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A connection between Gompertz diffusion model and Vasicek Interest Rate model

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Abstract

The main goal of this paper is to establish a new connection between the Gompertz diffusion model and the Vasicek Interest Rate model. These connection focus on elementary stochastic calculus and It$\hat{o}$'s calculus. Firstly, we prove that the exponential of the Vasicek Interest Rate model is a Gompertz diffusion process. Secondly, we prove that the logarithm of the Gompertz diffusion process is a Vasicek Interest Rate model. New computations of the probability transition density function and the mean functions of the processes have quite simple formulations.

Keywords: Vasicek interest rate, Gompertz diffusion process, Stochastic diffusion process, Stochastic differential equation, Itô's calculus.

1. Introduction

Two kinds of the point stochastic diffusion process are considered. The first is the Vasicek Interest Rate (VIR) model and the last is the stochastic Gompertz diffusion process (SGDP).

It is well-known that the VIR model is a mathematical method of modeling interest rate movements. The model describes the movement of an interest rate as a composite of market risk, time and equilibrium value, where the rate tends to revert to the average of these factors over time. In essence, it predicts where interest rates will end up at the end of a given period, given current market volatility, the long-term average value of the interest rate and a given market risk factor. This stochastic model is often used in the valuation of interest rate futures and is sometimes used

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to solve for the price of various hard-to-value bonds. Weilin Xiao et al. [1] proposed the fractional Vasicek model to describe the dynamics of the short interest rate.

The SGDP have been studied with respect to specific theoretical aspects and trend analysis, and they have been successfully applied to real cases in Gutiérrez et al. [2, 3, 4]. Gompertz [5] introduces the curve to model the law of human mortality and expressed it as a double exponential. Since then, the curve has undergone several modifications and has been expressed in various forms to facilitate its study. Francisco Torres et al. [6] presented a new SGDP with application to random growth. Tan [7] shows that the stochastic Gompertz birth-death process is a special case of inhomogeneous birth and death processes. El Azri and Nafidi [8] introduced a new computational aspects and results related to the power of the Lundqvist-Korf diffusion process.

In the present paper, using a transformation in diffusion process and by applying It$\hat{o}$'scalculus, we treat two possibilities of equivalence between certain processes are studied. Firstly, we study the VIR model by a SGDP. Finally, we study the SGDP by a stochastic the VIR model.

This paper is organized as follows. In the second section, we present the definitions of the VIR model and the SGDP. In the third section, we consider the between link the VIR model and the SGDP and to determine their probabilistic characteristics. Finally, the briefly conclusion from this study.

2. The VIR model and the SGDP

We shall start from the definition of a VIR model and the definition of a SGDP.

Definition 2.1. The stochastic model proposed is based on the VIR model, which is defined as a diffusion process $\{x(t); t \in [t_0; T]\}$, with values in $(0; \infty)$, and sample paths that are almost surely continuous and with infinitesimal moments that are given by

$$A_1(t,x) = a_1(b_1 - x); \qquad A_2(t,x) = \sigma^2$$

where a_1 is speed of reversion of the mean, b_1 is long-term level of the mean, and σ standard deviation that determine the volatility and a_1, b_1 and σ are real parameters.

We can consider the following SDE

$$dx(t) = a_1\left(b_1 - x(t)\right)dt + \sigma dw(t); \quad x(t_0) = x_0. \tag{2.1}$$

Let $\alpha = a_1 b_1$ and $\beta = -a_1$. After substitution in equation (2.1), we can consider the following SDE

$$dx(t) = (\alpha + \beta x(t))\, dt + \sigma dw(t); \quad x(t_0) = x_0 \tag{2.2}$$

where $\{w(t), t \in [t_0; T]\}$ is a standard Wiener process.

Definition 2.2. The SGDP model, which is defined as a diffusion process $\{x(t); t \in [t_0; T]\}$, with values in $(0; \infty)$, and sample paths that are almost surely continuous and with infinitesimal moments that are given by

$$B_1(t, x) = ax + bx \log(x); \qquad B_2(t, x) = c^2 x^2$$

where $c > 0$, a and b are real parameters.

We can consider the following SDE

$$dx(t) = (ax(t) + bx(t) \log(x(t)))\, dt + cx(t)dw(t); \quad x(t_0) = x_0. \tag{2.3}$$

3. The link between the VIR model and the SGDP

The main goal of this section is to establish a link between VIR model and SGDP and determine their probabilistic characteristics.

3.1. Transformation the VIR model into the SGDP

Theorem 3.1. *Let $\{x(t); t \in [t_0; T]\}$ be a diffusion process with values in $(0; \infty)$. If $\{x(t); t \in [t_0; T]\}$ is a VIR Model. Then the stochastic diffusion process $\left\{e^{x(t)}; t \in [t_0; T]\right\}$ is a SGDP.*

Proof. The VIR Model (2.2)

$$dx(t) = (\alpha + \beta x(t))\, dt + \sigma dw(t); \quad x(t_0) = x_0$$

By applying the Itô formula to the transformation $e^{x(t)}$, we obtain the following equation

$$d\left(e^{x(t)}\right) = \left(\left(\alpha + \frac{\sigma^2}{2}\right) e^{x(t)} + \beta x(t) e^{x(t)}\right) dt + \sigma e^{x(t)} dw(t); \quad x(t_0) = x_0. \tag{3.1}$$

Then, the equation (3.1) can be written as following form

$$dy(t) = \left(\left(\alpha + \frac{\sigma^2}{2}\right) y(t) + \beta y(t) \log(y(t))\right) dt + \sigma y(t) dw(t); \quad y(t_0) = y_0, \tag{3.2}$$

where $y(t)=e^{x(t)}$, $a=\alpha+\dfrac{\sigma^2}{2}$, $b=\beta$, $c=\sigma$ and $y_0=e^{x_0}$. The diffusion process $\{y(t); t\in[t_0;T]\}$, in the last equation is the SGDP with infinitesimal moments

$$a(t,y)=\left(\alpha+\frac{\sigma^2}{2}\right)y+\beta y\log(y); \qquad b(t,y)=\sigma^2y^2.$$

$\square$

3.2. Transition probability density function of the VIR model using the SGDP

Let Φ and φ be respectively the cumulative probability distribution function (PDF) and the transition probability density function (TPDF) of the SGDP, and F and f be respectively the cumulative PDF and the TPDF of the VIR Model. Then we have

$$\begin{aligned}
f(x,t|y,s) &= \frac{dF(x,t|y,s)}{dx}\\
&= \frac{dp(x(t)|x(s)=y\leq x)}{dx}\\
&= \frac{dp(e^{x(t)}|e^{x(s)}=e^y\leq e^x)}{dx}\\
&= \frac{d\Phi(e^x,t|e^y,s)}{dx}\\
&= e^x\varphi(e^x,t|e^y,s)
\end{aligned}$$

Finally, the TPDF of the VIR Model $x(t)$, given $x(s)$ for $s<t$ is

$$f\left(x,t|y,s\right)=\frac{1}{\sqrt{2\pi\sigma^2\upsilon^2(s,t)}}\exp\left(\frac{-\left[x-\mu(s,t,y)\right]^2}{2\sigma^2\upsilon^2(s,t)}\right), \tag{3.3}$$

where $\mu(s,t,y)=e^{\beta(t-s)}y+\dfrac{\alpha}{\beta}\left(e^{\beta(t-s)}-1\right)$ and $\upsilon^2(s,t)=\dfrac{1}{2\beta}\left(e^{2\beta(t-s)}-1\right)$. This function is the density function of the normal distribution $\Lambda\left(\mu(s,t,y),\sigma^2\upsilon^2(s,t)\right)$.

3.3. Trend function of the VIR model

By the properties of normal distribution, the conditional trend function (CTF) is, for $s<t$

$$E(x(t)|x_s)=e^{\beta(t-s)}x_s+\frac{\alpha}{\beta}\left(e^{\beta(t-s)}-1\right), \tag{3.4}$$

and by considering the initial condition $P\left(x(t_0)=x_0\right)=1$, the trend function (TF) of the process is

$$E(x(t))=e^{\beta(t-t_0)}x_0+\frac{\alpha}{\beta}\left(e^{\beta(t-t_0)}-1\right). \tag{3.5}$$

Remark 3.2. We can also study the asymptotic behaviour in time of the TF, if $\beta<0$, thus obtaining

$$\lim_{t\to\infty}E(x(t))=\frac{-\alpha}{\beta}.$$

3.4. Transformation the SGDP into the VIR model

Theorem 3.3. *Let $\{x(t); t \in [t_0; T]\}$ be a diffusion process with values in $(0; \infty)$. If $\{x(t); t \in [t_0; T]\}$ is a SGDP. Then the stochastic diffusion process $\{\log(x(t)); t \in [t_0; T]\}$ is a VIR Model.*

Proof. The SGDP (2.3)

$$dx(t) = (ax(t) + bx(t)\log(x(t)))\,dt + cx(t)dw(t); \quad x(t_0) = x_0$$

Hence, the equation (2.3) can be written as

$$\frac{dx(t)}{x(t)} = (a + b\log(x(t)))\,dt + cdw(t); \quad x(t_0) = x_0$$

By applying the Itô formula to the transformation $\log(x(t))$, we obtain the following equation

$$d(\log(x(t))) = \left(a - \frac{c^2}{2} + b\log(x(t))\right)dt + cdw(t); \quad x(t_0) = x_0 \tag{3.6}$$

Then, the equation (3.6) can be written as following form

$$dy(t) = (\alpha + \beta y(t))\,dt + \sigma dw(t); \quad y(0) = y_0, \tag{3.7}$$

where $y(t) = \log(x(t))$, $\alpha = a - \frac{c^2}{2}$, $\beta = b$, $\sigma = c$ and $y_0 = \log(x_0)$. The diffusion process $\{y(t); t \in [t_0; T]\}$, in the last equation is the VIR Model with infinitesimal moments

$$a(t, y) = a - \frac{c^2}{2} + by; \qquad b(t, y) = c^2.$$

□

3.5. TPDF of the Gompertz model using the VIR model

Let Φ and φ be respectively the cumulative PDF and the TPDF of the SGDP, and F and f be respectively the cumulative PDF and the TPDF of the VIR Model. Then we have

$$\begin{aligned}
\varphi(x, t|y, s) &= \frac{d\Phi(x, t|y, s)}{dx} \\
&= \frac{dp(x(t)|x(s) = y \leq x)}{dx} \\
&= \frac{dp(\log(x(t))|\log(x(s)) = \log(y) \leq \log(x)}{dx} \\
&= \frac{dF(\log(x(t)), t|\log(y), s)}{dx} \\
&= \frac{1}{x} f(\log(x(t)), t|\log(y), s).
\end{aligned}$$

Finally, the TPDF of the Gompertz process $x(t)$, given $x(s)$ for $s < t$ is

$$\varphi(x,t|y,s) = \frac{1}{x\sqrt{2\pi c^2 \upsilon^2(s,t)}} \exp\left(\frac{-\left[\log(x) - \mu(s,t,y)\right]^2}{2c^2\upsilon^2(s,t)}\right), \tag{3.8}$$

where $\mu(s,t,y) = e^{b(t-s)}\log(y) + \dfrac{2a-c^2}{2b}\left(e^{b(t-s)} - 1\right)$ and $\upsilon^2(s,t) = \dfrac{1}{2b}\left(e^{2b(t-s)} - 1\right)$. This function is the density function of the log-normal distribution $\Lambda_1\left(\mu(s,t,y), c^2\upsilon^2(s,t)\right)$.

3.6. TF of the Gompertz process

By the properties of log-normal distribution, the kth CTF is, for $s < t$

$$E(x(t)^k|x_s) = \exp\left(k\mu(s,t,y) + \frac{k^2\sigma^2\upsilon^2(s,t)}{2}\right),$$

for $k = 1$, the CTF is, for $s < t$

$$E(x(t)|x_s) = \exp\left\{e^{b(t-s)}\log(y) + \frac{2a-c^2}{2b}\left(e^{b(t-s)} - 1\right) + \frac{c^2}{4b}\left(e^{2b(t-s)} - 1\right)\right\}, \tag{3.9}$$

and by considering the initial condition $P\left(x(t_0) = x_0\right) = 1$, the TF (The Gompertzian curve) of the process is

$$E(x(t)) = \exp\left\{\log(x_0)e^{b(t-t_0)} + \frac{2a-c^2}{2b}\left(e^{b(t-t_0)} - 1\right) + \frac{c^2}{4b}\left(e^{2b(t-t_0)} - 1\right)\right\}. \tag{3.10}$$

Remark 3.4. We can also study the asymptotic behaviour in time of the TF, if $b < 0$, thus obtaining

$$\lim_{t\to\infty} E(x(t)) = \exp\left(\frac{c^2 - 4a}{4b}\right).$$

4. Conclusion

Using the elementary stochastic calculus and It$\hat{o}$'s formula, we show two results. First, we have shown that the Vasicek Interest Rate Model is a Gompertz diffusion process. The other hand, we have shown that the stochastic Gompertz diffusion process is a Vasicek Interest Rate Model.

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