On the Impact of Technology on University Analysis

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Dynamic Mathematics Systems provide a way to make mathematical objects such as sequences and functions accessible to students, and therefore one may hope that they can improve the learning even of advanced mathematics as taught in university-level analysis courses. During a course on analysis, students were given additional exercises with GeoGebra, and it was also used for demonstrations in the lecture. This paper reports both on the concept and context, as well as on the results.

Keywords: GeoGebra; Analysis; Task design; Intervention.

Introduction and Objective

The first year of university mathematics is currently of broad interest in Germany (e.g., Hoppenbrock et al., 2016) and internationally (e.g., Gonzales-Martin et al., 2017). In Germany, students enter university after having gained a high school diploma. The curriculum of these schools includes some basic calculus (derivatives and integrals) but on a very informal level where proofs play almost no role at all. Thus, when starting at the university, they experience a substantial gap that results in high failure rates in examinations after the first term (typically 70-80%, sometimes more), do not pass these examinations, and have to repeat them, often several times. In this paper, the use of the dynamic mathematics software GeoGebra (Hohenwarter, 2019) to improve the situation is investigated.

Literature Review

The learning of calculus has been investigated by many researchers. A recent overview is given by Bressoud et al. (2016). Insight has been gained into many problems students face when learning university-level calculus, e.g., problems with logic (e.g., Selden <& Selden (1995), Shipman (2016)) and proofs (e.g., Stavrou (2014), Selden (2012)). A wider overview is also given in (Winslow, 2018).

The use of technology is discussed in a variety of papers as well. Tall (2003) has argued that technology allows for an embodied approach to teaching calculus by making notions dynamic and visible. Similarly, Moreno-Armella (2014) argued that the traditional teaching approach cannot bridge the tension between intuition and formalism. He suggests some dynamic activities that illustrate limiting processes and involve differentials as small changes. This is inspiring, but his report gives no empirical evidence, and his approach tends to detour from the standard approach to calculus.

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Other uses of discrete structures that fit nicely with digital tools are described, e.g., by Weigand (2014).

A lot of research has investigated the use of dynamic math software such as GeoGebra (Hohenwarter, 2019) for the learning of mathematics in general and, although of calculus, specifically. However, the majority of research concentrates on the high school level. Beyond high school, college calculus is investigated to some extent, but there are only a few investigations about using GeoGebra at the university level of analysis. In Tall et al. (2008), an overview is given that aims mainly at the high school level, but it presents also ideas beyond that. One paper that focuses on university level analysis is Attorps et al. (2016). They find a positive effect in teaching Taylor approximations using a variation-theory-based approach. D'Azevedo Breda and dos Santos (2015) investigated complex numbers. Nobre et al. (2016) have investigated the use of GeoGebra in a calculus course for computer science students, and they find some significant results. However, the topics touched are more of the college-style calculus (or high school calculus in Germany), not of the university analysis content. Much the same can be said about Machromah et al. (2018).

There are some other papers, but most of them are centered on functions and vector structures, and the present paper is broader in this respect. Thus, the contribution of the present paper is new, as it addresses the learning of rigorous university-level analysis courses. The research question is: Can technology-based tasks have an effect on the understanding of rigorous aspects of analysis?

The course and participants

The course "Analysis I" was taught by the author over 14 weeks. 180 students were enrolled in the course, with 141 taking the examination at the end. Students' age and sex were not recorded for reasons of privacy, but the mean age is estimated to be 20, and there was an equal sex distribution.

The main learning objective of this course is to introduce students to the rigor of mathematics. This course is taken mainly by students aiming for a Bachelor in mathematics, but also by students from physics and trainee teachers for high schools. The content includes logic, axiomatic theory of natural, rational, real, and complex numbers, sequences, series and convergence, limits, continuity, differentiability, sequences of functions, Taylor series, and integrals. The approach is rigorous, i.e., all statements are proven, and exercises for students include many proof tasks. The course consists of 4 hours per week of lecture, 2 hours of exercises in a huge group, and homework exercises which are graded and discussed in small groups (2 hours per week).

This setup implies that many concepts have to be re-learned by the students, e.g., in high school, the sine and cosine functions are defined geometrically (coordinates of a point on a unit circle), while in this course, they are defined as the real and complex parts of $exp(ix)$, which itself is defined as a series.

Traditionally, computers are practically absent from such courses. However, for the redesigned course reported here, computers were used to some extent (Mathematica, Python, and GeoGebra). This paper concentrates on the use of GeoGebra. About half of the students reported that they knew GeoGebra from high school. Thus, it was natural to use this tool. The use of GeoGebra was twofold:

- Demonstrations in the lectures. Many concepts, e.g., addition and multiplication of complex numbers, epsilon-strip concept of convergence, convergence of function sequences (in general and particular for Taylor series), epsilon-delta definition of continuity, local linearity of differentiable functions, etc., were visualized.
- Non-mandatory homework. Every week, a set of homework assignments was given, and some of them were mandatory and graded; however, for legal reasons, the computer assignments were voluntary.

We will first give some impression of the tasks the students had to work on and will only after this justify the didactical approach from a theoretical point of view.

The first example illustrates the kind of demonstrations used in the lecture hall (but the files have been provided to the students, and they have been encouraged to use them for gaining their own experience). The following figure shows the logic of the epsilon-delta definition of continuity. This clarifies that for an arbitrary ε, one must be able to find a suitable δ, and this is impossible in the example from fig. 1 when x is moved to 2, where a jump occurs.

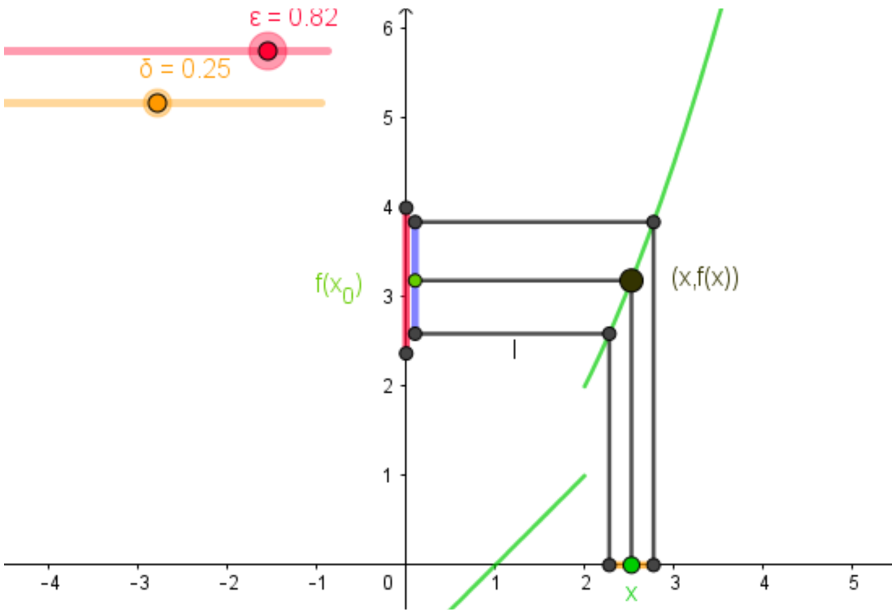

Figure 1. Applet that explains the definition of continuity

The idea behind this use of GeoGebra is to build a bridge between the formal approach and intuition, as has been asked for by Moreno-Armella (2014). In performing the demonstration, I linked the order in which the sliders are used to the order of the quantifiers in the formal definition of continuity. Thus, I first announced that if f is continuous for the chosen value x, then one may choose ε arbitrarily, and then it must be guaranteed that it is possible to find δ such that the image of the δ-interval around x is contained in the pre-chosen ε-interval. This should make the transition from the intuitive notion of a continuous function to its formal version and back as smooth as possible. That this may help in understanding is in line with the theory of Duval (2006), who emphasized the importance of conversions between registers. Two examples of homework tasks illustrate this, too:

Exercise: Investigate where the function $f\colon \left(-\frac{1}{2},\frac{1}{2}\right) \to \mathbb{R}, x \mapsto {}^{1}\!\big/\!\left\lfloor \frac{1}{x} \right\rfloor$ has a derivative.

In doing this, it is very useful to plot the function, as this gives the idea that it may be differentiable at the origin with $f'(0) = 1$. (This nice example is due to Peter Quast, Augsburg).

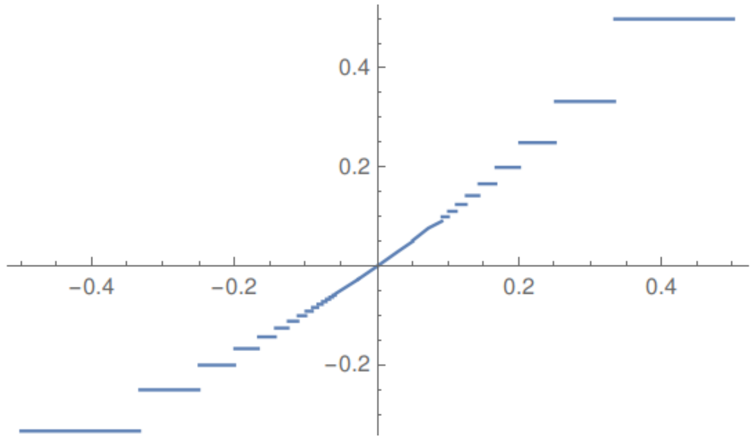

Figure 2. Has this function a derivative at 0?

The didactical principle behind this task is a kind of variation theory (Marton & Booth, 1997). In mathematics education, this theory has been mainly applied in elementary school mathematics. A very typical example is the use in a teaching experiment on logarithms (O'Neil & Doerr, 2015). In my own conceptualization, the theory says that learning materials should be arranged to allow the individual genesis of a concept by contrasting examples and counterexamples, experiencing relations to hold of a variety of examples, identifying single aspects, excluding counterexamples, and fusing several aspects to the general concept. Applied to the concept of differentiability, this led to the following learning trajectory: Students learned the concept definition $f'(x) \coloneqq \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ in high school, which emphasizes the aspect of rate of change and is applied to determine the slope of tangents. Thus, in my course, the derivative was introduced as by Caratheodory (1961) in the following manner: f is differentiable in x_0 if there is a function $q\colon\ U \to \mathbb{R}$ defined on some open neighborhood of x_0 and continuous in x_0 such that for all x in this neighbourhood

$$f(x) - f(x_0) = q(x) \cdot (x - x_0), \qquad i.e.\quad \Delta y = q(x) \cdot \Delta x$$

This definition emphasizes local linearity (as a continuous function can be approximated by a constant, so that the relation is locally linear). Hence, students were demonstrated in the lecture that graphs of differentiable functions appear straight when zoomed in at a sufficient scaling factor. Variation theory then suggested exploring a bunch of functions to sharpen the concept. The example given is the most challenging in this series. It counters possible misconceptions by showing that a function may be differentiable at a point yet have infinitely many points of discontinuity in each neighborhood, and yet, it confirms that the concept of local linearity, with its visual clue, is valid.

This aspect can be explored using GeoGebra effortlessly by zooming and plotting the difference to the approximating linear function.

A second example:

a) Investigate continuity and convergence (pointwise, uniform) of the family of functions $k \in \mathbb{N}, k > 1, f_k: \mathbb{R} \to \mathbb{R}, f_k(x) = x^2 - \frac{k^2x^3 - x^2}{k^2-1}$.

b) Plot sine and cosine functions, and also various partial sums of the series that define them. How to get an approximate value of π from this?

This example illustrates how working with GeoGebra is hoped to support students to gain an object view of functions, i.e., to master the most advanced level as described by Demarois & Tall, D. (1996), namely to see functions as a whole, as an object, and even sequences over such objects.

Assessment and results

The general research question I would like to investigate is if this kind of use of GeoGebra helps students to master the course. In this generality, of course, the question cannot be answered empirically, and more precise questions will be posed later on.

In general, empirical intervention studies at the university level are not easy to carry out. Ideally, one would randomly split courses into groups with different treatments and measure results. However, splitting a course requires teaching resources that are rarely available, and spitting also raises the ethical issue of whether some students are offered better conditions than others. In this situation, the problem that computer exercises could not be made mandatory turned out to offer a new possibility for research: Students themselves decided whether they did the computer exercises or not. Hence, this provided two groups without ethical problems. However, one should not assume these two groups to be equivalent. It seems likely that students doing the exercises might be more interested, more motivated, and thus stronger overall. The methodological trick to solve this problem was to give two different kinds of tasks: One that could potentially profit from the computer exercises because the mathematical content was related, and another that was not expected to benefit from doing the computer exercises. The categorization of the tasks into these groups was done by my own expertise; the tasks are explained below.

First study

The first small study was carried out at the beginning of the third week of the course. The mathematical topics dealt with in the lecture were logic and sets. During the first week, the following (non-mandatory) computer exercise was given:

Task (voluntary): Logic with GeoGebra

a) GeoGebra can plot the set of solutions of certain (not too complex) inequalities in the two variables x, y. Try this out using the following inequalities:

$x + 1 > y - x/2$ $\quad$ $x * y < 4$ $\quad$ $x^2 + y^2 < 9$ $\quad$ $2 * x^2 + y^2 > 5$

b) One may also plot logical combinations of inequalities (Hint: Logical symbols can be found …). Try out: $x > 1 \land x < 4$, $x > 1 \lor x < -1$, $x > 0 \land \neg y > 0, x > 0 \rightarrow y > 0$

c) Find ways to describe these sets:

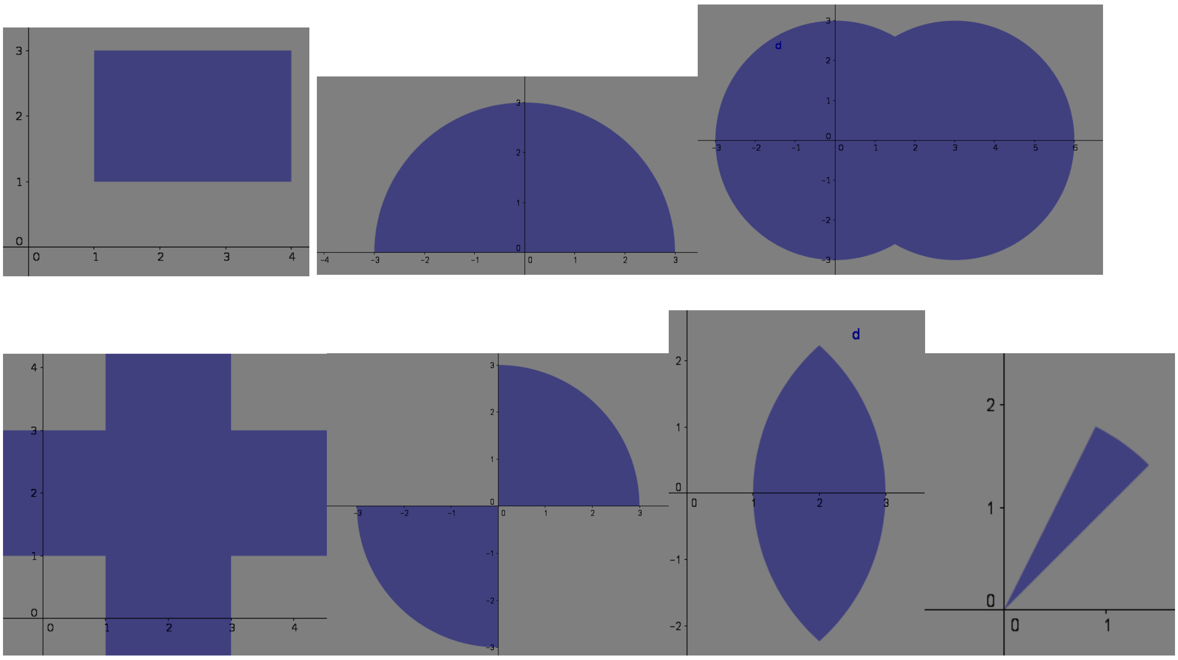

Figure 3. Figures that can described by logical combinations of inequalities

The rationale behind this task should be obvious: Students should have the opportunity to work in a visually appealing setting with logical operators that give direct feedback. The importance of feedback is widely acknowledged (e.g., Hattie & Timperley, 2007), so this should be effective.

During the second week, the students had to work on homework exercises that had to be done on paper and were graded. Three of these mandatory exercises are given below:

Exercise 1: a) Prove: $M = N \Leftrightarrow M \subset N \wedge N \subset M$.

b) Prove both *de Morgan* laws for sets.

c) Illustrate the symmetric set difference $M \ominus N := (M \cup N) \setminus (M \cap N)$ and prove: $M \setminus N = M \ominus (M \cap N)$ and $M \ominus N = (M \setminus N) \cup (N \setminus M)$.

Exercise 2: Define for $n \in \mathbb{N}, a \in \mathbb{R}$ the closed interval: $I_n(a) := [a - 1/n\,, a + 1/n]$

a) Illustrate $I_n \setminus I_{n+1}$ and write it as a union of intervals.

b) Determine $\cap_{i\in\mathbb{N}} I_i(2)$ and prove the result using predicate calculus.

c) Determine $\cup_{i\in\mathbb{N}}[1/n\,, 1 - 1/n]$

Exercise 3: Find pairs of equal sets and prove equality resp. inequality:

$$M_1 = \{\,(x,y) \mid \neg(x > 2 \wedge x < 3)\,\}, M_2 = \{\,(x,y) \mid x \cdot y > 0\,\},$$
$$M_3 = \{(x,y) \mid x > 2 \wedge y > 0 \vee y < 0\}$$
$$M_4 = \{(x,y) \mid x > 0 \wedge y > 0 \vee x < 0 \wedge y < 0\}, M_5 = \{(x,y) \mid x \leq 2 \vee x \geq 3\}$$
$$M_6 = \{(x,y) \mid \neg((x \leq 2 \vee y \leq 0) \wedge y \geq 0)\}$$

My expert classification was that exercise 3 might benefit from doing the voluntary GeoGebra task, while little effect of GeoGebra use on exercises 1 and 2 was to be expected.

Thus, the hypothesis was that students who decided to do the GeoGebra task would perform substantially better on exercise 3 and either not better or only slightly better on exercises 1 and 2.

Two methods were applied to assess which students took the voluntary GeoGebra task. They were asked explicitly to indicate if they did the GeoGebra task, and then they were asked to rank the intensity on a Likert scale from 0 (not done) to 5 (intensely).

Unfortunately, several of the master students who ranked the students' papers forgot to write down these engagement variables, and due to privacy issues, it was not possible to get this information. Hence, the usable data set consists of a rather small sample of n=23 students, 11 of them indicated that they had worked on the GeoGebra task (group G), 12 indicated that they didn't (group N). Statistics (all done in R) is thus limited, but here are the results:

E1 and E2 are the sum scores of students achieved at exercises 1 and 2, respectively. These variables can be considered to be normally distributed as the Shapiro test gives p-values for E1 of 0.33 for the whole group and of 0.48, respectively, 0.45 for the N and G groups. For E2, p= 0.05 for the whole group and 0.22 and 0.06 for the subgroups. However, E3 cannot be considered to be distributed normally. Thus, the Wilcox test is applied to discover group differences between the N and G groups.

Exercise	Wilcox-Test	Cohen d: G-N
E1	0.27	-0.371
E2	0.40	0.146
E3	0.014 *	0.842

Table 1. Test results for the first study

Conclusion: The students who worked on the GeoGebra task scored significantly better on the third task, as expected. The fact that they performed worse (although not significantly) on exercise 1 came as a surprise, and there is no good explanation yet. It is likely that this is just due to the small number of students, but it may also be that good and theoretically minded students did not do the computer exercises.

To support this result, I calculated D=normalize(E3)-normalize(E1+E2) and found that D is significantly different for CG and N groups with $p=0.012$ and $d=0.92$.

Another way to explore the findings statistically is to use a linear regression model that includes the information on the amount of time T students used for the GeoGebra task. Although this is not normally distributed, a linear model was devised: E3~T+A1+A2, and it turned out, that T is significant. However, the whole explained variance is $R^2=0.44$. Given the fact that many other issues influence performance on such tasks, this should be regarded as rather high.

Second study

The second study was conducted almost at the end of the course, in week 12. The methodology was the same as in the first study. While the first study focused on a very small intervention, the second study was more designed to account for the whole learning effect during the term.

A total of n=97 students' exercise responses could be used in the statistics.

First, there were two Likert-scale items to judge agreement with a statement from 0 to 10:

"I used GeoGebra regularly for this course.“ Mean: 3.4, Standard dev. 3.0

"GeoGebra is a useful tool for learning in this course.“ Mean: 6.7, Standard dev. 2.6

Those students who marked 5 or more on the first question were considered to be the GeoGebra user group (G, 37 students), the others the non-users (N: 60 students)

The marked mandatory exercises that were used in this study were the following:

Exercise 1 Prove for which $k \in \{1,2,3\}$ the functions $f_k: \mathbb{R} \to \mathbb{R}, f_k(x) := \begin{cases} \sin(x) + x^k \cdot \sin(1/x), x \neq 0 \\ 0, x = 0 \end{cases}$ are differentiable and if the derivatives are continuous.

Exercise 2: Prove: If $f: \mathbb{R}^+ \to \mathbb{R}$ is differentiable and $\exists c > 0: \exists d > 0: \forall x \geq d: f'(x) > c$, then $\lim_{x\to\infty} f(x) = \infty$.

Give an example that shows that the conclusion is not valid if one only demands: $\forall x \geq d: f'(x) > 0$.

Exercise 3:

a) A function is said to have a symmetric derivative if in x_0 if $f^{\sim}(x_0) := \lim_{h\to 0} \frac{f(x_0+h)-f(x_0-h)}{2h}$ exists.

a1) Calculate the symmetric derivative of $f: \mathbb{R} \to \mathbb{R}, f(x) = x^2$.

a2) Decide which implications are valid: $\exists f^{\sim}(x_0) \Rightarrow \exists f'(x_0)$ and $\exists f'(x_0) \Rightarrow \exists f^{\sim}(x_0)$.

b) Another kind of derivative is: $f^*(x_0) := \lim_{q\to 1} \frac{f(q\cdot x_0)-f(x_0)}{q\cdot x_0 - x_0}$

b1) Calculate this for two functions: $f: \mathbb{R} \to \mathbb{R}, f(x) := x^2, \log: \mathbb{R}^+ \to \mathbb{R}$

b2) Check equivalence with the ordinary derivative.

c) A function $f: D \to \mathbb{R}, D \subset \mathbb{R}$ is called m-continuous in x_0 with $f(x_0) \neq 0$ if $\forall \epsilon > 0: \exists \delta > 0: \forall x \in D: |x - x_0| < \delta \Rightarrow |\frac{f(x)}{f(x_0)} - 1| < \epsilon$

c1) Prove that $f: \mathbb{R} \to \mathbb{R}, f(x) = x^2$ is m-continuous for all real numbers and that besides zero points m-continuity and ordinary continuity are equivalent.

c2) Prove that $f: D \to \mathbb{R}, D = (1, \infty) \subset \mathbb{R}, f(x) = x^2$ is uniformly m-continuous.

The choice of these tasks was mainly driven to match the topics of the lecture in that week, but some theoretical considerations came into play: Given the application of GeoGebra to explore the concept of derivative given above, I assumed that GeoGebra could help affine students use the tool to foster their intuition about what is going on here. Thus, it was expected that this task would benefit from using GeoGebra. However, it seems not that obvious that the transfer from the graphical setting to giving a written proof, as was required here, can be made. The second task does not invite plotting, as no concrete function is given. Moreover, it deals with quantifiers that are not touched on in any GeoGebra activity. The third task is designed to test the transfer hypothesis further.

As above, students' solutions were marked and graded by points by master students. The maximum points possible were E1: 11, E2: 5, E3: a: 2+4, b: 4+2, c: 3+2.

For all these variables, the hypothesis of normal distribution was checked using the Shapiro test and had to be rejected.

Our general hypothesis is that students who used GeoGebra regularly performed better than others. More specifically, the use of GeoGebra should boost results for Exercise 1 because students who used GeoGebra regularly could be expected to investigate these functions' graphs and use zooming in to investigate the limit empirically. For Exercise 2, I didn't expect a benefit of using GeoGebra besides the baseline effect caused by the fact that GeoGebra use was likely to correlate with motivation and engagement. For Exercise 3, the hypothesis was that plotting helps to find counterexamples for a2, b2, c2, but less impact was supposed for a1, b1, c1.

Wilcox's tests were performed to test these hypotheses.

Exercise	Wilcox-Test	Cohen d: G-N
E1	0.00 **	0.52
E2	0.63	0.05
E3	0.007 **	0.35

Table 2. Test results for the second study

These results nicely confirmed the hypotheses.

To investigate E3 in more detail, the tests were also issued on the sub-items, and the results were in accordance with expectations as well:

Exercise	Wilcox-Test	Cohen d: G-N
E3a1	0.23	0.16
E3a2	0.03 *	0.31

E3b1	0.57	0.10
E3b2	0.02 *	0.37
E3c1	0.12	0.26
E3c2	0.03 *	0.32

Table 3. Detailed test results for the second study

A regression model was tested as well, namely E1 ~ R + S + E2, where R is the Likert scale of regular use of GeoGebra and S is the Likert scale of finding GeoGebra a suitable tool. Results are that the paths from R (p=0.01*) and E2 (p=0.00**) are significant, while the one from S is not significant (p=0.86). For the model E3 ~ R + S+ A2, things look quite similar.

It is interesting to note that not the idea of suitability but the actual use of GeoGebra explains better results!

Conclusion

Both studies show that using GeoGebra can improve students' understanding of the mathematics of the Analysis I course. Especially the finding in the second study seems to be important, that the use of GeoGebra can boost performance even in tasks that are not directly related to graphical aspects of mathematics. A strong impact on such non-graphical areas of math can also be made by the use of symbolic computation, and this will be detailed in another paper.

Looking back, the result is that GeoGebra is a good tool to enable thinking not only about functions but also about logic and inequalities.

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