A discussion on our universe boundaries

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Abstract

By formulating the direct integral equation for the Gaussian scalar gravitational potential, we were able to generalize the Newtonian law of gravity. Hence, the obtained integral equation is differentiated to obtain another integral equation for the gravitational force. A new indicator (R_i) is then defined. By the application of a suitable fundamental solution, it was demonstrated that both Gauss's and Newton's gravities were equivalent only in the case of having the (R_i) indicator equal to zero. This proves that our universe is topologically 3D infinite (with no external boundary). Other cases of having values for the (R_i) indicator due to nearby black holes demonstrated that such black holes create internal boundaries in our universe. The developed integral equations are then generalized to 4D spatial space to account for possible nearby universes. With the proposed generalized integral equations, together with the help of suitable measurements, a proposal is given for a computational methodology that could help in inversely locating the internal boundaries of our universe or give us a clue about places where nearby universes might be located.

1-Introduction

Gravity is one of the most mysterious phenomena in nature. There are two main interpretations of gravity [1]. Either to be represented as a force as in Newtonian gravity [2] or as a warp of the space-time continuum as in Einstein's gravity [3]. Several gravitational theories have been reported in the literature [4][5].

The well-known gravitational law of Newton is simple and due to the symmetry of all points in our universe, theoretically, it should be working everywhere. However, by observations, it was understood that it cannot represent the real gravitational field near black holes or in other places in the universe such as those where the flat rotation curve is observed. This might be due to the existence of the so-called dark matter. Recently, there are several works to modify Newton's law of gravity to resolve a few previously unsolved problems, see for example the work in [6][7][8][9].

As a historical evolution of Gauss' representation [10] of Newtonian gravity, according to the nice review article by Prof. Cheng [11], the original derivation of the Laplace equation was based on the study of Newtonian gravitational attraction in 1773. Lagrange recognized the fundamental solution to this problem as a potential function of O(1/r) [12], and the gravitational force could be represented as the gradient of this potential function. In 1782 and

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1787, Laplace [13] was the first to formulate the Laplace equation in polar and Cartesian coordinates for this potential function, respectively. In 1813, Poisson [14] formulated the equation of the gravitational potential in the well-known (till now) form of Poisson's equation (equation (1) in this paper).

The integral representation of Gauss' equations, in most of the work in the literature, was reported under the umbrella of the indirect integral form as in [8][10]. This form always deals with the effect of local gravity caused by local celestial objects, ignoring the effect of the universe boundaries (if any) and depending on the fundamental solutions which decay in the far field.

There are many unresolved questions in modern cosmology; among them is the following question: Does our universe have a physical boundary? This question was raised in several debates on the internet; see, for example, [15][16][17][18]. In most cases, the answer is "no" because measurements inform us that our universe has a flat curvature, so it is impossible to warp! This is not a satisfactory answer, especially for the case where the large-scale flat curvature might not be flat with respect to a much larger scale.

Up to this point, there are a few questions that need to be answered:

1. Despite the symmetry of points in our universe, why does the well-known Newtonian law of gravity sometimes fail near black holes or when having what is called dark matter?
2. Does our universe have a boundary? And what is its shape (if any)?

In this work, we will derive the direct integral equation for the Gaussian potential formulation. Hence, this integral equation will be differentiated to obtain the gravitational force integral equation. To this end, such a new integral equation will be discussed for three cases of points in our universe:

1. A point at which the well-known Newtonian law of gravity is satisfied.
2. A point nearby a black hole (a dark star); at which the well-known Newtonian law of gravity is not satisfied.
3. A point away from a black hole and at which the well-known Newtonian law of gravity is not satisfied.

Based on the symmetry of our universe, if the well-known Newtonian law of gravity is verified at a certain point (one of the above-mentioned cases), it should be valid for any other points (all other cases). Therefore, what we need is only to generalize this law. Throughout the paper, and considering the three cases, we will draw conclusions, and we will modify the derived integral equation to validate the well-known Newtonian law of gravity, making it a real universal law of gravity.

2-The proposed direct integral formulation

The potential form of Gaussian gravity could be formulated in the following Poisson's equation [10]:

$$\nabla^2 \Phi = 4\pi G \varrho$$

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(1)

Where Φ is the gravitational potential, G is the universal gravitational constant, and ϱ i is the density of the surrounding celestial objects (planets, dust, stars, etc.). The corresponding direct integral equation could be formulated by weighting the potential Φ by a function Φ* and integrating by parts (applying Green's identity) twice. This could be written for a point ξ (a source point) inside our universe domain Ω as follows:

$$\Phi(\xi) + \int_{\Gamma(\mathrm{x})} \frac{\partial \Phi^*(\xi,\mathrm{x})}{\partial n(\mathrm{x})} \Phi(\mathrm{x})\, d\Gamma(\mathrm{x}) * = \int_{\Gamma(\mathrm{x})} \Phi^*(\xi,\mathrm{x}) \frac{\partial \Phi(\mathrm{x})}{\partial n(\mathrm{x})}\, d\Gamma(\mathrm{x}) + \int_{\Omega(\mathrm{x})} \Phi^*(\xi,\mathrm{x})[4\pi G \varrho(\mathrm{x})]\, d\Omega(\mathrm{x})$$

(2)

In which x is a field point. It has to be noted that the last domain integral in equation (2) represents the particular integral (solution) of equation (1). It will be denoted in this paper by PI. Differentiating equation (2) with respect to the coordinates of the source point ξ:

$$\Phi_{;i}(\xi) + \int_{\Gamma(\mathrm{x})} \frac{\partial \Phi^*_{;i}(\xi,\mathrm{x})}{\partial n(\mathrm{x})} \Phi(\mathrm{x})\, d\Gamma(\mathrm{x}) = \int_{\Gamma(\mathrm{x})} \Phi^*_{;i}(\xi,\mathrm{x}) \frac{\partial \Phi(\mathrm{x})}{\partial n(\mathrm{x})}\, d\Gamma(\mathrm{x}) + \int_{\Omega(\mathrm{x})} \Phi^*_{;i}(\xi,\mathrm{x})[4\pi G \varrho(\mathrm{x})]\, d\Omega(\mathrm{x})$$

(3)

In which the $()_{;i}$ denotes the differentiation with respect to the coordinate of the source point ξ. n(x) is the normal at a boundary field point x. Γ and Ω are the boundaries of our universe. \(ϱ(\mathrm{x})\) is the mass density of the celestial object at the internal field point x. Roman lower case indices range from 1 to 3; unless otherwise stated. It has to be noted that the energy-mass equivalence could be used in the case of having a star at the internal field point x. For three-dimensional spatial space (as in our universe's case) and choosing Φ* to be the fundamental solution of equation (1), i.e.:

$$\nabla^2 \Phi^*(\xi,\mathrm{x}) = \delta(\xi,\mathrm{x})$$

(4)

Where δ(ξ,x) is the Dirac delta distribution. The solution of equation (4) could be obtained as follows [19]:

$$\Phi^*(\xi,\mathrm{x}) = \frac{1}{4\pi r(\xi,\mathrm{x})}$$

(5)

And its normal derivative with respect to the normal at the field point (x) could be obtained as follows:

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$$\frac{\partial \Phi^*(\xi, \mathrm{x})}{\partial n(\mathrm{x})} = \frac{-1}{4\pi r^2(\xi, \mathrm{x})} r_{,n}$$

(6)

Where the comma denotes the spatial derivative with respect to the coordinate of the field point (x). Noting that:

$$r_{;i}(\xi, \mathrm{x}) = -r_{,i}(\xi, \mathrm{x})$$

(7)

Hence

$$\Phi^*_{;i}(\xi, \mathrm{x}) = \frac{1}{4\pi r^2(\xi, \mathrm{x})} r_{;i}(\xi, \mathrm{x})$$

(8)

Equation (3) could be re-written as follows:

$$\Phi_{;i}(\xi) + R_i(\xi) = \int_{\Omega(\mathrm{x})} \Phi^*_{;i}(\xi, \mathrm{x})[4\pi G \varrho(\mathrm{x})] \, d\Omega(\mathrm{x})$$

(9)

Where the indicator R_i is defined as follows:

$$R_i(\xi) = \int_{\Gamma(\mathrm{x})} \frac{\partial \Phi^*_{;i}(\xi, \mathrm{x})}{\partial n(\mathrm{x})} \Phi(\mathrm{x}) d\Gamma(\mathrm{x}) - \int_{\Gamma(\mathrm{x})} \Phi^*_{;i}(\xi, \mathrm{x}) \frac{\partial \Phi(\mathrm{x})}{\partial n(\mathrm{x})} \, d\Gamma(\mathrm{x})$$

(10)

The gravitational force g_i could be defined as follows [12]:

$$\mathrm{g}_i(\xi) = \Phi_{;i}(\xi)$$

(11)

Substituting from equation (11) into (9) and changing the last integral in equation (9) to be a discrete summation over the celestial objects (k), to give:

$$\mathrm{g}_i(\xi) + R_i(\xi) = \sum_k \frac{-1}{4\pi r^2(\xi, \mathrm{x})} r_{;i}(\xi, x)[4\pi G \varrho(\mathrm{x}_k)]$$

(12)

Defining the radial vector e_i between the field and the source point as:

$$e_i(\xi, \mathrm{x}) = r_{;i}(\xi, \mathrm{x})$$

(13)

Substituting into equation (12) to give:

$$g_i(\xi) + R_i(\xi) = \sum_k \frac{-G\varrho(x_k)}{r^2(\xi, x)} e_i(\xi, x)$$

(14)

Equation (14) represents the general form of the Newtonian gravitational force. It has to be noted that:

1. Recalling equation (2), the term on the right-hand side of equation (14) is the PI part.
2. The Newtonian law of gravity in equation (14) is different from the well-known form in the literature as the R_i term is now included.

To this end, we have three cases of the point ξ, as will be presented in the next sections.

3-Case 1: Our universe's external boundary

The first case is when ξ is an internal point at which the well-known form of Newton's gravitational law is satisfied. This case represents most of the points inside our universe where no black hole is nearby or where there is no effect of dark matter. Referring to equation (14), in this case, the value of R_i should be equal to 0. With regards to Ref [20] Chapter 2, page 85, equation (2.134), this implies that Γ is infinite, or in other words, our universe has no external boundaries, or, more precisely, it is topologically infinite. This is the first conclusion of this paper.

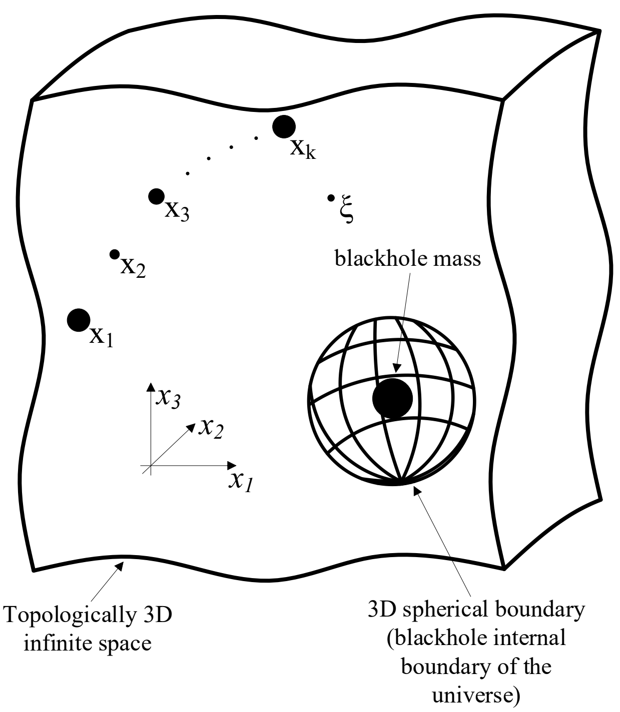

Figure (1): The internal boundary of our universe caused by the presence of a black hole.

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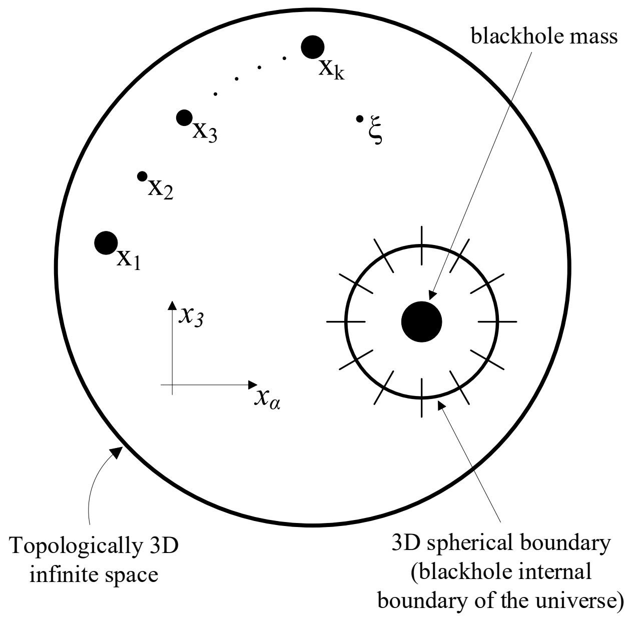

Figure (2): An alternative 2D representation of the 3D definitions in figure (1).

4- Case 2: Our universe's internal boundaries

The second case is when ξ is in the vicinity of a black hole. By observation, the well-known form of Newton's law of gravity does not work in this case. However, from the symmetry of our universe, all points are similar; therefore, such a law should work everywhere. In this case, we should think about adding an extra term (not modifying the order of terms as in MOND [7]). By referring to equation (14), the only way to make Newton's law of gravity work is if the R_i term should have a value. This means the extremely dense star (or black hole) makes such dense material get out of our universe, creating an internal spherical boundary as shown in figure 1 (up to this moment, it is a closed boundary; however, in section 5, we will show that this is an open boundary). Figure (2) represents the same information as that in figure (1) but with collapsing dimensions by one using index notations, i.e., representing a 3D domain as a 2D domain for further use in the next section.

To this end, equation (14) is still valid with R_i having a certain value. This is the second conclusion of this paper. A proposal is made in section (6) of this paper on how to use observational values to compute the radius of such an internal boundary.

It must be noted that, despite the dense material now being outside the boundaries of our universe, its effect could still be included in the direct integral equation. This fact is one of the great advantages of direct boundary integral equations.

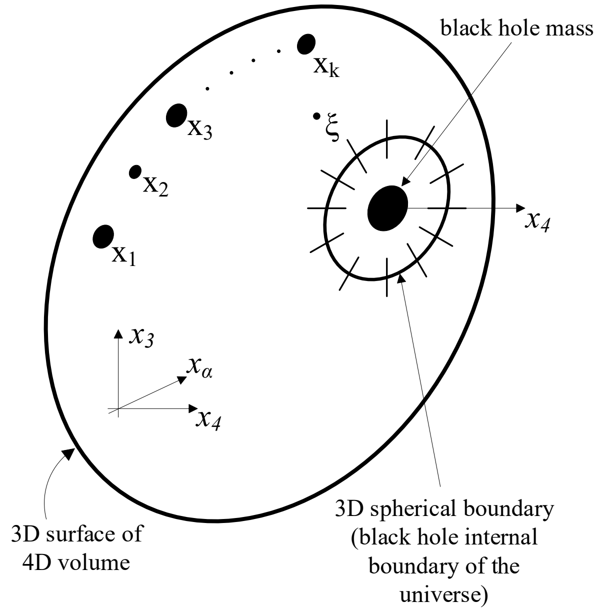

Figure (3): Representation of the 4th spatial dimension with respect to our universe and a certain black hole.

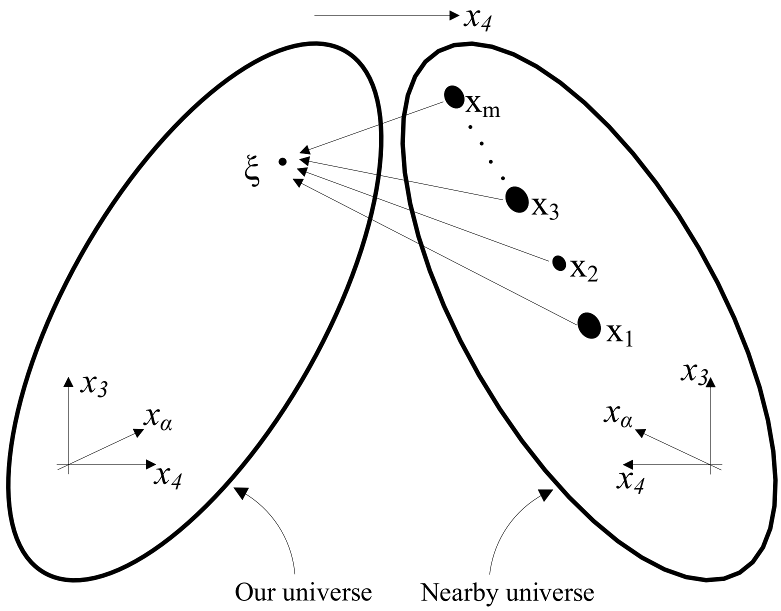

Figure (4): Case of having a nearby universe.

5- Case 3: Nearby universes and gravity in 4D spatial space

In some other cases, even when ξ is located away from black holes, still, the well-known Newtonian gravitational law is not accurate. For example, the case which is referred to as the existence of dark matter and that led to the well-known flat rotation curve [7]. Again, from the symmetry of our universe, equation (14) needs to be modified; noting that in such a case R_i is

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equal to zero (provided that ξ is far away from any black hole). The only way to generalize equation (14) now is to propose the existence of an additional gravitational field caused by another nearby universe. This suggests that both our universe and other nearby universes are embedded in 4D spatial space. The following consequences are raised:

1. The spherical internal boundary that surrounds a black hole is not a closed boundary as previously defined in section (4). It is open in the 4th spatial direction (which we do not feel as human beings). Therefore figure (2) should be re-represented as in figure (3).
2. Figure (4) demonstrates a possible representation of our universe and a nearby universe. Remembering that integral equations can feel the effect of body forces (gravitational fields) that are located outside their relevant boundaries (as previously mentioned at the end of section (4)), we need to modify the integral equation in (14) to account for any gravitational sources in nearby universes. In this case, we must re-formulate the gravitational integral equations in a 4D spatial space as follows: recalling equation (4) in the four spatial dimensions of space:

$$\nabla^2 \bar{\Phi}^*(\xi, \mathrm{x}) = \delta(\xi, \mathrm{x}) \tag{15}$$

Where ∇^2 is the 4D Laplacian, $\delta(\xi, \mathrm{x})$ is the 4D Dirac delta distribution, and $\bar{\Phi}^*(\xi, \mathrm{x})$ is the fundamental gravitational potential in 4D spatial space. The solution of equation (15) could be obtained as follows [21]:

$$\bar{\Phi}^*(\xi, \mathrm{x}) = \frac{1}{4\pi^2 R^2(\xi, \mathrm{x})} \tag{16}$$

Where $R(\xi, \mathrm{x})$ is the Euclidean distance in 4D spatial space. Differentiating (16) with respect to the coordinates of the source point ξ to give:

$$\bar{\Phi}^*{}_{;I}(\xi, \mathrm{x}) = \frac{-1}{2\pi^2 R^3(\xi, \mathrm{x})} R_{;I}(\xi, \mathrm{x}) \tag{17}$$

Where the capital indices range from 1 to 4. The PI term, in the case of 4D gravitational force, could be obtained in a similar way to that of the 3D case, to give:

$$PI = \sum_m \frac{-e_I(\xi, \mathrm{x}_m)}{2\pi^2 R^3(\xi, \mathrm{x})} [4\pi G \varrho(\mathrm{x}_m)] \tag{18}$$

In which the summation is carried out over (m) celestial objects in the nearby universe. Hence, the additional body force term could be written as follows (recall equation (14)):

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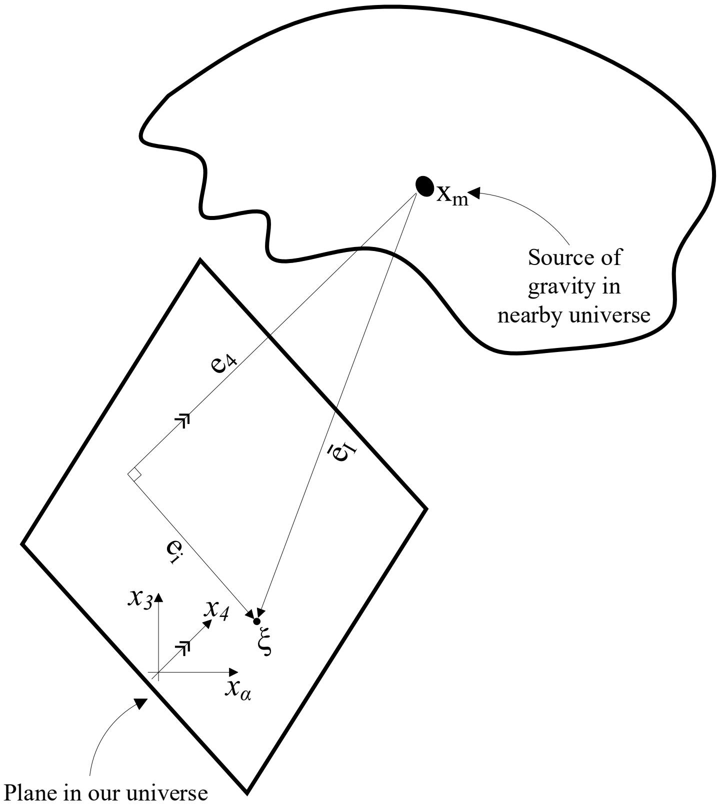

Figure (5): Adjusting vectors between 3D and 4D spaces.

$$g_i(\xi) + R_i(\xi) = \sum_k \frac{-G\varrho(x_k)}{r^2(\xi, x)} e_i(\xi, x) + \sum_m \frac{-2G\varrho(x_m)}{\pi R^3(\xi, x_m)} e_i(\xi, x_m) \tag{19}$$

It must be noted that e_I is changed to e_i (recall figure (5)) as the integral equation in (19) is written within our universe's 3D domain; however, it can account for external effects, even those influenced by higher dimensions.

3. As observations demonstrated that light cannot escape from black holes, this could be because light might be a 4D fluid influenced by gravitational effects in the 4th spatial dimension only. Therefore, it falls into black holes. This also conforms with observations which demonstrated that light is affected by dark matter (or the gravitational field along the 4th spatial dimension). However, this statement needs further research.

4. It seems that there should be a dynamic movement between different universes in the 4D spatial space. This is because of the gravitational effects in the 4th spatial space. This, over time, will lead to a stable configuration which stabilizes the gravitational effect between them. A possible configuration is a galaxy-galaxy interface (but not always, as this is according to the distribution of matter in the two nearby universes). This could demonstrate why most of (but not all) galaxies have the effect of dark matter.

5. The idea of the container 4D domain could be generalized to 5D, 6D, and so on. However, in such a case, we will have a very small gravitational effect as gravity becomes too weak together with large distances. Therefore, this point is outside the scope of this paper.

6- A Proposal for Computation

Up to this point, we mainly have two clear conclusions, which are:

1. Our universe has no external boundary, i.e., it is topologically infinite. In mathematical terms, for any ξ point away from the vicinity of black holes, R_i vanishes.
2. Black holes mainly form internal boundaries, i.e., R_i is no longer equal to zero.

Despite our previous mathematical illustrations, the following two points are still not clear:

1. How to compute the radius of an internal boundary created by a black hole? Moreover, what would be the shape of such a boundary in the case of two adjacent black holes rotating around each other?
2. How to imagine the shape and the distribution of celestial objects in any nearby universe?

A similar idea to the gravitational anomaly [22] which is mainly used to detect the formation of mountains, the existence of certain materials within the earth's crust, etc., is proposed to be extended herein. The idea is to detect the difference between the computed (from equation (14) or equation (19)) and the measured gravitational potentials. Hence, we could inversely predict all relevant information. To apply a similar idea to the above-mentioned two points, suitable measurements are needed. Unfortunately, such measurements are not available to the author. Therefore, the purpose of this section is to propose a possible computational procedure which, together with relevant measurements, could make these two points clear.

It is important to realize that all equations demonstrated in this paper are time-dependent; however, each frame of time could be considered individually without affecting the previous or the next frame in terms of the relevant gravitational field.

First, we will consider having measurements at point ξ near a black hole:

1. A series of gravitational force measurements should be available at several time intervals and at several points ξlocated at equal radial distances from the black hole's center.
2. The value of the computed 3D PI should be subtracted from the value of the measured gravitational force (recall equation (14)).
3. If, at each time interval, all the new gravitational forces at points having the same radial distance are equal, this means such a value is the value of R_i (recall equation (14)).
4. Due to symmetry, the problem could now be simplified to a one-dimensional problem in polar coordinates; hence, the value of the radius of the internal boundary created by the black hole could be computed.

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5. If the values of all the new gravitational forces (point 2) at points having the same radial distance are not equal, this means we have the existence of a form of nearby universe gravitational effects (recall equation (19)).

6. In the case of having two adjacent black holes, similar procedures are followed; however, the problem in such a case could not be simplified to a one-dimensional problem. In this case, the surface of the internal universe boundary should look like an interaction of two 3D spheres. To compute the shape of such a surface, we should propose a set of numerical values for the radii of these intersecting spheres and, using optimization techniques and/or machine or deep learning [23], we could inversely compute the shape of such a boundary. It must be noted that, in this case, the proposed boundary shape needs to be discretized into boundary elements to compute the value of R_i numerically from equation (14).

To this end, we have proposed a computational methodology to make the first point clear. Considering the second point, we will consider having measurements at a point ξ away from the vicinity of black holes. Moreover, at this point, the well-known Newtonian gravitational law does not work. In such a case, equation (19) should be considered. Moreover, we need to suggest or imagine the shape (and distances in the 4th dimension) of the celestial objects in the proposed nearby universe (which might be the gravitational effects of dark matter). In such a case:

1- Several proposed sets of celestial objects should be proposed first.
2- For each set, we should consider a proposed set of locations of the proposed objects at a series of time intervals corresponding to the given measurements.
3- At each time interval, first the value of the computed 3D PI from equation (19) should be subtracted. This is simply to remove the gravitational effects of the visible 3D objects in our universe. Hence, the value of the 4D PI should be computed from equation (19).
4- The previous 3 steps should be repeated for series of points ξ. Hence, with the help of optimization techniques and/or machine or deep learning [23], we could inversely compute the shape, distributions, and locations (in the 4th dimension) of celestial objects in a nearby universe.

7-Conclusions

Based on the symmetry of our universe, we postulated the necessity of the applicability of the well-known traditional Newtonian law of gravity. By formulating the direct integral equation of the Gaussian version of the gravitational field, we were able to generalize the Newtonian law of gravity to make it suitable to be applied at any point in our universe. Moreover, we had four clear conclusions:

1- Our universe has no external boundary, i.e., it is topologically infinite.
2- Black holes mainly form internal boundaries to our universe.
3- Our 3D universe is embedded in 4D spatial space where nearby universes could be located.
4- The gravitational effects of the nearby universes are possible, and this could be an explanation of what is called dark matter.

Due to the lack of available measurements to the author, a proposal is made to:

1- Compute the radius of an internal boundary created by a black hole.
2- Compute the shape of the internal boundary surface in the case of two adjacent black holes.
1- Imagine the shape and distribution of celestial objects in any nearby universe.

The author welcomes the collaboration of other research groups to provide measurements to continue these proposed research points.

Moreover, there are still several open questions proposed in this paper; among them are:

1. What is the correlation between the radius of the internal boundary caused by a black hole and the well-known Schwarzschild radius [24] in the general theory of relativity?

2. This paper proposes the light to be fluid that flows in the 4D dimensional spatial universe and is affected by the gravitational field in the 4^{th} dimension only. Therefore, it falls inside black holes.

3. Due to the gravitational field in the 4D space, there should be movement between universes and movements in the celestial objects in each 3D universe until reaching a steady state configuration.

All these points are open to discussion in future research.

The paper also highlights the strength of the use of the direct boundary integral equation as it can account for the effect of source (load) terms as particular integrals in the following two cases:

1. Such a source is outside the boundary of the relevant problem.

2. Such a source is outside the dimensionality of the considered problem.

The author hopes that the present paper opens a new area of using the boundary integral equations and the boundary element methods (in their discretized form) in cosmology.

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