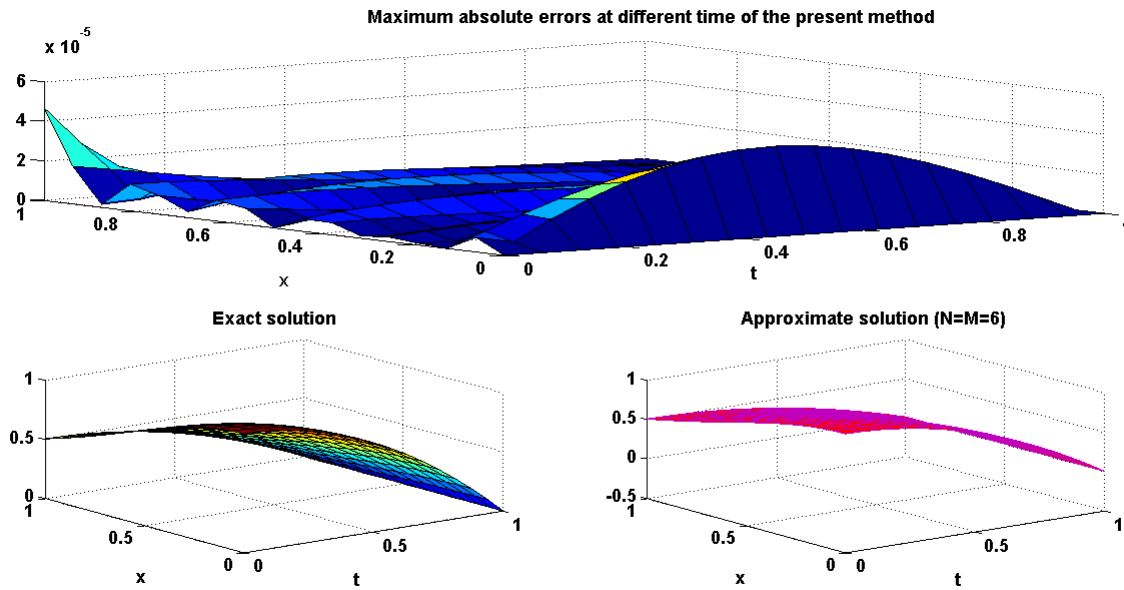

Figure 3: Errors, Exact (left) and approximate (right) solutions for example 4 for ($N = M = 6$).

14

Example 5. *Le given a linear problem as follows [3]*

$$u_t(x,t) = u_{xx}(x,t) + g(x,t) - \int_0^t u(x,s)ds, \quad x,t \in [0,1], \tag{42}$$

subject to the following conditions: $u(x,0) = \dfrac{1-x^2}{2}$, $u(0,t) = \dfrac{\cosh(t)}{2+\sinh^2(t)}$ *and* $u(1,t) = 0$, *where* $g(x,t)$ *is determined such that the analytical solution is* $u(x,t) = \dfrac{(1-x^2)\cosh(t)}{2+\sinh^2(t)}$. *The results for this example are given in tables 4-5 and figure 4. The numerical experiments obtained for this example are better than that given by Legendre multi-wavelets collocation method [3].*

Table 4: Errors of the present method using OBBP method for test (5)

t	N=4	N=5	N=8
0.0625	2.3407E-4	5.1581 E-6	3.1127 E-7
0.1250	2.7731 E-4	3.7329 E-6	1.5271 E-7
0.1875	2.2190 E-4	2.1161 E-6	5.2969 E-8
0.2500	1.2981E-4	1.5113 E-6	4.0246 E-8
0.3125	5.9121 E-5	1.3529 E-6	4.0669 E-8
0.3750	6.5468E-5	9.7856 E-7	1.8271 E-8
0.4375	5.3368 E-5	2.8124 E-7	1.6847 E-8
0.500	4.0608 E-5	1.0203 E-6	1.7874 E-8
0.5625	4.0269E-5	1.6929 E-6	1.7609 E-8
0.6250	6.2131 E-5	1.3648 E-6	1.7117 E-8
0.6875	5.4164 E-5	7.2680 E-7	1.0399 E-8
0.7500	3.7198 E-5	1.8706 E-6	2.1713 E-8
0.8125	3.7310 E-5	2.2765 E-6	1.8944 E-8
0.8750	7.1458E-5	1.6466 E-6	2.6507 E-8
0.9375	4.7623 E-5	2.8961 E-6	2.1335 E-8

Example 6. *Here, we take the following PIDE*

$$u_t(x,t) + u_{xx}(x,t) = g(x,t) - \int_0^t e^{s-t}u(x,s)ds, \tag{43}$$

15

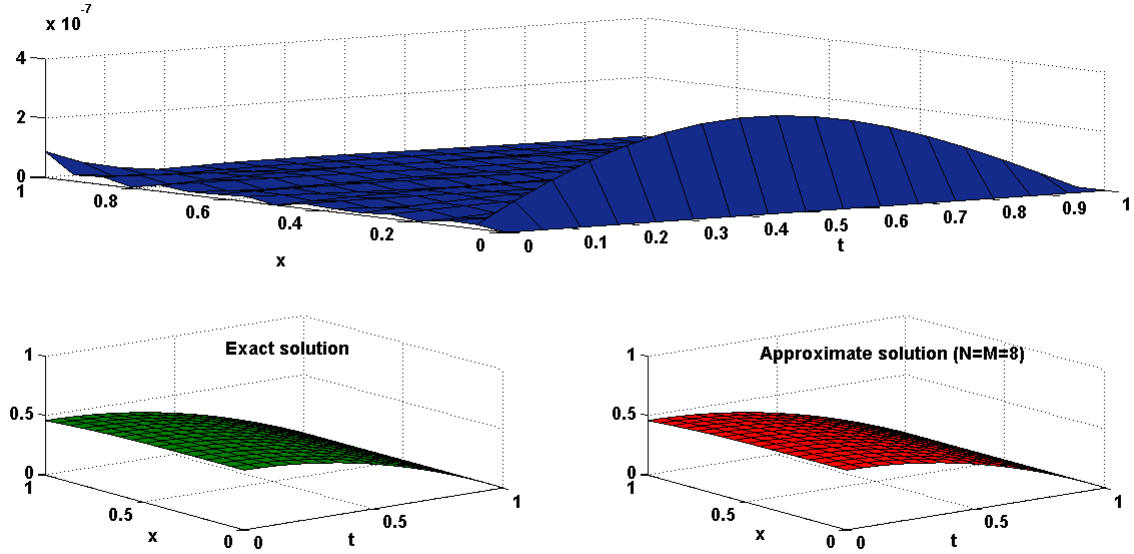Figure 4: Errors, Exact (left) and approximate (right) solutions for example 5 for ($N = M = 8$).

Table 5: Errors using OBBP method for test (5)

	Legendre multiwavelets Method [3]			Present Method
t	$N = 8 \times 8$	$N = 16 \times 16$	$N = 64 \times 64$	Present method N=8
0.1	1.8049 E-6	9.2110 E-7	1.1342 E-7	2.2094 E-7
0.2	1.3464 E-5	2.3295 E-6	4.3291 E-8	4.5662 E-8
0.3	7.2956 E-5	1.9806 E-6	1.8598 E-7	4.2211 E-8
0.4	4.4007 E-5	1.7830 E-5	5.6575 E-7	9.7235 E-9
0.5	3.7671 E-4	7.7404 E-5	4.0517 E-6	1.7920 E-8
0.6	4.5192 E-5	9.5238 E-6	1.0850 E-6	1.7605 E-8
0.7	2.3648 E-5	2.0197 E-5	1.1987 E-6	7.2310 E-9
0.8	8.3275 E-5	2.3185 E-5	1.5333 E-6	2.0556 E-8
0.9	1.0790 E-4	1.1975 E-5	7.5777 E-7	1.8646 E-8
1.0	3.4302 E-4	8.6424 E-5	4.5429 E-6	2.5563 E-8

16

with $u(x, 0) = x$, $u(0, t) = 0$, $x \in [0, 1]$, $u(1, t) = e^{-t}$, $t \in [0, 1]$, where $g(x, t)$ is determined such that $u(x, t) = x \exp(-xt)$ is the analytical solution. We remark that when N increases, the error decreases. The errors obtained by our method for $N = 10$ are presented in 5 and gives a better results than that given by Hermite-Taylor matrix method for $N = 12$ [8] and radial basis functions $N = 40$ [33].

Figure 5: Errors, Exact (left) and approximate (right) solutions for example 6 for ($N = M = 10$).
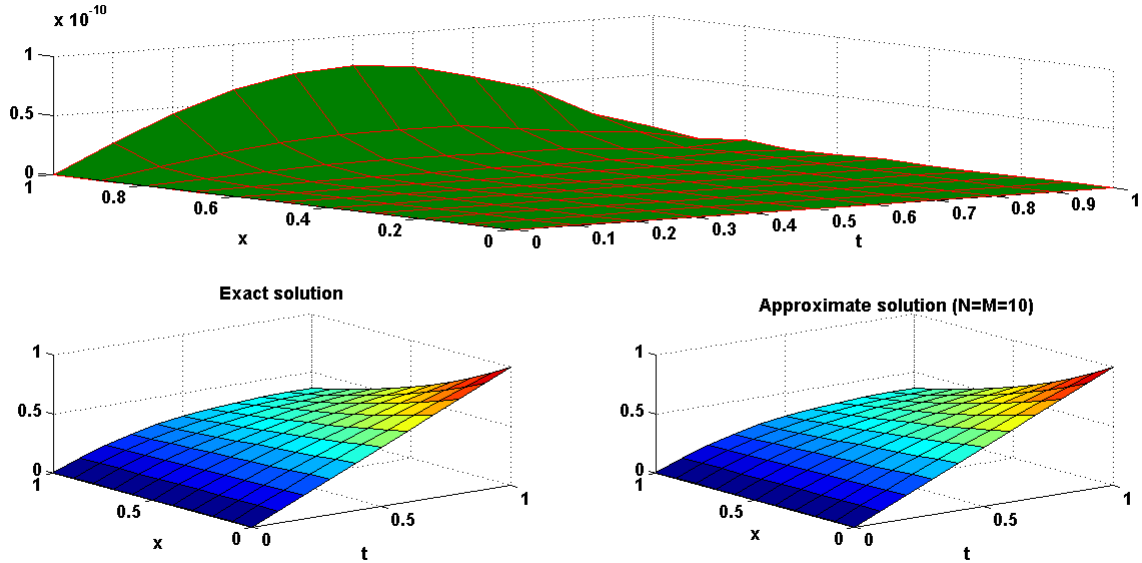
6. Conclusion

In this article, a new numerical approach was proposed. This approach was utilized to solve partial integro-differential equations with Volterra and Fredholm types. The matrices of orthonormal Bernoulli polynomials were derived and used to obtain the approximate solution of PIDEs. After we take Gauss-Legendre nodes in the intervals $[0, b]$ and $[0, T]$ as collocation points. The approach was applied to obtain numerical solutions of some test problems. The numerical results show the high accuracy of the scheduled algorithm. The presented method is easily implementable and simple and can be used for different types of PIDEs and also for differential equations. Many test problems were inserted and compared with other algorithms to appreciate the good efficiency of the proposed methodology. The proposed algorithm can be employed to more dimensions.

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