Qeios

Quaternion Quantum Mechanics: Unraveling the Mysteries of Gravity and Quantum Mechanics within the Planck-Kleinert Crystal

Marek Danielewski* and Chantal Roth[2]

[2]Independent Researcher, Sägeweg 12, 3283, Kallnach, Switzerland
*****Correspondence: daniel@agh.edu.pl

Abstract: We present quaternion quantum mechanics and its ontological interpretation. The theory combines the Cauchy model of the elastic continuum with the Planck-Kleinert crystal hypothesis. In this model, the universe is an ideal elastic solid where the elementary particles are soliton-like waves. Tension induced by the compression and twisting of the continuum affects its energy density and generates the force of gravity, as density changes alters the wave speed and hence gravity could be described by an index of refraction.

Keywords: quantum mechanics; gravity; Dirac equations**;** quaternion; Planck-Kleinert crystal

1. Introduction

Quaternion quantum mechanics answers two central ontological questions of its interpretation:

- The being: the Cauchy elastic continuum and
- The categories of being and their relations: the Planck oscillator, the quaternion algebra and gravity.

We present the above concepts in popular manner and show the key formulae only. Our simplifications should not give the impression that we did not obey the math rules and that we are not aware of the enormous prospects of quaternion quantum mechanics, QQM. The extended presentations of the Planck-Kleinert Crystal hypothesis, the quaternion quantum mechanics and the elements of the quaternion algebra can be found in the attached paper [1] and references [2].

Quantum mechanics, QM. From its beginning the imaginary number *i* has been common in QM. But, the complex number algebra is inadequate to quantify the observed relations and over the years QM was fortified with special operators and theorems. Since Dirac QM is called the operator quantum mechanics and promotes the developing of new mathematics in operator algebra. It has been very effective, but, due to the complexity and the non-intuitive definitions, it is problematical even for the mathematicians.

In simple words QM in the Max Born interpretation includes the idea that the particle wave function ψ doesn't exist in reality. It's just a mathematical convenience that we use to describe the probabilities for where we might find a subatomic particle. QM is indeterministic, the probabilities of particle position can be calculated. But, there are particles certain pairs of properties which cannot all be measured simultaneously. Consequently, no reality can be attributed to the particle. During the measurement the ψ function mysteriously collapses in the whole space. The present explanations assume that the ψ collapse has no observable consequences and they do not explain what happens with energy. The instant collapse is not expected in General Relativity and "string theory". This "spooky action" irritated Einstein; also

2023-08-04 - https://doi.org/10.32388/RDW13U

Schrödinger never accepted the "probability" interpretation of the ψ function and considered the wave to be a real wave:

"I am opposing not a few special statements of quantum physics held today, I am opposing as it were the whole of it, I am opposing its basic views that have been shaped 25 years ago, when Max Born put forward his probability interpretation, which was accepted by almost everybody."[3]

Bell who verified the nonlocal phenomena was also dissatisfied with conceptual status of QM [4]:

"Either the wavefunction, as given by the Schrödinger equation, is not everything, or it is not right."

Murray Gell-Mann in his lecture at the 1976 Nobel Conference [5]:

"*Niels Bohr brainwashed the whole generation of theorists into thinking that the job* (of finding an interpretation of quantum mechanics) *was done 50 years ago* (by Born)*"*.

There have been minor advances in our understanding of QM and the widely known remark by Feynman is valid in 2023: *"It is safe to say that no one understands quantum mechanics"* [6].

The Cauchy model was already published [7] when Maxwell spoke about the æther hypothesis [8]:

"*On our theory, it* (energy)*… may be described according to a very probable hypothesis, as the motion and the strain of one and the same medium* (elastic æther)*"* [9].

Entirely forgotten is the Maxwell explicit statement on gravity:

"…assumption… that gravitation arises from the action of the surrounding medium leads to the conclusion that every part of this medium possesses, when undisturbed, an enormous intrinsic energy. As I am unable to understand in what way a medium can possess such properties, I cannot go any further in this direction in searching for the cause of gravitation."

Maxwell's idea of the solid showing "*enormous intrinsic energy*" was considered absurd in the 19th century. The defected solid by Kleinert [10] and the ideal elastic solid crystal [11] were considered only one and half century later. In our model, the macro properties of crystalline æther are approximated by the Cauchy model of the ideal elastic crystal continuum [7]. At the Planck scale the building blocks of the *fcc* crystal are Planck particles that obey the laws of mass, momentum and energy conservation, Table 1.

Table 1. The physical constants of the crystalline æther [12].

Label Used in This Work	**Planck Constants**	**Symbol for Unit**	**Value**	**SI Unit**
Lattice parameter	Planck length	l_P	$1.616229(38)\times10^{-35}$	m
Poisson ratio		ν	0.25	-
Mass of the Planck particle	Planck mass	m_P	$2.176470(51)\times10^{-8}$	kg
Planck crystal density		ρ_P	2.062072×10^{97}	$kg\cdot m^{-3}$
Duration of the internal process	Planck time	t_P	$5.39116(13)\times10^{-44}$	s^{-1}
Transverse wave velocity	Light velocity	c	2.99792458×10^{8}	$m\cdot s^{-1}$

Personal note. I visited Kleinert in 2008 in Berlin (MD). I was taken by surprise and presented a seminar on the Planck-Kleinert Crystal. I was expecting a mild compliments on the gravity concept in the first paper [11]. Unpredictably, Hagen praised the concept of the imaginary potential and… that was it, "**the**

quaternion hint". In 2008 we both considered the æther as the solid in three-dimensional space. But, the imaginary number q in complex algebra: $q = q_0\mathbf{1} + q_1 i \in \mathbb{C}$ is visualized in the two-dimensions: two perpendicular axes. The relations in a crystal call for the generalization to a four-dimensional number: $q = q_0\mathbf{1} + q_1 i + q_2 j + q_3 k$. Hamilton created quaternions, the $\mathbb{R}^4$ analog of the complex numbers: *"Time is said to have only one dimension, and space to have three dimensions. The mathematical quaternion partakes of both these elements;… and in this sense it has… a reference to, four dimensions."*[13]

The beauty of quaternions was immediately recognized by James Clerk Maxwell [14]:

"*The invention of the calculus of quaternions is a step towards the knowledge of quantities related to space which can be compared for its importance, with the invention of triple coordinates by Descartes*"

Quaternion quantum mechanics today. The Hurwitz theorem states that the real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, quaternions $\mathbb{Q}$ and octonions $\mathbb{O}$, are the only normed division algebras over the real numbers. Simply, only $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$ and $\mathbb{O}$ can be used in the models where the energy (real number $\mathbb{R}$) is conserved, e.g., in the models where the energy of the ψ function (say a $\mathbb{C}$ valued function) can be split between three or more $\mathbb{C}$ valued wave functions. The algebra rules will provide the consistency of such model with the energy conservation. The QQM has logical consistency and its equations carry more information than their complex counterparts [15]. Recently we combined the Cauchy model with the Planck–Kleinert Crystal hypothesis and derived the Klein–Gordon [16] and the quaternion Schrödinger equations [2].

Besides that, we try to convince reader that the quaternions $(q_0, q_1, q_2, q_3) = (q_0, \hat{q})$ can be considered a physical reality in the same sense as the four-dimensional time-space continuum $(t, x, y, z) = (t, \vec{\mathbf{u}})$. What is more, the reformulation of the basic principles in terms of quaternion algebra, due to the noncommutativity of quaternion-valued functions: $\psi_1 \cdot \psi_2 \neq \psi_2 \cdot \psi_1$ allows understanding the QQM. A relevant visualization of the noncommutativity is the twist of an arbitrary point P in $\mathbb{R}^3$. In the case of the different twist angles the final position of point P depends on the sequence of twists.

2. Quaternion representation of the Cauchy classical theory of elasticity

The proofs of uniqueness and completeness of solutions of the Cauchy model are done so, it constitutes the unique, reliable foundation for the extensive study [17]. The equation of motion relates the acceleration $\ddot{\mathbf{u}}$, with the displacements of $\mathbf{u}$. It is the vector equation, the equivalent of three scalar equations

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} = 3c^2 \,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} - c^2 \,\mathrm{rot}\,\mathrm{rot}\,\mathbf{u}.$$

$$\begin{bmatrix}\text{acceleration of}\\ \text{displacement}\end{bmatrix} = \begin{bmatrix}\text{gradient of displacement}\\ \text{due to the compression}\end{bmatrix} - \begin{bmatrix}\text{rotation of displace-}\\ \text{ment due to the twist}\end{bmatrix} \quad (1)$$

When cracks and/or defects are not allowed, the displacement can be expressed by the curl-free component and a divergence-free component. Explicitly, in an ideal elastic continuum: $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_\phi$, where $\mathrm{rot}\,\mathbf{u}_0 = 0$ and $\mathrm{div}\,\mathbf{u}_\phi = 0$ [18]. By acting on Eq. (1) by the divergence and rotation operators, e.g., $\mathrm{div}\,\mathbf{u} = \mathrm{div}\,\mathbf{u}_0 + \mathrm{div}\,\mathbf{u}_\phi = \mathrm{div}\,\mathbf{u}_0$, we get four equations: the transverse and the longitudinal wave equations:

$$\operatorname{div}\left(\frac{\partial^2\mathbf{u}}{\partial t^2}=3c^2\operatorname{grad}\operatorname{div}\mathbf{u}-c^2\operatorname{rot}\operatorname{rot}\mathbf{u}\right)\;\Rightarrow\;\frac{\partial^2}{\partial t^2}\operatorname{div}\mathbf{u}_0=3c^2\Delta\operatorname{div}\mathbf{u}_0\;\overset{\operatorname{div}\mathbf{u}_0=\sigma_0}{\Leftrightarrow}\;\frac{\partial^2\sigma_0}{\partial t^2}=3c^2\,\Delta\sigma_0,$$
$$\operatorname{rot}\left(\frac{\partial^2\mathbf{u}}{\partial t^2}=3c^2\operatorname{grad}\operatorname{div}\mathbf{u}-c^2\operatorname{rot}\operatorname{rot}\mathbf{u}\right)\overset{\operatorname{rot}\operatorname{rot}\mathbf{u}=-\Delta\mathbf{u}}{\Rightarrow}\frac{\partial^2}{\partial t^2}\operatorname{rot}\mathbf{u}_\phi=c^2\Delta\operatorname{rot}\mathbf{u}_\phi\;\overset{\operatorname{rot}\mathbf{u}_\phi=\hat{\phi}}{\Leftrightarrow}\;\frac{\partial^2\hat{\phi}}{\partial t^2}=c^2\,\Delta\hat{\phi}. \tag{2}$$

In (2) we split the acceleration in Eq. (1) and obtained four scalar equations. The Hamilton algebra $\mathbb{Q}$ combines the curl-free and divergence-free equations in (2) into the single quaternionic equation:

$$\begin{array}{l}\dfrac{\partial^2\sigma_0}{\partial t^2}=3c^2\,\Delta\sigma_0 \text{ in } \mathbb{R}^1\\ \quad + \\ \dfrac{\partial^2\hat{\phi}}{\partial t^2}=c^2\,\Delta\hat{\phi} \text{ in } \mathbb{R}^3\end{array}\;\overset{\sigma=\sigma_0+\hat{\phi}\in\mathbb{Q}}{\Longleftrightarrow}\;\frac{\partial^2\sigma}{\partial t^2}=c^2\Delta\sigma+2c^2\Delta\sigma_0 \text{ in } \mathbb{R}^4, \tag{3}$$

where $\operatorname{div}\mathbf{u}_\phi=0$. In Eq. (1) the acceleration and displacements are in $\mathbb{R}^3$. Equation (3) logically relates acceleration $\ddot{\sigma}$ and q-potential σ in $\mathbb{R}^4$. The 4D deformations $(\sigma_0,\phi_1,\phi_2,\phi_3)$ occur in the 4D time-space continuum (t,x,y,z). The energy density of the deformation field in the quaternion form equals [2]

$$\begin{array}{ccccccc} e & = & \frac{1}{2}\hat{\dot{u}}\cdot\hat{\dot{u}}^* & + & \frac{1}{2}c^2\sigma\cdot\sigma^* & + & c^2\sigma_0^2 \\ \begin{bmatrix}\text{overall energy per}\\ \text{mass unit: } m^2/s^2\end{bmatrix} & = & \begin{bmatrix}\text{kinetic}\\ \text{energy}\end{bmatrix} & + & \begin{bmatrix}\text{energy of twist}\\ \text{and compression}\end{bmatrix} & + & \begin{bmatrix}\text{energy of}\\ \text{compression}\end{bmatrix}\end{array} \tag{4}$$

where $\sigma=\sigma_0+\hat{\phi}$ and $\sigma^*=\sigma_0-\hat{\phi}$. The imaginary units obey the relation $i^2=j^2=k^2=-1$ and consequently the product of quaternion σ and its conjugate σ^* is a real number, the energy e in relation (4). This property takes account of the fact that the direction of the vector $\hat{\phi}$ of the twisting element depends on the vector position on the element surface. In plain language, the static external observer of the twist measures the opposite directions of the twist vector at the front and back walls of element: $\hat{\phi}_{front}(t,x)=-\hat{\phi}_{back}(t,x)$. The relations between imaginary units grant the independence of the twist energy on the observer position.

A topic of primary importance is the Cauchy-Riemann derivative D. This operator "*returns us as the observers*" to $\mathbb{R}^3$. Under the constraint in (3), $\operatorname{div}\hat{\phi}=0$, the derivative D of the q-potential equals

$$\mathrm{D}\sigma=\operatorname{grad}\sigma_0+\operatorname{rot}\hat{\phi}. \tag{5}$$

$\mathrm{D}\sigma$ is a sum of two vectors in $\mathbb{R}^3$. Thus, in the elastic continuum, the Cauchy-Riemann derivative of the q-potential $\boldsymbol{\sigma}$ in $\mathbb{R}^4$ corresponds physically to the gradient in $\mathbb{R}^3$.

The quaternion Schrödinger equation was derived upon considering the particle wave ψ showing energy E_m . The rigorous derivation in [2] follows the straightforward schema.

1. By multiplying the energy density in (4) by the crystal mass density ρ_P, the energy of the particle wave in volume Ω equals: $E_m = \int_\Omega \rho_P \, e \, \mathrm{d}x$. Relation (4) is not an equation, it contains three unknowns: the q-potential, the velocity and the wave energy. It can be written in symmetrical form

$$E_m = \int_\Omega \rho_P \left(\tfrac{1}{2} \hat{u} \cdot \hat{u} + \tfrac{1}{2} c^2 \tilde{\sigma} \cdot \tilde{\sigma}^* \right) \mathrm{d}x \quad \text{where } \tilde{\sigma} = \sqrt{3}\sigma_0 + \hat{\phi} . \tag{6}$$

2. For the lattice deformation field it is reasonable to guess that the velocity $\hat{u}$ is related to the q-potential. Namely to the normalized Cauchy–Riemann derivative of deformation potential, $l_P D\tilde{\sigma}$, and to the wave propagation velocity c: $\hat{u} = -c\, l_P D\tilde{\sigma}$. Thus, the momentum of the Planck mass equals:

$$\hat{p} = m_P \hat{u} = -m_P c\, l_P D\tilde{\sigma} = -\hbar\, D\tilde{\sigma} \quad \text{where } \hbar = m_P c l_P . \tag{7}$$

There is also the particle wave momentum: $\hat{p}_m = \hat{u}\, m$. The equality of moments implies

$$\hat{u}\, m = \hat{p} \quad \Rightarrow \quad \hat{u} = -\hbar/m \; D\tilde{\sigma} . \tag{8}$$

By joining (6) and (8) the energy integral contains two unknowns, the q-potential and the wave energy:

$$E_m = \int_\Omega \rho_P \left[\frac{\hbar^2}{2m^2} \left(D\tilde{\sigma} \right) \cdot \left(D\tilde{\sigma}^* \right) + \tfrac{1}{2} c^2 \tilde{\sigma} \cdot \tilde{\sigma}^* \right] \mathrm{d}x . \tag{9}$$

3. The symmetry of the kinetic and the strain terms in (9) allows obtaining the functional [2]

$$Q[\psi] = \int_\Omega \left(\frac{\hbar^2}{2m} (D\psi) \cdot (D\psi)^* + V(x)\, \psi \cdot \psi^* \right) \mathrm{d}x , \tag{10}$$

where $V(x)$ denotes potential field and $\psi = \sqrt{\rho_P / m}\, \tilde{\sigma}$ is the quaternion valued rescaled wave energy density. Because ψ obeys relation $\int_\Omega \psi \cdot \psi^* \mathrm{d}x = 1$ it may be called the particle density function.

4. The integral in (10), was rigorously minimized with respect to a quaternion function ψ [2]. In simple words, we looked for a differential equation that has to be satisfied by the ψ function to minimize the energies allowed by (10). We have shown that the extremum problem leads to the quaternion analog of the time–independent Schrödinger equation satisfied by the particle in the ground state of the energy E

$$-\frac{\hbar}{2m} \Delta \psi + \frac{1}{h} \left[V(x) - E \right] \psi = 0 . \tag{11}$$

This result confirms the existence of the particle waves and justifies further exploration of the Eq. (3).

Second order wave equations. The 2nd order equations follow from the Eq. (3). The one-way schema for the quaternionic Klein-Gordon equation of the boson particle is shown below:

$$\begin{cases} \left(\dfrac{\partial^2}{\partial t^2} - c^2 \Delta \right) \sigma + G_0(m)\, \sigma \cdot \sigma^* = 0, \\ -2c^2 \Delta \sigma_0 - G_0(m)\, \sigma \cdot \sigma^* = 0, \end{cases} \quad \begin{matrix} \Rightarrow \\ \not\Leftarrow \end{matrix} \quad \left(\frac{\partial^2}{\partial t^2} - c^2 \Delta \right) \sigma - 2c^2 \Delta \sigma_0 = 0 , \tag{12}$$

where the kernel of the coupling function $G_0(m)\, \sigma \cdot \sigma^*$ is the scalar oscillator $G_0\left[s^{-2} \right]$.

Upon adding equations in the system (12) we get back the fundamental momentum balance (3). However, the system in (12) does not follow "directly" from (3). The real meaning of the coupling function $G_0(m)\, \sigma \cdot \sigma^*$ can be seen upon expressing (12) in the generalized form:

$$\begin{cases} \left(\frac{\partial^2}{\partial t^2} - c^2\Delta\right)\hat{\phi} = 0, \\ \left(\frac{\partial^2}{\partial t^2} - 3c^2\Delta\right)\sigma_0 = 0, \\ 2(n-1)c^2\Delta\sigma_0 - G_0(m)\sigma\cdot\sigma^* = 0 \text{ where } n = 0,2,3,... \end{cases} \tag{13}$$

Upon adding two wave equations in (13) we get back the Eq. (3). Yet, the real importance is the proof that by introducing the function $G_0(m)\sigma\cdot\sigma^*$ in (12) we postulated the existence of the harmonic oscillator $G_0(m)$ that implies the more general relation:

$$2(n-1)c^2\Delta\sigma_0 - G_0(m)\sigma\cdot\sigma^* = 0 \text{ where } n = 0,2,3,... \tag{14}$$

For $n = 0$ the coupling for bosons follows, Eq. (12). For coupling: $n = 2,3,...$ the q-potentials $\tilde{\sigma}_n$ equal

$$\tilde{\sigma}_n = \sigma - n\sigma_0 = (1-n)\sigma_0 + \hat{\phi} \text{ where } n = 2,3,... \tag{15}$$

Upon $\tilde{\sigma}_n$ substitution in the system (13), again the particle 2nd order wave equation is visible:

$$\begin{cases} \left(\frac{\partial^2}{\partial t^2} - c^2\Delta\right)\tilde{\sigma}_n + G_0(m)\sigma\cdot\sigma^* = 0, \\ \left[n\frac{\partial^2}{\partial t^2} - (n+2)c^2\Delta\right]\sigma_0 - G_0(m)\sigma\cdot\sigma^* = 0. \end{cases} \tag{16}$$

The quaternionic oscillator couples the transverse and longitudinal waves into the q-potential wave. The coupling take place in the crystal elementary cell, i.e., at the Planck scale. The oscillator grants that:

- the accelerations at the Planck scale of the q-potential components are equal, $\ddot{\sigma}_0 = \ddot{\phi}_1 = \ddot{\phi}_2 = \ddot{\phi}_3$,
- the oscillations energies obey the equipartition theorem,
- the overall momentum change in the particle volume Ω is determined by the particle wave energy E_m,
- in the systems (12) and (16), the wave propagation depends on the transverse wave velocity *c*.

The common acceleration within the particle wave implies the equal periods of the compression cycle, $\sigma_0(t)$, as well as all the twists cycles, $\phi_1(t), \phi_2(t)$ and $\phi_3(t)$. Both, the displacement $\mathbf{u}(t,x)$ as well as the deformation potential $\sigma(t,x)$ are generated by the coexisting harmonic processes: the particle wave, f, and the Planck wave, f_P. The duration of the particle cycle $T = 1/f$, exceeds the Planck cycle by orders of magnitude: $f_P \gg f$. We consider stable particles only and don't analyze processes at $t < T$.

In both cycles we do assume the harmonic approximation [1], that implies a simple relation between the average velocity of the displacement, e.g., during the Planck cycle, $\bar{\dot{u}}_P = c$, and its magnitude $|\dot{u}|_P$:

$$|\dot{u}|_P = 1/2\,\pi\,\bar{\dot{u}}_P = 1/2\,\pi\,c. \tag{17}$$

During each Planck cycle the velocity changes four times in the range $[-1/2\,\pi\,c, 1/2\,\pi\,c]$. Thus, the sum of velocity changes at the Planck distance equals

$$\Delta_P \left|\dot{u}\right|_P = 4\frac{1}{2}\pi c = 2\pi c\,. \qquad (18)$$

Upon dividing the sum of the changes by the Planck length we get the rescaled frequency

$$f_P^* = 2\pi c / l_P = 2\pi f_P \qquad (19)$$

The momentum change during the particle wave cycle follows the same harmonic schema. The average and magnitude of the particle wave velocity equal: $\bar{u}_\lambda = f\lambda$ and $\left|\dot{u}\right|_\lambda = 1/2\,\pi f \lambda$, respectively. The sum of the velocity changes solely due to the particle cycle equals $\Delta_\lambda \left|\dot{u}\right|_\lambda = 4\left|\dot{u}\right|_\lambda = 2\pi f \lambda$, which upon dividing by the wave length λ results in rescaled frequency solely due the particle cycle

$$f^* = 2\pi f \qquad (20)$$

The Planck and particle cycles are simultaneous and the average displacement acceleration is a product

$$\bar{\bar{u}} = f^* f_P^* = 4\pi^2 f\ f_P\,. \qquad (21)$$

By noting that: $\sigma_0 = \operatorname{div}\mathbf{u}_0 = \lim_{\Delta x \to l_P} \Delta\left|\mathbf{u}_0\right| / \Delta x$ we assume that (21) holds for the deformation by compression and the average acceleration of the compression equals:

$$\left\langle \frac{\partial^2 \sigma_0}{\partial t^2} \right\rangle = \left\langle \frac{\partial^2}{\partial t^2}\left(\frac{u}{l_P}\right)\right\rangle = 4\pi^2 f_P f\ . \qquad (22)$$

The equipartition allows us extending the relation (22) for all q-potential components: $\sigma_0, \phi_1, \phi_2, \phi_3$ in $\mathbb{R}^4$:

$$\left\langle \frac{\partial^2 \sigma}{\partial t^2} \right\rangle = 4\left\langle \frac{\partial^2 \sigma_0}{\partial t^2} \right\rangle = 16\pi^2 f_P f\ . \qquad (23)$$

Thus the estimated average acceleration of changes of the quaternionic oscillator is now:

$$G_0(f) = 16\pi^2 f_P f\ , \qquad (24)$$

where f is an unknown particle frequency that may be postulated or computed.

The particle wave frequency $f = f(m_0)$ follows from the $\mathbb{R}^1$ schema. The sum of moments of all Planck masses (at the arbitrary time and solely due to the particle wave), equals the momentum of particle m_0 itself. To simplify, we estimate the average moment of the arbitrary Planck mass m_P, during the particle twist cycle $T = f^{-1}$. The cycle implies that Planck mass returns to its initial conditions: $u_P(t) = u_P(t+T)$ and $\dot{u}_P(t) = \dot{u}_P(t+T)$ The overall distance on which the moment of the mass m_P changes equals $2\pi l_P$. Consequently the average momentum of a Planck mass m_P is given by

$$p(m_P) = m_P \frac{2\pi l_P}{T} = 2\pi m_P l_P f\,. \qquad (25)$$

The momentum of the particle m_0 is due to the particle propagation velocity:

$$p(m_0) = m_0 c\,. \qquad (26)$$

Both moments (25) and (26) must be equal and the frequency of particle wave becomes:

$$f = \frac{m_0 c}{2\pi m_P l_P} = \frac{m_0 c^2}{2\pi m_P c l_P} = \frac{m_0 c^2}{2\pi \hbar} \text{ where } \hbar = m_P c l_P\,. \qquad (27)$$

Combining Eq. (27), relation $f_P = 1/t_P$ and (24) the total power of the quaternionic oscillator equals:

$$G_0 = 8\pi\, m_0 c^2 \big/ \hbar t_P\ , \tag{28}$$

and the Poisson equation in system (12) becomes

$$c^2 \Delta\sigma_0 = -4\pi\rho\, l_P^3 \Big/ \left(m_P t_P^2\right) = -4\pi\,\rho\, G. \tag{29}$$

Using data in Table 1, the gravitational constant equals: $G = l_P^3 \Big/ \left(t_P^2 m_P\right) = 6.674082 \times 10^{-11}$.

Upon replacing $m_0 c^2 = E_0$ in (27) the Planck-Einstein relation follows: $E_0 = h\, f$ where $h = 2\pi\hbar$.

The 1st order wave equation was success in explaining both the electron spin and the fine structure and the utmost importance of Dirac's discovery was evident. Dirac applied the operator algebra [19], we base on the concept of the medium as a solid. The schema of the secondary quantization is shown below.

2nd order equation in $\mathbb{R}^4$, variable :

q-potential $\tilde{\boldsymbol{\sigma}} = \sqrt{3}\boldsymbol{\sigma}_0 + \hat{\boldsymbol{\phi}}$

& $\Longrightarrow$ **1st order equation in $\mathbb{R}^3$, variable:**
displacement velocity: $\hat{u}(t, x)$

Planck frequency: $f_P = 1 \big/ t_P$

The system (16) for the deformation potential $\tilde{\sigma} = \tilde{\sigma}_0 + \hat{\phi} = \sqrt{3}\sigma_0 + \hat{\phi}$ equals:

$$\begin{cases} \left(\dfrac{\partial^2}{\partial t^2} - c^2\Delta\right)\tilde{\sigma} + G_0(m)\,\sigma\cdot\sigma^* = 0, \\ \left(1-\sqrt{3}\right)\left(\dfrac{\partial^2}{\partial t^2} - \dfrac{3-\sqrt{3}}{1-\sqrt{3}}c^2\Delta\right)\sigma_0 - G_0(m)\,\sigma\cdot\sigma^* = 0, \end{cases} \tag{30}$$

where c denotes the transverse wave velocity.

For the potential $\tilde{\sigma}$ the energy of the deformation field, $E_m = \rho_P/2\int_\Omega \left(\hat{u}\cdot\hat{u}^* + c^2\tilde{\sigma}\cdot\tilde{\sigma}^*\right)\mathrm{d}x$, and the derivative of the potential, $\mathrm{D}\tilde{\sigma} = -m/\hbar\,\hat{u}$, hint at the displacement velocity as an alternative variable:

$$\hat{u} = -\hbar/m\,\mathrm{D}\tilde{\sigma}\,. \tag{31}$$

The changes of q-potential $\tilde{\sigma}$ in (30) are only due to the wave propagation within the particle itself. We know the propagation velocity c, thus the time derivative of the potential $\tilde{\sigma}$ in (30) we express by:

$$\frac{\partial\tilde{\sigma}}{\partial t} = \frac{\partial\mathbf{x}}{\partial t}\cdot\left(\frac{\partial\tilde{\sigma}}{\partial\mathbf{x}}\right). \tag{32}$$

The first term on the right hand side is the propagation velocity c and the second term is the Cauchy-Riemann derivative. The rate of potential changes can be substituted with the velocity using Eq. (31). In a similar manner the 2nd order space derivative of the q-potential is reduced to a 1st order derivative:

$$\begin{gathered} \frac{\partial\tilde{\sigma}}{\partial t} = c\,\mathrm{D}\tilde{\sigma} = -mc\big/\hbar\times\hat{u}, \\ DD\tilde{\sigma} = -m\big/\hbar\times D\hat{u}. \end{gathered} \tag{33}$$

The symmetry of the relation between the densities of the deformation and kinetic energies in (6) implies:

$$\hat{u}\cdot\hat{u}^* = c^2\tilde{\sigma}\cdot\tilde{\sigma}^*\,. \tag{34}$$

Introducing (33), (34) and $G_0 = 8\pi mc^2/\hbar t_P$ in system (30) results in the 1st order particle wave equation

$$\left(\frac{1}{c}\frac{\partial}{\partial t} - D\right)\hat{u} - \frac{8\pi}{t_P c^2}\hat{u}\cdot\hat{u}^* = 0\,. \tag{35}$$

Using the relation for the normalized wave energy density, $\psi(t,x) = \sqrt{\rho_P/m}\,\hat{u}$, we get

$$\left(\frac{1}{c}\frac{\partial}{\partial t} - D\right)\psi - \frac{8\pi}{l_P}\sqrt{\frac{m}{\rho_P}}\psi^*\cdot\psi = 0\,. \tag{36}$$

The quaternion form of the 1st order wave Eq. (36) allows an insight into the Dirac equation and there-fore spin ½. In order to visualize this concept, a simple interactive simulation of a periodically twisting and compressing 3D grid illustrating spin ½ in an elastic solid for two particles is presented [20,21].

Summary QQM has an ontological interpretation. In simple words, the QQM permits considering the æther as an ideal elastic solid. Elementary particles would have to be a soliton-like waves. Tension induced by the compression and twisting of the solid would increase energy density, consequently generate the force of gravity because it affects the wave propagation speed. Therefore, gravity could be described by an index of refraction [22].

The model allows deriving the Schrödinger equation, the 2nd order wave equation systems for different particles and their potentials and the 1st order quaternionic wave equation. Moreover, the fundamental constants: the Planck constant $\hbar$ and gravity constant G are predicted and computed. A simple interactive simulation of a periodically twisting and compressing 3D grid illustrating spin ½ in an elastic solid for two particles is presented [20].

References

[1] Danielewski, M., Sapa, L. and Ch. Roth, Quaternion Quantum Mechanics: Unraveling the Mysteries of the Gravity and Quantum Mechanics within the Planck-Kleinert Crystal, (2023)

[2] Danielewski, M. and Sapa, L. " Foundations of the Quaternion Quantum Mechanics, *Entropy* 22 (2020) 1424; DOI:10.3390/e22121424.

[3] Schrödinger, E. The Interpretation of Quantum Physics (Ox Bow Press, Woodbridge, CN, 1995).

[4] Bell, J.S., *Schrödinger–Centenary Celebration of a Polymath*; Kilmister, C.W., Ed.; Cambridge Univ. Press: Cambridge, UK, 1987; pp. 41–52.

[5] Gell-Mann, M. *The Nature of the Physical Universe*; Huff, D., Prewett, O., Eds., John Wiley & Sons: New York, NY, USA 1979; p. 29.

[6] Feynman, R.P. *The Character of Physical Law*, 2nd ed.; The MIT Press: Cambridge, MA, USA, 2017.

[7] Cauchy, A.L., De la Pression ou Tension dans un Corps Solide. *Exerc. Math.* 1827, *2*, 60–81

[8] Maxwell, J.C., A Dynamical Theory of the Electromagnetic Field. *Phil. Trans. R. Soc. London* 1865, *155*, 459–512; doi:10.1098/rstl.1865.0008.

[9] Maxwell, J.C., Introductory lecture on experimental physics, in *The Scientific Papers of James Clerk Maxwell, vol. II,* edited by W.D. Niven (Dover, New York, 1965), pp. 241-255.

[10] Kleinert, H., Emerging Gravity from Defects in World Crystal, Brazilian Journal of Physics, vol. 35. no. 2A, June, 2005

[11] Danielewski, M., The Planck–Kleinert Crystal. Z. Naturforsch. 2007, 62a, 564-568.

[12] National Institute of Standards and Technology, Available online: http://physics.nist.gov (accessed on Nov 10th 2018).
[13] Graves, R.P., Life of Sir William Rowan Hamilton, (Hodges, Figgis, & Co., Dublin 1989).
[14] Maxwell, J.C., Remarks on the Mathematical Classification of Physical Quantities. *Proc. London Math. Soc.* 1869, *3*, 224–233.
[15] Birkhoff, G.; von Neumann, J. The Logic of Quantum Mechanics. *Ann. Math*. 1936, *37*, 823–843.
[16] Danielewski, M. and Sapa, L., Nonlinear Klein–Gordon equation in Cauchy–Navier elastic solid. Cherkasy Univ. Bull. Phys. Math. Sci. **2017**, 1, 22–29.
[17] Neumann, F., *Vorlesungen über die Theorie der Elasticität der Festen Körper und des Lichtäthers*; B.G. Teubner: Leipzig, Germany, 1885.
[18] Helmholtz, H.v., *Über Integrale der hydrodynamischen Gleichungen, welche den Wirbel-bewegungen entsprechen*, Crelle J. **55** (1858) 25-55.
[19] Dirac, P.A.M., The Quantum Theory of the Electron, Proc. Roy. Soc. London A 117, 610 (1928).
[20] Snoswell, M., personal communications 2022.
[21] Roth, C., Simulation of electron spin: https://elastic-universe.org/.
[22] Evans, J.C., Alsing, P.M., Giorgetti, and Nandi, K.K. ”Matter waves in a gravitational field: An index of refraction for massive particles in general relativity,” *Am. J. Phys.* **69** (2001) 1103-1110.