Commentary

E=mc² Is Not a Relativistic Formula

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The mass-energy formula $E = mc^2$ is thought to be derived by Einstein from special relativity. The present study shows that Maxwell's electromagnetic momentum $P = E/c$ and the Newtonian momentum $P = mv$ imply this formula. It can be derived from classical physics with *c* as the constant velocity of light in its medium, ether. The present study demonstrates that this classical physics-based formula is also correct in other inertial frames that move relative to the ether frame. In contrast, Einstein's derivation in 1905 is logically flawed as a relativistic proof because 1) it ignored that the difference rather than the sum of the emitted energy between the opposite directions affects the kinetic energy of the emitting object and made incorrect assumptions; 2) its mass and energy are measured in different reference frames whereas the mass-energy equivalence should be for mass and energy measured in the same reference frame; 3) its result is an approximation and valid only at low velocity whereas the term relativistic usually means "also correct at high velocity." Einstein's nonrelativistic derivation in 1946 is incorrect from a relativistic point of view because it ignores the relativistic effects in the moving (observed) frame. It is unnecessary from a classical point of view because it uses the two classical equations $P = mv$ and $P = E/c$, from which $E = mc^2$ can be obtained directly. Therefore, $E = mc^2$ is a classical rather than a relativistic formula. The relativistic formula that Einstein should have derived from his thought experiments is $E = E_0/\sqrt{1 - v^2/c^2} = m_0c^2/\sqrt{1 - v^2/c^2}$ derived by Laue and Klein, which corresponds to the relativistic mass-velocity equation derived by Lorentz.

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1. Introduction

The mass-energy formula $E = mc^2$ has a prominent role in physics research and public perception of science. The formula explains the power of nuclear bombs and the energy source of stars[1][2][3] and stimulates the general public's imagination. It also underlies key components of the Dirac equation, which has accounted for the fine details of the hydrogen spectrum and implied the existence of antimatter[4]. Although Einstein[5] initially derived mass-energy equivalence as a velocity-dependent approximation, the formula's accuracy has been confirmed by experiments to a high level of precision[6].

The explicit expression $E = mc^2$ was first proposed by Planck[7][8][9] or De Pretto[10], but it is generally believed that Einstein[5] derived the mass-energy formula $E = mc^2$ from special relativity. Fernflores[11] asserts in *Stanford Encyclopedia of Philosophy*: "Einstein correctly described the equivalence of mass and energy as 'the most important upshot of the special theory of relativity'[12], for this result lies at the core of modern physics". Although there are still some disputes on whether Einstein discovered the mass-energy equation first and whether Einstein's derivation might be logically flawed[13][14][15][16][17], few people question whether the mass-energy equation is a relativistic result.

It has long been known that the mass-energy equation appears to be implied in Maxwell's electromagnetic theory[18][19][20], and Lewis has provided a derivation within the framework of classical physics[21][22]. Einstein[23] also gave a nonrelativistic derivation of $E = mc^2$. Since the mass-energy equation might be derived within the framework of classical physics, it could be a classical physics result rather than a relativistic one. This study proves that $E = mc^2$ is a classical physics formula. The relevant relativistic formula should be $E = E_0/\sqrt{1 - v^2/c^2}$. Even though Einstein's first derivation only obtained an approximate mass-energy relationship at low velocity, it is logically invalid because it made incorrect assumptions, and the obtained relationship is between mass measured in one reference frame and energy measured in another reference frame. The correct mass-energy equivalence should be for mass and energy measured in the same reference frame.

The rest of this paper is structured as follows: Section 2 shows that $E = mc^2$ is a natural result of Maxwellian electromagnetism and Newtonian mechanics; Section 3 analyzes the logical validity of Einstein's first derivation of the mass-energy relation in 1905; Section 4 demonstrates what Einstein should have derived is $E = E_0/\sqrt{1 - v^2/c^2}$; Section 5 examines the logical validity of the last derivation by Einstein[23] and provides a logically more consistent corresponding derivation; Section 6 concludes.

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2. Derivation from $\mathbf{P = E/c}$ and $\mathbf{P = mv}$ in classical physics

Many derivations, including Einstein's derivation in 1946, rely on the formula $P = E/c$ of classical electromagnetic theory. $P = E/c$ was first derived as a classical formula from Maxwell's classical theory of electromagnetism[24]. This expression has also been obtained through applying laws of thermodynamics[25][26][27][28]. Nichols and Hull[29] experimentally demonstrated the radiation pressure. In classical physics, light is a type of electromagnetic wave propagating in its medium, ether, and *c* is the constant velocity of light in its medium (frame). The formula $P = E/c$ describes the momentum of wave packets in the ether frame in Maxwell's classical electromagnetic theory. Before the Michelson-Morley experiment[30] and the Lorentz ether theory[31][32], electromagnetism, like mechanical waves in their media, was considered covariant under Galilean transformations. After the advent of special relativity, physicists reinterpreted the meaning of *c*, making it the speed of light in any inertial frames rather than only in the ether frame. This reinterpretation of *c* does not invalidate $P = E/c$ in the ether frame as a classical physics formula.

Since $P = E/c$ in the ether frame is a classical physics formula and we also have $P = mv$ in Newtonian mechanics, we can obtain $E = mc^2$ for light wave packets in their ether frame directly. The velocity of light wave packets is *c*, so $m = P/v = P/c$ in the ether frame, and Maxwell's electromagnetic momentum $P = E/c$ implies

$$m = \frac{P}{v} = \frac{E/c}{c} = \frac{E}{c^2},$$

$$E = mc^2 \tag{1}$$

The above derivation was first formulated by Lewis[21]. This explains why Preston[33], Poincaré[20], De Pretto[10], and Hasenöhrl[34] had proposed or derived similar mass-energy relations well before Einstein postulated the constancy of the speed of light. Becquerel used the conversion of mass into energy to explain the radioactive energy of radium in 1900, and the conversion ratio that he used is in the same order of magnitude as the mass-energy equation[35]. Rutherford[36] and Soddy[37] also proposed the conversion of mass into energy as a source of radioactive energy before special relativity.

As $E = mc^2$ for light wave packets in their ether frame is implied in classical physics, we can ask what the relationships between mass and energy in other reference frames should be. Following the design of the Michelson-Morley experiment, we can consider first the scenario where the direction of light rays is perpendicular to the direction of the velocity of the reference frame in question. Here the

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velocity of the reference frame is relative to the ether frame, as in the Michelson-Morley experiment. Since in classical physics the velocity of light follows the Huygens principle, we have the velocity of light when the direction of light rays is perpendicular to the direction of the velocity of the reference frame c_N,

$$c_N = \frac{d}{t} = \sqrt{c^2 - v^2} \tag{2}$$

In equation (2), *d* is the length of the light path, *t* is the time interval needed for the light ray to cover the length *d*, and *v* is the velocity of the reference frame relative to the ether frame. The two-way velocity of light is used here for convenience, and the actual impact of the frame's velocity on momentum depends on the one-way velocity of light.

Using the classical momentum formula $P = mv$, we obtain the momentum of light wave packets when the direction of light rays is perpendicular to the direction of the velocity of the reference frame,

$$P_N = mc_N = m\sqrt{c^2 - v^2} = mc\sqrt{1 - \frac{v^2}{c^2}} = P\sqrt{1 - v^2/c^2} \tag{3}$$

We may draw an analogy from the influence of the frame velocity on kinetic energy from the classical kinetic energy formula $K = \frac{1}{2}mv^2$. As energy is proportional to the square of velocity while momentum is proportional to the velocity, for a velocity change from *c* to $\sqrt{c^2 - v^2}$ when the direction of light rays is perpendicular to the direction of the velocity of the reference frame, we have the energy of light wave packets

$$E_N = E\left(\frac{c^2 - v^2}{c^2}\right) \tag{4}$$

In equations (3) and (4), *m* and *E* are the mass and the energy implied by the momentum of light wave packets in the ether frame respectively, and P_N and E_N are the momentum and energy of the light wave packets in the frame moving relative to the ether frame at *v* respectively. If we use the values of momentum, energy, and velocity of light measured in this frame, and $P_N = E_N/c_N$, we obtain the relationship between mass and energy

$$mc_N = E_N/c_N$$

$$E_N = mc_N{}^2 \tag{5}$$

When the direction of light rays is parallel to the direction of the velocity of the reference frame, the two-way velocity of light measured by the moving frame is

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$$c_P = \frac{d}{t} = \frac{c^2 - v^2}{c} \tag{6}$$

We have

$$P_P = m\frac{c^2 - v^2}{c} = mc\left(1 - \frac{v^2}{c^2}\right) = P\left(1 - \frac{v^2}{c^2}\right) \tag{7}$$

For a velocity change from c to $\left(c^2 - v^2\right)/c$, we have

$$E_P = E\frac{\left(c^2 - v^2\right)^2}{c^4} \tag{8}$$

If we use the values of momentum, energy, and velocity of light measured in this frame, and $P_P = E_P/c_P$, we obtain the relationship between mass and energy

$$mc_P = E_P/c_P$$

$$E_P = m{c_P}^2 \tag{9}$$

Therefore, in classical physics, $E = mc^2$ is true in all inertial reference frames.

3. Einstein's derivation in 1905 is logically flawed as a relativistic proof

Einstein's first derivation links the mass-energy equation with special relativity[5]. The derivation is based on a thought experiment unlikely to be achievable in a laboratory[16][15]. Its key part is quoted here.

> "Let a system of plane waves of light, referred to the system of co-ordinates (*x*, *y*, *z*), possess the energy *L*; let the direction of the ray (the wave-normal) make an angle $\varphi$with the axis of *x* of the system. If we introduce a new system of co-ordinates (ξ, η, ζ) moving in uniform parallel translation with respect to the system (*x*, *y*, *z*), and having its origin of co-ordinates in motion along the axis of *x* with the velocity *v*, then this quantity of light—measured in the system (ξ, η, ζ)—possesses the energy
>
> $$L^* = L\frac{1 - \frac{v}{c}\cos\varphi}{\sqrt{1 - v^2/c^2}} \tag{10}$$
>
> where *c* denotes the velocity of light. We shall make use of this result in what follows.

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Let there be a stationary body in the system (x, y, z), and let its energy—referred to the system (x, y, z) be E_0. Let the energy of the body relative to the system (ξ, η, ζ) moving as above with the velocity v, be H_0.

Let this body send out, in a direction making an angle $\varphi$with the axis of x, plane waves of light, of energy ½L measured relatively to (x, y, z), and simultaneously an equal quantity of light in the opposite direction. Meanwhile, the body remains at rest with respect to the system (x, y, z). The principle of energy must apply to this process, and in fact (by the principle of relativity) with respect to both systems of co-ordinates. If we call the energy of the body after the emission of light E_1 or H_1 respectively, measured relatively to the system (x, y, z) or (ξ, η, ζ) respectively, then by employing the relation given above we obtain

$$E_0 = E_1 + \frac{1}{2}L + \frac{1}{2}L \tag{11}$$

$$H_0 = H_1 + \frac{1}{2}L\frac{1-\frac{v}{c}\cos\varphi}{\sqrt{1-v^2/c^2}} + \frac{1}{2}L\frac{1+\frac{v}{c}\cos\varphi}{\sqrt{1-v^2/c^2}} = H_1 + \frac{L}{\sqrt{1-v^2/c^2}} \tag{12}$$

By subtraction, we obtain from these equations

$$H_0 - E_0 - (H_1 - E_1) = L\left(\frac{1}{\sqrt{1-v^2/c^2}} - 1\right). \tag{13}$$

The two differences of the form $H - E$ occurring in this expression have simple physical significations. H and E are energy values of the same body referred to two systems of coordinates which are in motion relatively to each other, the body being at rest in one of the two systems (system (x, y, z)). Thus, it is clear that the difference $H - E$ can differ from the kinetic energy K of the body, with respect to the other system (ξ, η, ζ), only by an additive constant C, which depends on the choice of the arbitrary additive constants of the energies H and E. Thus, we may place

$$H_0 - E_0 = K_0 + C \tag{14}$$

$$H_1 - E_1 = K_1 + C \tag{15}$$

since C does not change during the emission of light."

Equations (14) and (15) are the key in Einstein's derivation, which is equivalent to a statement that (the change in) non-kinetic energy has the same value in all reference frames, i.e., the difference in energy values of an object measured in two reference frames is only the difference in its values of kinetic energy. This assertion by Einstein has been a major source of controversy regarding the validity of Einstein's derivation in 1905. Ives[13], Jammer[14], and Arzeliès and Tordjman[38] think that the mass-energy equation is implied by equations (14) and (15); without justifying them, Einstein's derivation is invalid. Stachel and Torretti[39] and Ohanian[16] think using equations (14) and (15) is not a *petitio principii*. From equations (14) and (15), Einstein derived an approximate mass-energy equivalence. In Einstein's thought experiment, the system (x, y, z) is the observed frame, and the system (ξ, η, ζ) is the observing frame[40][17][41].

> "So we have
>
> $$K_0 - K_1 = L\left(\frac{1}{\sqrt{1 - v^2/c^2}} - 1\right) \tag{16}$$
>
> The kinetic energy of the body with respect to (ξ, η, ζ) diminishes as a result of the emission of light, and the amount of diminution is independent of the properties of the body. Moreover, the difference K_0 - K_1, like the kinetic energy of the electron (§ 10), depends on the velocity.
>
> Neglecting magnitudes of fourth and higher orders, we may place
>
> $K_0 - K_1 = \frac{1}{2}\frac{L}{c^2}v^2$."[5] (17)

Equation (16) is a logical consequence of (14) and (15), which states that $K_0 - K_1$, the difference of an object's kinetic energy measured at two time points in a reference frame H, equals the difference between the change of total energy measured in frame H (i.e. $H_0 - H_1$) and that measured in the object-stationary frame E (i.e. $E_0 - E_1$) at these two time points. The right-hand side of equation (17) approximates the right-hand side of equation (16) at low velocity, which gives an appearance of the classical expression of kinetic energy. From this approximation, Einstein concluded that "if a body gives off the energy L in the form of radiation, its mass diminishes by L/c^2."

Although Ohanian[16] agrees with Stachel and Torretti[39] that Einstein's derivation is not a *petitio principii*, he thinks that Einstein's conclusion is a *non sequitur*. "Einstein's mistake lies in an unwarranted extrapolation: he assumed that the rest-mass change he found when using a nonrelativistic, Newtonian approximation for the internal motions of an extended system would be equally valid for relativistic motion." Ohanian's criticism seems pertinent. When v is larger, such as $v = 0.8c$, magnitudes of fourth and higher orders cannot be neglected. So $E = mc^2$ derived implicitly by Einstein in 1905 is only an approximation when v is relatively small (nonrelativistic); it is not a universal relation applicable to objects at all velocities. Einstein[1] acknowledged the imprecision of his mass-energy equation by noting that "It is customary to express the equivalence of mass and energy (though somewhat inexactly) by the formula $E = mc^2$". However, as we know now, the formula is fairly precise[6], so Einstein's velocity-dependent approximation could not be accepted as a correct derivation of the mass-energy equivalence.

Besides the logic issue of *non-sequitur*, Einstein also made a subtle but fatal mistake in assuming equations (14) and (15), which has not been identified until now. Equations (11) and (12) can be rewritten as

$$\Delta E = E_0 - E_1 = \frac{1}{2}L + \frac{1}{2}L = L, \tag{11a}$$

$$\Delta H = H_0 - H_1 = \frac{1}{2}L\frac{1 - \frac{v}{c}\cos\varphi}{\sqrt{1 - v^2/c^2}} + \frac{1}{2}L\frac{1 + \frac{v}{c}\cos\varphi}{\sqrt{1 - v^2/c^2}} = \frac{L}{\sqrt{1 - v^2/c^2}}. \tag{12a}$$

Einstein's equations (14) and (15) stipulate $\Delta K = \Delta H - \Delta E = L/\sqrt{1 - v^2/c^2} - L$, which is incorrect but deceptively difficult to recognize due to the nature of kinetic energy. We will dissect this mistake.

Suppose the object emits a pulse of light waves with energy L/2 only in one direction. In that case, it will gain kinetic energy in the system (x, y, z), and E_1 will include non-kinetic and kinetic energy. However, when it also emits a pulse of light waves with energy L/2 in the opposite direction, there is no gain in kinetic energy, and E_1 includes no kinetic energy. Therefore, what affects the kinetic energy of an object is the difference of emitted energy between the opposite directions. When there is no difference, there is no change in the object's kinetic energy. This also applies to the system (ξ, η, ζ). When the object emits two pulses of light waves with energy $\frac{L/2}{\sqrt{1-v^2/c^2}}$ in opposite directions in the system (ξ, η, ζ), there is no change in the object's kinetic energy in the system (ξ, η, ζ) because there is no difference in emitted energy between the opposite directions. Therefore, it is incorrect to assume by Einstein that the change in kinetic energy in system (ξ, η, ζ) is due to the difference in the emitted total energy between systems (x, y, z) and (ξ, η, ζ).

Since the change in kinetic energy is caused by the difference in emitted energy between the opposite directions, we have when $\varphi = 0$,

$$\Delta K = K_0 - K_1 = \frac{1}{2}L\frac{1 + \frac{v}{c}}{\sqrt{1 - \frac{v^2}{c^2}}} - \frac{1}{2}L\frac{1 - \frac{v}{c}}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{\frac{v}{c}L}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{18}$$

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As the electromagnetic kinetic energy $K = Pc$, we have

$$\Delta P = \frac{\frac{v}{c^2} L}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{19}$$

The change ΔP is both the momentum carried away by the light waves and the loss of momentum by the object in the system (ξ, η, ζ). Since $P = mv$ and the emission of light waves has not changed the velocity of the object in the system (ξ, η, ζ), we obtain from equation (19)

$$\Delta m = \frac{L/c^2}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{20}$$

Hence, a change in its mass $\Delta m = \frac{L/c^2}{\sqrt{1 - v^2/c^2}}$ causes the change in its momentum. Let $\Delta m_0 = L/c^2$, we have

$$\Delta m = \frac{\Delta m_0}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{21}$$

This is the Lorentz mass-velocity formula. Multiplying both sides of equation (20) by c^2, we obtain

$$L^* = \Delta m c^2 = \frac{L}{\sqrt{1 - \frac{v^2}{c^2}}} \text{ or } E = \frac{E_0}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{22}$$

As indicated by equation (22), if an object's energy is E_0 measured by a reference frame in which the object is stationary (the observed frame), its energy is $E = \frac{E_0}{\sqrt{1 - v^2/c^2}}$ measured by a frame moving at a velocity of v (the observing frame) relative to the object and the first frame. This is the relativistic formula that Einstein should have obtained with two reference frames. We have used the classical expression $P = mv$, $P = E/c$, or $E = Pc$, which implies $E = mc^2$, in deriving equations (19)-(22). Special relativity contributes $E = \frac{E_0}{\sqrt{1 - v^2/c^2}}$ but not $E = mc^2$.

4. Derivation of the mass-energy equation from conservation of momentum

Einstein's equations (14) and (15) are among the main controversial points regarding the validity of Einstein's derivation[13][14][38]. The two equations ignore that the difference in the emitted energy between the opposite directions affects the kinetic energy of the emitting object, so they are incorrect in Einstein's derivation. We have derived the relativistic formula $E = \frac{E_0}{\sqrt{1 - v^2/c^2}}$ in the preceding section using that fact and Einstein's thought experiment. Since some researchers might doubt the validity of equation (18), we use the conservation of momentum to derive the mass-energy relation from Einstein's thought experiment to prove our current result further. In the frame (x, y, z) where the radiating body is at rest, we have

$$P_{S0} = P_{S1} + \frac{E_S}{2c} - \frac{E_S}{2c} = P_{S1} = 0 \tag{23}$$

In equation (23), P stands for momentum, the subscript S indicates the frame where the radiating body is stationary; $\frac{E}{2c}$ is the momentum of a light wave packet in one direction (as in Maxwell's classical electromagnetic theory, here Einstein's L is replaced with the more conventional E for energy).

In the frame (ξ, η, ζ) where the radiating body is moving at velocity v,

$$P_{V0} = P_{V1} + \frac{E_S}{2c}\frac{1+\frac{v}{c}\cos\varphi}{\sqrt{1-v^2/c^2}} - \frac{E_S}{2c}\frac{1-\frac{v}{c}\cos\varphi}{\sqrt{1-v^2/c^2}} = P_{V1} + E_S\frac{\frac{v}{c^2}\cos\varphi}{\sqrt{1-v^2/c^2}} \tag{24}$$

In equation (24), the subscript V indicates the moving frame. When $\varphi = 0$,

$$\Delta P_V = P_{V0} - P_{V1} = \frac{\frac{v}{c^2}E_S}{\sqrt{1-v^2/c^2}} \tag{25}$$

Since $P_V = m_V v = m_S v/\sqrt{1-v^2/c^2}$ and there is no velocity change of the object in the frame (ξ, η, ζ), we have

$$\Delta m_V = \frac{\frac{E_s}{c^2}}{\sqrt{1-v^2/c^2}}, \tag{26}$$

Equation (26) is the same as equation (20)

$$\Delta m_S = E_S/c^2. \tag{27}$$

In the frame where the radiating body is stationary, when energy E is emitted, there is a loss of mass $\Delta m = E/c^2$. This mass-energy equivalence in the same reference frame is exact rather than approximate. Except for using the relativistic mass formula to remove $\sqrt{1-v^2/c^2}$ from equation (25), this is the same derivation as equation (1) with Maxwell's electromagnetic momentum and Newtonian momentum.

From equation (25) and $P_V = m_V v$, we can also obtain

$$\Delta m_V c^2 = E_S/\sqrt{1-v^2/c^2}$$

Let $\Delta m_V c^2 = E_V$, we obtain

$$E_V = E_S/\sqrt{1-v^2/c^2} = \Delta m_S c^2/\sqrt{1-v^2/c^2}. \tag{28}$$

Equation (28) is the same as equation (22), the relativistic formula describing the relationship between values of the same energy measured in two reference frames, which depends on their relative velocity *v*.

If we use subscript 0 to indicate measurements obtained in the frame where the radiating body is stationary, our new derivation reveals what Einstein should have proved is the equation derived by Laue[42] and Klein[43],

$$E = E_0/\sqrt{1-v^2/c^2}. \tag{29}$$

Equation (29) corresponds to the relativistic mass equation[32]

$$m = m_0/\sqrt{1-v^2/c^2}.$$

The essence of Einstein's derivation in 1905 is to approximate $\Delta E = E - E_0$ i.e., the measured energy difference between moving and stationary frames, based on equation (29) and the classical kinetic energy expression $K = \frac{1}{2}mv^2$,

$$E - E_0 = \frac{E_0}{\sqrt{1-v^2/c^2}} - E_0 = E_0(\frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \cdots) \approx \frac{1}{2}\frac{E_0}{c^2}v^2. \tag{30}$$

Einstein's approximation works only when *v* is small, whereas all relativistic functions should work well at high velocity.

Similarly, expanding the difference between the relativistic mass and the rest mass and using the classical kinetic energy expression $K = \frac{1}{2}mv^2$ can give the same relationship when *v* is small,

$$m - m_0 = \frac{m_0}{\sqrt{1-v^2/c^2}} - m_0 = m_0(\frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \cdots) \approx \frac{1}{2}\frac{m_0 v^2}{c^2} = \frac{E_0}{c^2} \tag{31}$$

However, both equation (31) and Einstein's derivation in 1905 need a classical kinetic energy formula and low velocity, which are a *non sequitur* to a relativistic conclusion[16]. Besides the *non sequitur* issue and the fatal mistake examined in the preceding section, there is another logical mistake in Einstein's derivation by taking approximation, which researchers have overlooked. The mass-energy equivalence should be for mass and energy measured in the same frame. Still, the mass-energy relationship derived by Einstein is between the energy measured in the object-stationary frame (observed frame) and the (change in) mass measured in the object-moving frame (observing frame). This mismatch is also logically incorrect.

Therefore, the relativistic result from Einstein's thought experiment in 1905 should be $E = E_0/\sqrt{1 - v^2/c^2}$, which is just a different expression of the relativistic mass equation $m = m_0/\sqrt{1 - v^2/c^2}$ that was first derived by Lorentz[32]. This relationship between energy values measured in two reference frames has been shown by Laue[42], using the conservation of energy-momentum tensor and assuming that there is no energy flow in the rest frame. Klein[43] extended Laue's results to closed systems with or without energy flow.

5. Einstein's derivation in 1946

Einstein[23] gave his last derivation of the mass-energy equivalence, based on the conservation of momentum and Maxwell's classical theory of electromagnetism. Since the derivation is short, its key part is quoted here (Fig.1).

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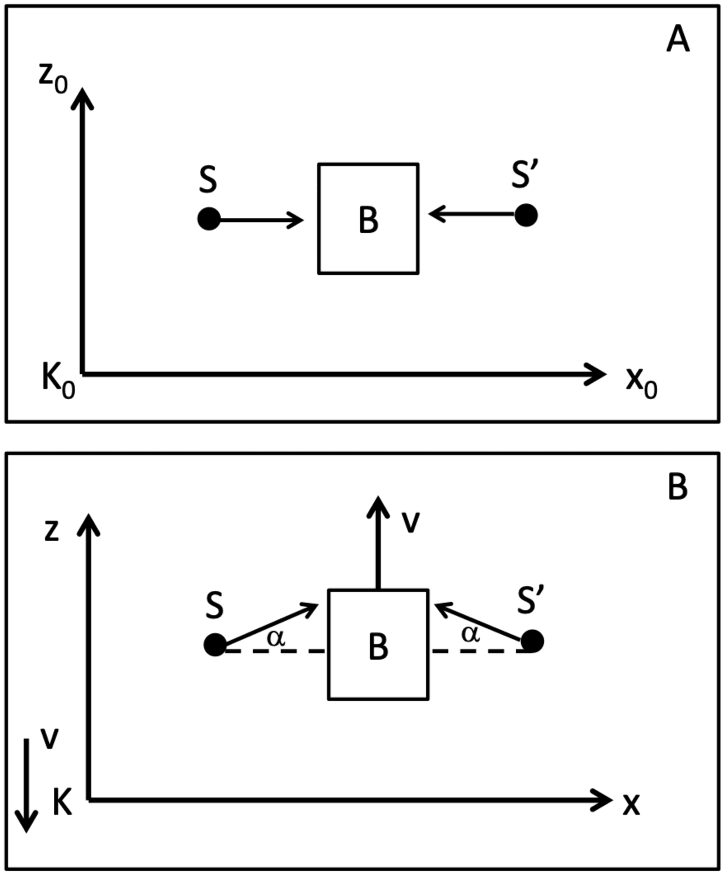

Fig.1. An object B absorbing two wave complexes (S and S') from opposite directions with energy $E/2$ each. A. Object B is at rest in frame K_0. B. In frame K, which moves along the z-axis negative direction of frame K_0 with velocity v, object B is moving in the **z**-axis positive direction with velocity v, and the two wave complexes have an angle α with the x-axis,$\sin\alpha = v/c$.

"We now consider the following system. Let the body B rest freely in space with respect to the system K_0. Two complexes of radiation S, S' each of energy $E/2$ move in the positive and negative x_0 direction respectively and are eventually absorbed by B. With this absorption the energy of B increases by E. B stays at rest with respect to K_0 by reasons of symmetry. Now we consider this same process for the system K, which moves with respect to K_0 with the constant velocity v in the negative Z_0 direction. With respect to K the description of the process is as follows:

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The body B moves in positive Z direction with velocity v. The two complexes of radiation now have directions with respect to K which make an angle α with the x axis. The law of aberration states that in the first approximation $\alpha = \frac{v}{c}$, where c is the velocity of light. From the consideration with respect to K_0 we know that the velocity v of B remains unchanged by the absorption of S and S'.

Now we apply the law of conservation of momentum with respect to the z direction to our system in the coordinate-frame K.

I. Before the absorption let m be the mass of B; mv is then the expression of the momentum B (according to classical mechanics). Each of the complexes has the energy $E/2$ and hence, by a well-known conclusion of Maxwell's theory, it has the momentum $\frac{E}{2c}$. Rigorously speaking this is the momentum of S with respect to K_0. However, when v is small with respect to c, the momentum with respect to K is the same except for a quantity of second order of magnitude ($\frac{v^2}{c^2}$compared to 1). The z-component of this momentum is $\frac{E}{2c}\sin\alpha$or with sufficient accuracy (except for quantities of higher order of magnitude) $\frac{E}{2c}\alpha$or $\frac{E}{2} \cdot \frac{v}{c^2}$. S and S' together therefore have a momentum $E\frac{v}{c^2}$ in the z direction. The total momentum of the system before absorption is therefor

$$mv + \frac{E}{c^2} \cdot v. \tag{32}$$

II. After the absorption let m' be the mass of B. We anticipate here the possibility that the mass increased with the absorption of the energy E (this is necessary so that the final result of our consideration be consistent). The momentum of the system after absorption is then

$m'v$

We now assume the law of the conservation of momentum and apply it with respect to the z direction. This gives the equation

$$mv + \frac{E}{c^2} \cdot v = m'v. \tag{33}$$

or

$$m' - m = \frac{E}{c^2}. \tag{33a}$$

This equation expresses the law of the equivalence of energy and mass. The energy increase E is connected with the mass increase $\frac{E}{c^2}$. Since energy according to the usual definition leaves an additive constant free, we may choose the latter that

$E = mc^2$." (34)

Special relativity is not involved in Einstein's derivation in 1946; hence, deriving $E = mc^2$ does not require special relativity. However, Einstein's derivation in 1946 has the shortcoming of not distinguishing different values of energy or mass measured in the two reference frames by ignoring "a quantity of second order of magnitude." In a relativistic setting, this quantity cannot be ignored. A wave complex has different energy values in two frames K_0 and K with relative motion. In equations (33) and (34), the energy values of the wave complexes are those measured in frame K_0. In contrast, the momentum and the mass are measured in frame K. Einstein introduced the same logical mistake in 1946 as in 1905, i.e., proving the equivalence of energy measured in one frame with mass measured in another frame.

Since the energy or mass of an object measured in two reference frames with relative motion has different values, we must show the equivalence of mass and energy measured in the same reference frame. Given the Lorentz mass-velocity formula $m = m_0/\sqrt{1 - v^2/c^2}$, the mass measured in frame K_0 is $m_0 = m\sqrt{1 - v^2/c^2}$. Labeling the energy measured in frame K_0 as E_0, we can write Einstein's equation (33) with mass and energy measured in the same frame as,

$$mv\sqrt{1 - v^2/c^2} + \frac{E_0}{c^2}v = m'v\sqrt{1 - v^2/c^2}. \tag{35}$$

From equation (35), we obtain

$$(m^{'} - m)c^2 = \frac{E_0}{\sqrt{1 - v^2/c^2}}, \tag{36}$$

which is not a straightforward $E = mc^2$. There is an additional term, $1/\sqrt{1 - v^2/c^2}$, on the right-hand side of the equation.

The mass-energy relationship should be between mass and energy measured in the same reference frame. Using the Lorentz relativistic mass formula to calculate the mass measured in frame K_0, m_0, we obtain the mass-energy formula for frame K_0 from equation (36)

$$E_0 = \left(m^{'} - m\right)c^2\sqrt{1 - v^2/c^2} = ({m_0}^{'} - m_0)c^2 = \Delta m_0 c^2 \tag{37}$$

Let the energy of the radiation complexes measured in frame K be $E = \left(m^{'} - m\right)c^2$, we have

$E = \frac{E_0}{\sqrt{1 - v^2/c^2}}$.

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Therefore, when we try to use the transformation between reference frames to derive the mass-energy relation, the relativistic energy formula is equation (29), i.e., what Laue[42] and Klein[43] have found. There is no need for Einstein's thought experiment in 1946 to use the transformation between reference frames to derive the mass-energy formula since the application of electromagnetic momentum $P = E/c$ and Newtonian momentum $P = mv$ by Einstein has implied $E = mc^2$ already. Using the transformation between reference frames can logically lead only to $E = E_0/\sqrt{1 - v^2/c^2}$ or $m = m_0/\sqrt{1 - v^2/c^2}$.

6. Discussions

Electromagnetic energy contributing to the mass or inertia of electrons was common knowledge or belief in the late nineteenth century. Thomson[44] first proposed that electromagnetic energy provided part of an electron's inertia or mass. Lodge[45] also investigated electric inertia. Heaviside[46] improved Thomson's theoretical calculation of electromagnetic inertia. Searle[47] derived a formula for the electromagnetic energy of a charged sphere in motion. These scholars recognized that electrostatic energy behaves as having electromagnetic mass, which increases with its velocity. Based on the electromagnetic energy origin of the mass of electrons, Lorentz[31] derived mass-velocity formulae for the longitudinal and transverse mass, which differ from the later formulae[32] by only an indeterminate coefficient. Kaufmann provided the experimental evidence of electromagnetic mass[48][49]. Abraham also developed his theory of electromagnetic mass[50][51]. Becquerel used the conversion of mass into energy to explain the source of the energy emitted by radioactivity[35]. Hasenöhrl[34] obtained a mass-energy relation $m = \frac{4}{3}\frac{E}{c^2}$ in his study of the radiation of moving objects. Many researchers considered all mass to arise from electromagnetic energy[52][53]. Therefore, the idea of mass-energy equivalence was not as radical or revolutionary as believed by people nowadays who read modern textbooks or popular science books and are not familiar with physicists' ideas and thoughts in the late nineteenth century and the beginning of the twentieth century.

Strictly speaking, Einstein's first derivation and many others are more like hat tricks to package the mass-energy equivalence implied in classical physics as a relativistic formula. The mass-energy relationship, $E = mc^2$, is implied by Newtonian momentum $P \equiv mv$ and electromagnetic momentum $P = E/c$ resulted from Maxwellian electromagnetic theory[24] and thermodynamics[25]

[26][27][28]. If $m = P/v$, we can obtain the mass-energy equation directly from $P \equiv mv$ and electromagnetic momentum $P = E/c$. The electromagnetic energy contained in an object provides mass $m = E/c^2$; when a material object with mass *m* is converted completely into electromagnetic energy, the total energy released is $E = mc^2$. In contrast, Einstein's "relativistic" derivation using two reference frames is more suited to examine the relationship between the values of the same variable measured in two frames with a relative velocity, such as a mass in the rest frame and the moving frame. Understandably, Einstein's result is velocity-dependent (i.e., valid only at low velocity) because the key difference between two inertial frames is their relative velocity.

The Lorentzian ether theory, especially Einsteinian special relativity, deals with the relationship between a variable's values measured in two reference frames rather than between two variables within one reference frame. The mass-energy equivalence is an issue within the same reference frame instead of one across two reference frames, so it is not relativistic. Einstein packaged it as an issue between two reference frames, which unavoidably led to logical mistakes and resulted in a velocity-dependent mass-energy relation. Although $P = mv$ and $P = E/c$ in classical physics and $E = E_0/\sqrt{1 - v^2/c^2}$ in special relativity have been known to physicists for a long time, most physicists and the general public still strongly believe that $E = mc^2$ is an exclusively relativistic result, overlooking the deep-rooted connection of the mass-energy equation with classical physics. Therefore, establishing the true identity of $E = mc^2$ is not only important in physics but also philosophically and historically significant.

From the present study, we may draw the following conclusions:

First, the mass-energy equation $E = mc^2$ is contained in Maxwell's classical electromagnetic theory and the momentum definition of Newtonian mechanics. With the momentum definition in Newtonian mechanics $P \equiv mv$ and Maxwell's electromagnetic momentum $P = E/c$, the mass-energy equation $E = mc^2$ should be a logical consequence.

Second, all logically valid derivations of $E = mc^2$, where both mass *m* and energy *E* are measured in the same reference frame, rely on the two classical equations $P \equiv mv$ and $P = E/c$. No matter whether a derivation is under classical or relativistic conditions, the two equations must be held true. If the two equations are denied in any of those derivations, it is not possible to arrive at $E = mc^2$ logically. If these two equations are held true, the mass-energy equation $E = mc^2$ can be obtained directly without the special scenarios assumed for those derivations.

Third, $E=mc^2$ is a classical physics formula since it can be derived without resorting to relativistic results.

Fourth, Einstein's "relativistic" derivation in 1905 relies on the unjustified assertion of $H_0 - E_0 = K_0 + C$ and $H_1 - E_1 = K_1 + C$, and ignored that only the difference of the emitted energy between the opposite directions affects the kinetic energy of the object. Therefore, his derivation is logically invalid.

Fifth, the mass-energy equivalence should be for mass and energy measured in the same reference frame rather than in different reference frames. Einstein's "relativistic" derivation leads only to an approximation at low velocity for a velocity-dependent equation and an equivalence between energy and mass measured in different reference frames, demonstrating further that it is not a logically valid derivation.

Sixth, Einstein's nonrelativistic derivation in 1946 is incorrect from a relativistic point of view because it ignores the relativistic effects in the moving frame. It is unnecessary from a classical physics point of view because it uses the two classical equations, $P = mv$ and $P = E/c$, from which $E = mc^2$ can be obtained directly.

Seventh, the logically valid result of relating energy and mass measured in two reference frames is the relativistic transformation of energy between two reference frames $E = E_0/\sqrt{1-v^2/c^2}$, corresponding to the relativistic transformation of mass between two reference frames $m = m_0/\sqrt{1-v^2/c^2}$.

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