https://doi.org/10.32388/ROFQKG.2

The conservation laws in quantum mechanics revisited and extended

Richard Oldani
Illinois Institute of Technology
https://orcid.org/0000-0001-6884-5614
email: oldani@juno.com

Abstract

It is often claimed that Einstein is wrong about quantum mechanics. However, when comparisons are made with respect to theoretical foundations rather than experimental results Einstein's theories are found to be superior. Although quantum mechanics correctly describes energy exchange due to emissions with remarkable accuracy it ignores momentum exchange (the absorptions), thereby failing to recognize their causal relationship due to the conservation laws. The deficiencies in non-relativistic theory are corrected by introducing Hamilton's principle and deriving relativistic equations of motion. This allows wave function behavior to be described as the combined action of a particle and an associated localized field. The calculus of variations is applied to the Schrödinger wave equation to show that it is incomplete because it yields twice the allowable action minimum.

Keywords: Non-relativistic quantum mechanics; relativistic quantum mechanics; Hamilton's principle; energy; momentum; causality

1. Introduction

1.1 The conservation laws in quantum mechanics

We will not review previously proposed solutions for the conservation laws in quantum mechanics because there is no agreement. However, a recent treatment of the subject matter is particularly appropriate for it gives an honest assessment of the difficulties involved and serves as an excellent introduction [1].

"That paradoxical processes must arise in quantum mechanics in connection with conservation laws is to be expected. Indeed, on the one hand, physics is local: Causes and observable effects must be related. On the other hand, measurable dynamical quantities are identified with eigenvalues of operators, and their corresponding eigenfunctions are not, in general, localized. These questions lead us to uncover quantum processes that seem, paradoxically, not to conserve energy. The present paper presents the paradox and discusses various ways to think of conservation laws, but does not offer a resolution of the paradox."

"To summarize, in our paper, we present an effect that raises questions about what we actually mean by conservation laws. The standard approach is statistical, and it is good as far as it goes. However, our effect begs the question of what happens in individual cases. A particle, prepared in a superposition of low-energy states and with no high-energy component whatsoever, comes out of a box with great energy. It is legitimate to ask where the energy comes from. We showed that the mechanism used for extracting the particle—the only other system in the problem—did not provide this energy, so we are left with a puzzle. Our example concerned energy conservation. It is, however, clear that one can construct examples involving conservation of other quantities such as momentum or angular momentum. The phenomenon is, therefore, a general one.

Conservation laws are one of the most important aspects of nature. As such, they have been intensively studied and extensively applied, and are considered to be perfectly well established. We, however, raise fundamental questions about the very meaning of conservation laws in quantum mechanics. We argue that, although the standard way in which conservation laws are defined in quantum mechanics is perfectly valid as far as it goes, it misses essential features of nature and has to be revisited and extended."

1.2 Classical conservation laws

Classically defined conservation laws do not compromise the accuracy of the individual events. Energy is absorbed when a system heats up and energy is returned to the environment as it cools. The energy of absorption and emission may be examined to any desired level of accuracy. Thus it is believed that energy exchange holds accurately to the level of individual quanta despite being unobservable at that level.

The concept of momentum for extremely tiny amounts of energy exchange has a long history in science. It was first proposed by Kepler in 1607 to explain why the tail of Haley's comet faces away from the sun. He also imagined the practical application of momentum in outer space for "sails adapted to the heavenly breezes". Maxwell expanded upon the classical concept of momentum by extending its use to field applications. Like energy, momentum can neither be created nor destroyed, but only changes through the action of forces as described by Newton's laws of motion. We shall assume in the following that classical measurements of momentum may be analyzed as precisely as desired, in a way similar to energy, and are valid to the level of individual quanta.

2. Three quantum mechanical formulations

2.1 A statistical theory due to Einstein

Einstein's only attempt at a theory of quantum mechanics doubled as a derivation of Planck's law by statistical methods "in an amazingly simple and general manner" [2]. There he describes the dynamic equilibrium that exists between the thermal energy absorbed by molecules and its subsequent quantum mechanical emission as black body radiation. Heat energy that is absorbed according to classically defined Maxwell Boltzmann statistics is transformed at the molecular level and emitted according to the Planck radiation law. He then extends his statistical analysis to the microscopic level by directly analyzing molecular behavior. The absorption of a photon is in the direction of propagation, while the *emission* of a photon causes a recoil momentum -E/c that is directed in the opposite direction of propagation. The sudden reversals and random nature of the impulses cause molecular trajectories to be discontinuous, as observed in Brownian motion.

He next distinguishes between the absorption and emission of energy by taking into consideration the conservation laws. “If a radiation beam with a well-defined direction does work on a Planck resonator [quantum oscillator], the corresponding energy is taken from the beam. According to the law of conservation of momentum, this energy transfer corresponds also to a momentum transfer from the beam to the resonator.” He demonstrates here the close association that exists between momentum and energy. Molecular impulses transfer energy to the oscillator which then radiates the energy when an electron decays. If there are no impulses; for example, at absolute zero; the molecule will not radiate. Thus energy absorption is distinguished from energy emission.

He continues, “We now turn to the investigation of the motion which our molecules execute under the influence of radiation. In doing this we use a method which is well known from the theory of Brownian movement . . . . . Let a molecule of given kind be in uniform motion with speed **V** along the X-axis of the coordinate system K. We inquire about the momentum transferred on the average from the radiation to the molecule per unit time. To calculate this we must consider the radiation from a coordinate system K′ that is at rest with respect to the given molecule. For we have formulated our hypotheses about emission and absorption only for molecules at rest.”

Momentum exchange due to the impulse of neighboring atoms occurs in K. It causes the excitation of electrons in K' to a higher energy level. When the electrons decay additional contributions to momentum exchange occur due to recoil. The exchanges of momentum are important for they contribute to the energy of the gas molecules. He analyzes all sources of momentum both positive and negative by taking into account viscosity and temperature of the gas, frequency of the radiation, and radiation density. Absorption energy is due to a molecule’s kinetic energy in the coordinates of K and emission energy due to electron decay is described in the coordinate system K' which is “at rest with respect to the given molecule”. He does not describe energy only in K or only in K' for energy can neither be created nor destroyed. He describes energy as a transformation from K to K'.

2.2 Heisenberg’s non-relativistic interpretation of energy conservation

Matrix mechanics evolved from long-standing attempts to describe dispersion phenomena, the continuous change in the angle of refraction of different frequencies of light by a prism or other medium. Although light disperses continuously across the entire spectrum, at certain specific frequencies characteristic of the medium, it is completely absorbed forming lines. When complex sets of mathematical rules were discovered that describe the relationship between the observed frequencies and intensities of spectral lines it allowed Heisenberg to formulate a theory of quantum mechanics that reconciles the continuity of radiation fields with the discrete energy states of an atom by expressing electron transitions in the form of a matrix [3].

$$\sum_{k}(p_{nk}q_{km}-q_{nk}p_{km})=\begin{matrix} i\hbar \, for \; n=m \\ 0 \, for \; n\neq m \end{matrix} \qquad 1)$$

Although p in the above equation refers to momentum it is not the continuous, classically described momentum in K used by Einstein, rather it is the discrete momentum of a photon designated by two indices and emitted in K'. The diagonal elements of the matrix $m=n$ refer to all possible energy states of a quantum system. Because the theory only concerns observables, such as frequencies and transition probabilities, it is formulated exclusively in coordinates relative to K'.

Off-diagonal elements are assigned a value of zero in K' because the energy of an absorption offsets the energy of an emission except for a difference in phase. However exchanges of momentum by these same emissions and absorptions are ignored because they are unobservable. Although the collisions cannot be observed individually they are evident statistically in measurements of gas temperature. Heisenberg did not take these microscopic interactions into account because he believed that quantum mechanics should be "founded exclusively upon relationships between quantities which in principle are observable". This caused him to focus exclusively on emissions, the intensity and frequency of the spectral lines as measured in K'.

Describing energy by means of emission processes alone is a violation of energy conservation. The meaning of energy is not embodied in emission alone, nor in absorption alone for due to conservation laws energy can neither be created nor destroyed. It is correctly described as a transformation from one form to another.

2.3 Feynman's analytical approach to particle dynamics

In the path integral approach to quantum mechanics we can see violations of the conservation laws even more clearly [4]."If one solves the problem of an atom being perturbed by a potential varying sinusoidally with time, which would be the situation if matter were quantum mechanical and light classical, one finds indeed that it will in all probability eject an electron whose energy shows an increase of hν, where ν is the frequency of variation of the potential. When, however, we come to spontaneous emission and the mechanism of the production of light, we come much nearer to the real reason for the apparent necessity of photons. The fact that an atom emits spontaneously at all is impossible to explain by the simple picture given above. In empty space an atom emits light and yet there is no potential to perturb the systems and so for it to make a transition. The explanation of modern quantum mechanical electrodynamics is that the atom is perturbed by the zero-point fluctuations of the quantized radiation field. It is here that the theory of action at a distance gives us a different viewpoint. It says that an atom alone in empty space would, in fact, *not* radiate. Radiation is a consequence of the interaction with other atoms (namely, those in the matter which absorbs the radiation). We are then led to the possibility that the spontaneous radiation of an atom in quantum mechanics also, may not be spontaneous at all, but induced by the interaction with other atoms, and that all of the apparent quantum properties of light and the existence of photons may be nothing more than the result of matter interacting with matter directly and according to quantum mechanical laws."

Feynman's discussion begins in a way that is consistent with the conservation laws; that is, atoms absorb energy before radiating and atoms in empty space do not radiate at all. Although he realizes that energy conservation governs the whole radiation process, and not emission alone, energy measurement is all-important so he disregards considerations of momentum [5]. The paths in his path integral formulation contribute to the probability amplitude for an event so they are regarded as *possible* trajectories, but they are not like anything anyone has ever experienced for they are unrestricted by the conservation laws. They may form loops, extend to infinity, go backwards in time, or exceed the speed of light.

2.4 Stopped light

The best kept secret of theoretical physics is that photons are not a part of quantum mechanics. The original non-relativistic formulation that is due to Heisenberg, Schrödinger, and Dirac uses a Hamiltonian formulation of the atom. Energy is treated as a potential that is assigned to an electron when it is excited and released as a photon when the electron decays to a lower energy state. In theory the photon is not present inside the atom. Despite its non-relativistic foundations we do not have to look far to find a description of photons residing within an excited atomic state. Localized electromagnetic fields within an atom have been observed experimentally and they are referred to as "stopped light" [6]. After an exhaustive study of the effect and a discussion of its possible use in quantum computing the authors conclude, "Future efforts should aim at single photon storage." If photons are stored within the atom, not as potentials, but as discrete entities then the Hamiltonian model is not an appropriate way to describe them. In the next section we shall apply Hamilton's principle and use a Lagrangian instead.

3. Relativistic quantum mechanics

3.1 The absorption of energy by a classical system

To describe particle paths in a way that allows the local application of conservation laws we apply Hamilton's principle. It assumes that a particle, in this case the electron, begins at a particular position P_1 and point in time t_1 and travels at a certain velocity to arrive at a position P_2 after a given amount of time [7]. The tentative paths are not restricted in any way. Not only can we choose any path between the points but the particle's motion in time may change freely as well. All that we require is that motions begin at the observed time t_1 and end at the observed time t_2. The time integral of the energy extended over the entire motion between P_1 and P_2 is called "the action". The quantity that we use as a measure of the action S is the time integral of the difference between the kinetic and potential energies T-V.

$$S=\int_{P_1}^{P_2}\int_{t_1}^{t_2}(T-V)dt=0 \qquad 2)$$

Each of the paths between the same two end-points P_1 and P_2 will have a different action. The principle of least action asserts that the particle path actually taken is the one for which the action is a minimum. For constraints on a classical system of fixed end positions P_1 and P_2 and fixed time travel t_1 and t_2 the action minimum is equal to zero.

3.2 The relativistic absorption of energy by an atomic system

Following Einstein (see 2.1) we wish to describe the energy that is absorbed from K. Even the simplest atomic system, the hydrogen atom, includes an infinite number of paths that connect the electron's initial and final points, from the ground state to an excited state. Hamilton's principle expresses the meaning of the entire set of differential equations describing the paths. It calls for minimizing a single physical quantity, the action, in order to obtain the path actually taken. The principle of least action gives a relativistically correct description for it is independent of any particular system of coordinates and it is invariant for all coordinate transformations in a manner that is consistent with general relativity.

Hamilton's principle is valid for systems upon which work is performed; that is, non-conservative systems. Radiating atoms that absorb and emit energy are *non-conservative systems*. Work is performed on them causing energy to be taken from the environment and transferred to the system. Neither Heisenberg's nor Feynman's methods include energy absorption so they treat the atom incorrectly as a *conservative system*. Therefore to properly describe the energy absorption of an atom we utilize the symmetry of an atomic system and apply Hamilton's principle 2) in generalized coordinates. The transition of an electron from the ground state to an excited state is characterized in generalized coordinates with six dimensions, three to describe its position on the electron shells R_1 and R_2, and three to describe its trajectory. We can think of the Schrödinger equation as a collection of all possible paths of an electron between electron shells, each of which has a different probability of happening and describes a path with a different action. The actual path of the electron, the one chosen by nature, is the one whose action is minimum.

Energy is absorbed by atoms from a radiation field in K due to momentum exchange. The path taken by the electron initiates its motion at a fixed point on the equipotential surface R_1 of the ground state at time t_1, proceeds along a path r, and upon arriving at the excited state R_2 adopts the circular path $2\pi r$ of an orbital thereby assuming orbital angular momentum. The action minimum S[r(t)] for a path between stationary points on R_1 and R_2 yields the reduced Planck's constant ħ.

$$S[r(t)]=\int_{R_1}^{R_2}\int_{t_1}^{t_2}(T-V)dt=\hbar \qquad 3)$$

The action, S[r(t)], is a functional that describes the absorption process in four dimensions. It has as its argument an infinite number of functions, the possible electron trajectories r(t). The path of the electron that is actually followed, has an action minimum not equal to zero as in eqn 2) of classical mechanics, but ħ.

3.3 Relativistic emission of energy by an atomic system

At the relativistic or high end of the energy spectrum in quantum field theory, particles are treated as excited states of the more fundamental underlying quantum fields. As Nobel laureate Frank Wilczek noted [8], "In quantum field theory, the primary elements of reality are not individual particles, but underlying fields." Particles are described by an action functional $S[\psi(x_i)]$ where the fields $\psi(x_i)$ of particles are defined throughout space. Each of the elementary particles has a field and the action depends on all of the fields. We will follow these same practices in order to extend the field interpretation to the low energy emissions currently described by non-relativistic theory. An alternative derivation based on Dirac's methods appears elsewhere [9].

The region of space-time that is of interest is located in K' (see 2.1) and lies between the two states of an atomic oscillator. Within the space-time region between the excited and ground states we define a Lagrangian density of the fields and their first derivatives $£(\phi_i, \phi_{i,\mu})$ which allows for a complete accounting of the energy interactions, where ϕ_i is the current density and $\phi_{i,\mu}$ is the electromagnetic field strength. The action integral for a quantum oscillator with an outer electron that occupies either of two allowable energy states may now be formulated, with emission initiating from the excited state $R_2 = (x_2,y_2,z_2)$ at time t_2 and finalizing at the ground state $R_1 = (x_1,y_1,z_1)$ at time t_1. Applying Hamilton's principle we require the integral of the Lagrangian density over the region of space-time between the excited and ground states to be a minimum for all small variations of the coordinates inside the region, where the action minimum for an arbitrary quantum system is defined in angular measure to be the reduced Planck's constant ħ.

$$S[\phi_i(t)] = \int_{R_2}^{R_1}\int_{t_2}^{t_1} £(\phi_i \phi_{i,\mu}) d^3x dt = \hbar \qquad 4)$$

The energy states |2> and |1> coincide with the electron shells and determine invariant field boundaries where fields vanish. The change in action yields a relativistic formulation of emission that is invariant, the same for all observers. The action $S[\phi_i(t)]$ is a functional, a function of the values of coordinates on the *discrete* boundaries of the space-time surfaces R_2 and R_1 which are in turn functions of the *continuous* space-time variables of the fields within the surface. The field boundaries are uniquely fixed in four dimensions by the volume d^3x and the time interval t_2-t_1 causing the photon to be described as a four-dimensional localization of fields. Experimental verification can be found by examining an atomic clock which defines time with respect to fixed time periods and fixed trajectory end-points [10]. Thus the initial and final points of an electron's path can be confirmed experimentally with nearly perfect accuracy.

3.4 Comparison of the relativistic and non-relativistic models

The principal difference between relativistic and non-relativistic models is in their underlying physical assumptions. In relativistic theory the emission of energy is described with a Lagrangian (T-V) as a four-dimensional localization of fields £(ϕ_i, $\phi_{i,\mu}$), while in non-relativistic theory it is described with a Hamiltonian (T+V) as the potential of a point electron. To demonstrate their equivalence we will show how the two methods relate physically. In non-relativistic theory a particular type of wave function or "spinor", is used to describe energy emission. It defies explanation in ordinary space-time for it acts like an ordinary vector for infinitesimal rotations, but transforms to its negative for complete rotations and requires two complete rotations to return to its original state. To simultaneously describe translations and rotations in a single expression is an impossibility in ordinary space-time. Thus attempts to visualize spinor behavior make use of imaginary geometries such as the Möbius strip. A simpler explanation is possible by using physical arguments from the relativistic model.

Rather than treat quantum mechanical energy emission as an event carried out by a single particle we use equations 3) and 4) to describe it with field sources, an electron of field ϕ_i and a photon of field $\phi_{i,\mu}$, during two distinct physical processes, absorption and emission. The field ϕ_i of an electron has spin described by Pauli matrices oriented in three-dimensional space and the field $\phi_{i,\mu}$ of a photon is oriented due to polarization in four-dimensional space-time by Maxwell's equations. We interpret the spinor therefore as a superposition of two field geometries that undergo continuous change as they cycle through the physical processes of excitation, localization, and emission. There are two paths possible, spin-up or spin-down, and two rotations are necessary to complete an entire photon emission, where rotations represent changes in phase of the photon's electromagnetic field. Thus non-relativistic spinors are rotations in abstract "space-time", which we interpret relativistically as the real field geometries of particles.

The non-relativistic Hamiltonian model of atomic structure conceives of energy as a potential that is assigned to the electron. Experiments with "stopped light" indicate otherwise. A physical separation exists between the matter and energy of an excited state due to field boundaries. An electron does not absorb energy when it is excited, rather excitation causes field boundaries to be erected that localize energy within the atom and create a "bound" photon. In view of these experiments and the above physical arguments we may visualize the wave function as the combined action of a particle and an associated localized field.

4. Discussion

To determine the true evolution of a quantum system we use Hamilton's principle and take advantage of the symmetries of the system to specify generalized coordinates describing the electron shells R_2 and R_1. Then the actual path the electron follows between the initial and final stationary points of a complete cycle between t_1 and t_2 is the one which minimizes the action. However, the true path is not determinable through experimental means. We have no better way to describe the path of an electron other than to calculate the probability of its arrival at a given detector by using the Schrödinger equation. The path with the greatest probability does not always minimize the action. Only paths having action minimum ħ are realized. Electron excitation is one path and decay is another. The fact that two events cause one result, a photon emission, is to a large extent what makes quantum theory seem so mysterious. The electron experiences two transitions but it results in the emission of only a single photon.

The Schrödinger wave equation describes the absorption-emission process by using two rotations of the wave function. Each rotation of 2π represents the change in phase of a photon's field with action minimum ħ. Thus it includes an incoming quantum of energy of action ħ given by 3) and an outgoing quantum of energy of action ħ given by 4). In other words, the emission of energy is a combination of two physical processes, each of which obeys the uncertainty principle and is determined by an action minimum. Wave functions describing the changes in state of an electron cycle, or of any fermion for that matter, yield twice the action minimum. Therefore they describe two paths.

5. Conclusion

The significance of Einstein's 1917 paper, "On the quantum theory of radiation", is now clear. He begins by noting that when atoms are excited to a higher energy state by a radiation field they absorb momentum in the direction of propagation (see 2.1). And when the same atoms decay, an equal amount of energy is emitted which causes recoil momentum in a direction opposite to that of propagation. Absorptions result from positively directed momentum, and emissions lead to negatively directed momentum. He then notes that due to momentum conservation the relationship between energy and momentum is causal. "To this transfer of energy there also corresponds a momentum transfer from radiation bundle to resonator, by momentum conservation." And summing up he concludes, "energy and momenta are always intimately linked together". Thus momentum and energy are causally related, either momentum causes energy transfer or energy causes momentum transfer.

The Aharanov study points out the need to "revisit and extend the standard way in which conservation laws are defined in quantum mechanics". There is no better authority on the fundamentals than Einstein. As he demonstrates in his 1917 study, causality due to the conservation laws requires that positive momentum exchange be linked to energy absorption (eqn 3); and negative momentum exchange due to recoil be linked to energy emission (eqn 4). Two equations are believed necessary because the relationship between momentum and energy is causal and momentum is a vector. Though the momenta associated with absorption and emission have the same magnitude they are directed oppositely in space.

References

[1] Aharanov, Y., Popescu, S., Rohrlich, D. "On conservation laws in quantum mechanics" PNAS 118 (1) e1921529118 (2020) doi:https://doi.org/10.1073/pnas.1921529118

[2] A. Einstein, "Quantum theory of radiation" *Phys Z* **18**, 121 (1917). W.

[3] Heisenberg W. "Quantum mechanical re-interpretation of kinematic and mechanical relations", *Z Phys* **33**, 879 (1925) in B.L. van der Waerden (ed.), *Sources of Quantum Mechanics*, (Dover, 1968).

[4] Brown, L.M., *Feynman's thesis: A new approach to quantum theory* (World Scientific Publishing Co., 2005), p. 4.

[5] Feynman, R. "A space-time approach to non-relativistic quantum mechanics" *Rev Mod Phys* **20**, 367-387 (1948).

[6] Heinze, G., Hubrich, C., and Halfmann, T., "Stopped light and image storage by electromagnetically induced transparency up to the regime of one minute", *PRL* **111**,2013, p. 033601. doi: 10.1103/PhysRevLett.111.033601

[7] Lanczos, C. *The Variational Principles of Mechanics* 4th edn. (Dover, 1970)

[8] Wilczek, F. "Mass Without Mass I: Most of Matter," *Physics Today* **52** (11), 11-13 (1999).

[9] Oldani, R. "Lagrangian quantum mechanics" (IntechOpen, London, 2019).

[10] T. Bothwell *et al*. "Resolving the gravitational red shift within a millimeter atomic sample". *Nature* **602**, Issue 7897, 420 (2022) arXiv:2109.12238