Research Article

Conclusions Not Yet Drawn from the Unsolved 4/3-Problem: How to Get a Stable Classical Electron

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It has been known for over 100 years that there is a discrepancy between Maxwell's electrodynamics and the idea of a classical electron as the "atom" of electricity. This incompatibility is known under the terms 4/3 problem of the classical electron and radiation reaction force and was circumvented in the currently most successful theories, the quantum field theories, by limit value considerations, by the mutual subtraction of infinities, i.e. by purely mathematical methods that eliminate obvious contradictions but are not really based on an intuitive understanding of its physical causes. The actual origin of the problems mentioned lies in the instability of the classical electron. Stabilization cannot be achieved within the framework of Maxwell's electrodynamics. This raises the question of what a minimal change to the fundamentals of electrodynamics should look like, which contains Maxwell's theory as a limiting case. A detailed analysis of the 4/3 problem points to models that fulfill these requirements.

I. Introduction

The discussion about the origin of the problems of the classical electron is essentially about the concept of particles. The question is whether the idea of assuming atoms of electricity to describe electrodynamic phenomena, as Helmholtz had already suggested, is expedient. Stoney suggested the name "electron" for these "atoms". In formulating this description, classical electrodynamics encountered two unsolvable problems, the 4/3 problem and the problem of radiation reaction force[1][2]. In this article, we focus in particular on the cause of the 4/3 problem and examine what conclusions can be drawn from the form of the discrepancy and what kind of models can solve both problems of classical electrodynamics.

Early on in the formulation of the dynamics of electrons, an idea emerged that is still generally accepted today: a distinction is made between the dynamics of electromagnetic fields, the dynamics of electrons and the interaction between particles and fields. According to the special theory of relativity, the mass of particles is expected to increase with velocity according to the well-known equation (4) with the γ factor. It is a characteristic of moving particles that are described in a "stationary" three-dimensional Euclidean reference system Σ, see section II and Fig. 1, that the velocity vector $u^\mu = \gamma(c, \vec{v})$ does not have to be orthogonal to the position vectors $x^\mu := (0, \vec{x})$, i.e. the scalar product[1]

$$u_\mu x^\mu := -\gamma \vec{v}\vec{x} \leq 0. \tag{1}$$

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does not have to vanish. The equal sign only applies to particles at rest.

For electrons, which are described by electromagnetic fields, such a particle behavior was already expected by Lorentz and Abraham[3][2]. Rohrlich clearly demonstrated in[4][5] that the formalism of special relativity only guarantees that the four-momentum of a field distribution behaves Lorentz-covariantly. However, the 4/3 problem already established by Abraham[1] shows that it was not possible to consistently represent electrons moving in Σ by fields. The reason lies in the instability of the classical electron. To discuss the 4/3 problem, it is sufficient to consider electrons moving at constant speed. The result of such a field description of the classical electron is given in Section II in order to be able to discuss in Section III which minimal changes lead to a stable model of classical electrons, so that the energy-momentum relationships also apply in reference systems that are not orthogonal to u^μ, as the inequality (1) allows. Such a model, which allows to formulate a stable classical electron, is presented in section IV. No divergences occur in it. This minimal extension of Maxwell's electrodynamics no longer contradicts Millikan's famous experiment, which proved a quantization of the electric charge before quantum mechanics moved quantization to the center of scientific interest.

II. Particle and field description of the classical electron

Particles are lumps of matter that remain undestroyed when scattered with sufficiently low energies. Such particles can be assigned an invariant mass m_0 and the concepts of kinematics can be applied without contradiction. To understand what this means, it is helpful to look at the definitions and relationships of relativistic kinematics and their relationship to the non-relativistic terms, see Appendix A.

From the assignment of the space-time coordinates x^μ to time t and the position vector $\vec{x}$

$$x := (c_0 t, \vec{x}) \tag{2}$$

and the four-momentum p^μ to the energy E and the spatial momentum $\vec{p}$

$$p := \left(\frac{E}{c_0}, \vec{p}\right) \tag{3}$$

it follows that the mass m of the particles depends on the ratio of their velocity v to the speed of light c_0

$$m(\beta) \overset{(A13)}{=} \gamma m_0 \text{ with } \gamma := \frac{1}{\sqrt{1-\beta^2}} \text{ and } \beta := \frac{v}{c_0} \tag{4}$$

so the four-momentum results in

$$p \overset{(A13)}{=} \gamma m_0 (c_0, \vec{v}). \tag{5}$$

A closer look shows that the definition (3) follows from the definition (2) if a suitable action function for a free particle is defined and the momentum is derived from it as the temporal component of the energy-momentum tensor

$$p^\mu \overset{(A34)}{=} \int_\Sigma \Theta^{\mu 0}(x) d^3\sigma. \tag{6}$$

The integration here takes place over that three-dimensional space-like volume Σ in which the velocities $\vec{v}$ are determined, i.e. in principle over any three-dimensional space-like volume. Precisely this arbitrariness of Σ is obviously one of the characteristics of a particle, see Fig. 1.

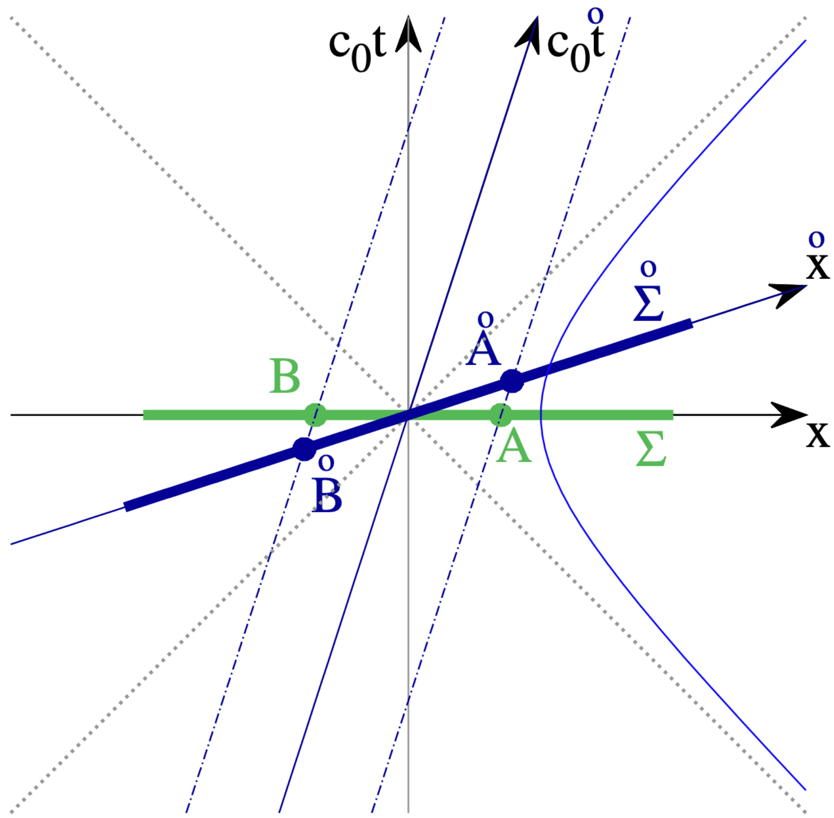

Figure 1. We describe an electron in the laboratory system Σ and in the comoving system Σ° according to the rules of special relativity, as shown in this Minkowski diagram. Note that the points A and A° connected by dashed lines have the same spatial coordinates $\vec{x}^\circ$ and only differ in time $\vec{t}^\circ$. Since the electron is at rest in the comoving system, the field strengths measured in the comoving system are the same at A as in A°. To determine the field strength measured in Σ at A, the coordinates x and t at A must be taken into account and the field strength tensor must be transformed according to the transformation rules of a tensor. The relationships valid for A apply accordingly for B.

In the field description we assume, like Abraham[1], that the mass of the electron is purely electromagnetic in nature, see Appendix B, and calculate energy and momentum according to Eq. (6) for the field of a charge $e = -e_0$ from the symmetric energy-momentum tensor[6]

$$\Theta^{\mu\nu} := -\frac{1}{\mu_0}\eta^{\mu\kappa}F_{\kappa\lambda}F^{\nu\lambda} + \frac{1}{4\mu_0}\eta^{\mu\nu}F_{\kappa\lambda}F^{\kappa\lambda}. \tag{7}$$

In a reference frame Σ, in which the electron moves at a speed $\vec{v} = c\vec{\beta}$, see Fig. 1, the field strengths transform according to the Lorentz transformation of the field strength tensor $F^{\mu\nu}$ to

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$$\vec{E}(\vec{\beta}) = \vec{E}_{\parallel}(0) + \gamma \vec{E}_{\perp}(0) \quad \text{mit} \quad \vec{E}_{\parallel} := \frac{\vec{\beta}(\vec{\beta}\vec{E})}{\beta^2}, \quad \vec{E}_{\perp} := \vec{E} - \vec{E}_{\parallel}, \\ c\vec{B}(\vec{\beta}) = \gamma \vec{\beta} \times \vec{E}(0). \tag{8}$$

For a charge at rest, each of the three electric field components contributes to the energy density (B3) with one third of the rest energy density $\mathcal{E}_0(\vec{r})$. This leads to an energy of the moving charge e

$$E_e(\beta) \overset{(B16)}{=} E_e(0) \frac{4\gamma^2 - 1}{3\gamma}, \tag{9}$$

that does not have the form expected for particles (5), $E_e(\beta) = \gamma E_e(0)$. The momentum

$$\vec{P}_e(\vec{\beta}) \overset{(B18)}{=} \frac{\varepsilon_0}{c_0} \int_{\mathring{\Sigma}} d^3 \mathring{\sigma} \left[-\vec{E}_{\perp}(0) \left(\vec{E}_{\parallel}(0) \vec{\beta} \right) + \gamma \vec{\beta} \vec{E}_{\perp}^2(0) \right] \overset{(B19)}{=} \vec{\beta}\gamma \frac{4}{3} \frac{E_e(0)}{c_0} \tag{10}$$

shows momentum densities normal to the velocity and thus internal stresses in the classical electron, which cancel each other out and therefore do not contribute to the total momentum. The factor 4/3 in $\vec{P}_e(\vec{\beta})$ means a discrepancy between the gravitational mass $E_e(0)/c_0^2$ and the inertial mass $\frac{4}{3} E_e(0)/c_0^2$ of the classical electron, which can be read from the expression (10). This is in obvious contrast to the particle description of a classical electron according to Eq. (5). The cause of this contradiction lies in the instability of the classical electron described by the electromagnetic fields of Maxwell's electrodynamics, as has been known for over 100 years[7] and as is explained in detail in section III. Furthermore, conclusions are drawn in this section as to how a model of a stable classical electron can be formulated.

The fact that the discrepancy discussed in this paper, which is over 100 years old, is not a violation of the rules of special relativity is confirmed by the calculation of the four-momentum of the moving electron, if the coordinates $(c_0 t, \vec{x})$ are used for the calculation, but integrated over the space-like volume $\mathring{\Sigma}$, which are simultaneous for the electron at rest. This integration leads to

$$P^{\mu}(\vec{\beta}) \overset{(B20)}{=} \frac{E_e(0)}{c_0} \left(\gamma, \vec{\beta}\gamma \right), \tag{11}$$

and relates to the four-momentum of the electron at rest as expected. That the rules of special relativity also apply to the unstable classical electron was emphasized by Rohrlich with his calculations in Ref.[5].

III. Conclusions from the problems

An interpretation of the frustrating result (9) for the energy of the moving electron is facilitated by a comparison with the Sine-Gordon model, a Lorentz-invariant, topologically interesting, field model in one space and one time dimension, which is illustrated very clearly in Ref.[8]. The rest energy $E_{SG}(0)$ of a Sine-Gordon soliton increases for a moving soliton to

$$E_{SG}(\beta) \overset{(6.23)[8]}{=} E_{SG}(0) \frac{\gamma^2 + \overbrace{1 + \gamma^2\beta^2}^{\gamma^2}}{2\gamma} = \gamma E_{SG}(0). \tag{12}$$

The stress energy $\propto \gamma$, the potential energy $\propto 1/\gamma$ and the kinetic energy $\propto \gamma\beta^2$ are listed here in sequence. This result, $E_{SG}(\beta) = \gamma E_{SG}(0)$, for the energy of the Sine-Gordon soliton fulfills the expectations of a particle that is subject to the laws of relativistic kinematics. It is also stable because the stress term broadening a particle $\equiv$ Sine-

Gordon soliton and the compressing potential energy keep each other in equilibrium. In the soliton at rest, these two energy contributions must be equal in order for stability to occur according to the Hobart-Derrick theorem[9][10]. This results from the one-dimensional integration over the real axis and the number of derivatives, as the stress term contains two derivatives and the potential term contains no derivative. The stress energy is therefore proportional and the potential energy indirectly proportional to the diameter of the soliton.

When comparing the energy expressions (9) and (12), it is noticeable that adding an energy contribution of one third of the rest energy and with a $1/\gamma$ behaviour, i.e. of $\frac{E_e(0)}{3\gamma}$, to the energy $E_e(\beta)$ in Eq. (9) leads to the behaviour expected for a stabilized classical electron

$$E_e(\beta) \to E_{stab}(\beta) := E_e(\beta) + \frac{E_e(0)}{3\gamma} \overset{(9)}{=} \frac{4}{3}\gamma E_e(0). \tag{13}$$

The addition of this contribution thus leads to the energy value required by the momentum calculation (10) of the moving classical electron. The added energy $\frac{E_e(0)}{3\gamma}$ is obviously the energy contribution required for stabilization. After taking it into account, the energy $E_e(0)$ of the electric field is only 3/4th of the rest energy of a stable classical electron. The size of the added contribution, one third of the electromagnetic field energy for a particle at rest, shows that this contribution must not have any Lorentz indices, i.e. it must be a potential energy. This is because only such a contribution

$$E_{pot}(0) := \int d^3r \mathcal{E}_{pot}(\vec{r}) \tag{14}$$

to the energy of a particle at rest scales under the substitution $r \to \lambda r$ as one third of the electric field energy of a particle at rest, $E_{pot}(0) \overset{(15)}{=} E_e(0)/3$,

$$\frac{d}{d\lambda}\left[\int d^3(\lambda r)\mathcal{E}_0(\lambda r)\right]_{\lambda=1} + \frac{d}{d\lambda}\left[\int d^3(\lambda r)\mathcal{E}_{pot}(\lambda r)\right]_{\lambda=1} = -E_e(0) + 3E_{pot}(0) = 0, \tag{15}$$

if the total energy $E_{stab}(\beta) = E_e(\beta) + E_{pot}(\beta)$ has a minimum at $\lambda = 1$. Such an energy contribution $E_{pot}(\beta)$ also contributes nothing to the momentum $\vec{P}_e(\vec{\beta})$ of a moving electron.

We conclude from this comparison with the Sine-Gordon model that the solution of the 4/3 problem requires a formulation of the degrees of freedom of an electron that allows the field values that occur in the center of the electron to have a high potential energy density and thus prevent an unlimited increase in the size of the central region. This is only possible with a formulation of electrodynamics in which the vector fields A_μ are not the fundamental fields, but with vector fields based on a scalar field (Higgs field) $Q(x)$, with which a suitable potential energy density $\mathcal{E}_{pot}(0)$ can be formulated, which disappears sufficiently quickly at infinity. The trick is to find a formulation in which the dynamics of the scalar field is formulated by the vector field A_μ in the usual form with four Lorentz indices.

We can draw another important conclusion from the internal stresses that we have determined in connection with the calculation of the momentum of a moving classical electron in Eq. (10). Their contributions are formulated in the first term in the square brackets in Eq. (10). Although they cancel each other out in the calculation of the total momentum, they reflect the instability of the classical electron. These stresses only disappear if the field strength components $\vec{E}_\parallel(0)$ and $\vec{E}_\perp(0)$ are everywhere orthogonal to each other in some additional, internal space. This can be achieved in

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non-abelian formulations, when the field strength tensors take values in the Lie algebra of some nonabelian group, as they occur in quantum chromodynamics or in the su(2) algebra, in which such field components can belong to orthogonal directions in the algebra.

This simplest solution of the 4/3 problem mentioned here implies that there is no division of the field degrees of freedom between degrees of freedom for electrons and degrees of freedom for electromagnetic fields, no division of the Lagrangian function into a dynamics of free fields, a dynamics of free particles and an interaction term between these free fields, as exemplified by the Sine-Gordon model. The Lagrangian function of the scalar field should consist of a dynamic term with four Lorentz indices and a potential term and generate the interactions through its non-linearity. In such a description, electrons are a concentrated electromagnetic field. One cannot separate these classical electrons from the electric and magnetic fields they generate. Moreover, this inseparability is a clear conclusion from the experiments in which electrons never appear without their electromagnetic fields. The field itself is uncharged, so that the problem of the instability of the classical electron, which has remained unsolved for 100 years, cannot be attributed to the repulsion of charged regions inside the electron, as is often assumed[11]. It is the structure of the field and the Hamiltonian function that lead to stability and to attractive or repulsive forces between different charges. For the formulation of electrodynamics proposed here, the second problem of classical electrodynamics, the radiation reaction problem, is irrelevant. A single fundamental non-Abelian field with a suitable Lagrangian function as proposed above cannot have a reaction on itself, but can only follow its dynamics as formulated in the Lorentz invariant Lagrangian function. As far as the Lagrangian function is a Lorentz scalar, no contradiction with special relativity can occur due to the consistency of the theory.

The above considerations show what minimal changes could be made to Maxwell's electrodynamics in order to eliminate these inconsistencies in Maxwell's formulation of electrodynamics, which are more than a hundred years old. Maxwell's electrodynamics should then turn out to be a clever linear approximation to the nonlinear theory. It should be interesting to investigate models with such properties[12].

IV. A Minimal Modification to Eliminate the Problems

Here we only give a brief outline of such a formulation, which can be read in detail in the original papers[13][12]. A suitable scalar field with 3 generators is SO(3). It enables a field distribution that assigns the unit of SO(3) to the center of the electron and the fields at infinity to rotations by π around the respective spatial direction of the position vector $\vec{r}$

$$\vec{\omega}(\vec{r}) := \omega(r)\frac{\vec{r}}{r}. \tag{16}$$

A suitable formulation of an arbitrary SO(3) field is an SU(2) field (unit quaternions on an S^3)[2]

$$Q(x) := \exp\{-i\vec{\omega}(x)\vec{\sigma}/2\} = \cos\frac{\omega}{2} - i\sigma_k\frac{\omega_k}{\omega}\sin\frac{\omega}{2} \tag{17}$$

for which it is only necessary to note that pairs of Q fields that differ only by a global sign represent the same SO(3) field. The expression for the potential energy must then be chosen so that the potential term of the Lagrangian density disappears for rotations around π and the potential increases monotonically as the angle of rotation decreases up to the maximum value at the angle of rotation zero (or 2π), see Eq.(9) in Ref. [12] and Eq. (20).

The non-Abelian vector fields then result from the geometry of R^4, in which the S^3 is embedded, see Eq. (4[12]), from the derivatives, the tangent vectors $\partial_\mu Q(x)$. The R^4 is spanned by the real and the three "imaginary" quaternion components in the direction $-i\sigma_k$, as we have used them in Eq. (17). σ_k are the three Pauli matrices. The electric and magnetic field strengths

$$^\star\vec{F}_{\mu\nu} := -\frac{e_0}{4\pi\varepsilon_0 c_0}\vec{R}_{\mu\nu} \tag{18}$$

then result from the ratio $\vec{R}_{\mu\nu}$ of infinitesimal areas in the space of rotations to the corresponding areas in the 3+1-dimensional space-time, from the "area densities" $\vec{R}_{\mu\nu}$, see Eq. (12[12]).

The dynamic term $\mathcal{L}_{dyn}$ of the Lagrangian density is to be chosen proportional to the square of these surface densities,

$$\mathcal{L}_{dyn} := -\frac{\varepsilon_0}{2}\{\vec{E}_k^2 - c_0^2\vec{B}_k^2\} = -\frac{\alpha_f\hbar c_0}{4\pi}\left(\frac{1}{4}\vec{R}_{\mu\nu}\vec{R}^{\mu\nu}\right), \quad \alpha_f := \frac{e_0^2}{4\pi\varepsilon_0\hbar c_0}, \tag{19}$$

whereby the proportionality factor was adapted to the international measurement system (SI) in order to describe a stable classical electron in the usual units.

Inclusion of the potential energy density

$$\mathcal{V} := -\frac{\alpha_f\hbar c}{4\pi}\frac{\cos^6(\omega/2)}{r_0^4}, \qquad \mathcal{L} := \mathcal{L}_{dyn} - \mathcal{V} \tag{20}$$

in the Lagrangian density $\mathcal{L}$ leads to a very simple solution of the variational problem for the profile function in Eq. (16)

$$\omega(r) \overset{(31[12])}{=} 2\arctan\frac{r}{r_0}, \tag{21}$$

an energy density that is finite everywhere and a total energy of the classical electron

$$E_0 \overset{(34[12])}{=} \frac{\alpha_f\hbar c_0}{r_0}\frac{\pi}{4}. \tag{22}$$

r_0 determines the size of the core of the electron and corresponds to the regularization parameter in Eq. (B9). Three quarters of E_0 are the contribution of the electric field energy and one quarter comes from the potential energy. The adjustment of the total energy E_0 to the experimentally determined self-energy $E_0 = 0.511$ MeV results in a radius parameter

$$r_0 = 2.21\mathrm{fm}, \tag{23}$$

which is of the order of the classical electron radius $r_{cl} := \frac{\alpha_f\hbar}{m_e c_0} = 2.82$ fm.

From Eqs. (16) and (21) it follows that the scalar field Q for $r \gg r_0$ is reduced to the degrees of freedom of an S^2 and the potential energy density (20) becomes zero. In this limiting case, the field strength tensor (18) describes the electric field of dual Dirac monopoles, which is U(1)-invariant against rotations of the bases on S^2. The corresponding non-Abelian formulation of Dirac monopoles was already proposed by Wu and Yang in [14][15].

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Further interesting conclusions from this model, in addition to the finite self-energy and the quantization of the electric charge, are the formulation of the spin quantum number as the topological quantum number of the number of coverings of the S^3 of the unit quaternions, see Eq. (41[12]), as well as a running coupling as a geometric effect of the finite extent of the electron. Initial calculations[16] show that the size of the effect is of the right order of magnitude. How far the experimental measurements can be reproduced can only be the subject of further investigations.

The model discussed in this section is limited to the topic of this paper, the classical description of the electron and the phenomena of electrodynamics. It says nothing about quantum mechanics, nothing about the stationary states of atomic physics and the quantum mechanical uncertainty of the measurements. The field quantization $\equiv$ second quantization) is described as topological quantization.

According to this model, electrons are the lightest stable building blocks of our world, the most symmetrical objects imaginable. They are "nodes" of space, topological solitons of the SO(3) group of orientations of spatial Dreibeins.

The model has no free real parameters that can be adjusted. It has only four parameters, for the four man-made scales of the SI, which relate to the classical electron and electrodynamics and establish the relationship to their natural scales. The speed of light c_0 relates the time scale to the length scale, the charge quantum e_0 relates the ampere to the second, the dielectric constant ε_0 relates the length scale to the voltage scale V and thus to the energy scale J = VAs = Ws and to the mass scale $[m] = [E/c_0^2]$ because of $[\varepsilon_0] =$As/Vm. Using the mass of the electron, we can finally fix the length scale r_0 and thus the size of the electron quantitatively.

All other properties[12] are consequences of the model, dynamic properties that come about through interaction and require further investigation.

Appendix A. Kinematics of point particles

A relativistic description of the motion of point particles requires that from the knowledge of the kinematic quantities in one inertial system, their values can be calculated in other inertial systems. It is therefore useful to formulate kinematic quantities as scalars, vectors, and tensors. This allows for a simple transformation of these quantities between the reference systems and an easy check of whether the principle of relativity is fulfilled, i.e., whether the same physical laws apply in all reference systems.

The transformation

$$x^{\mu'} := \Lambda^{\mu'}_{\nu} x^{\nu} \quad \Leftrightarrow \quad x' := \Lambda x, \tag{A1}$$

of the coordinates $x^\mu \overset{(2)}{=} (ct, \vec{x})$ defines a Lorentz transformation from the laboratory system Σ to the moving system Σ'. The condition

$$\Lambda^T \eta \Lambda \overset{!}{=} \eta \quad \text{with} \quad \eta := \mathrm{diag}(1, -1, -1, -1) \tag{A2}$$

on the Lorentz transformations Λ guarantees the invariance

$$x^\mu x_\mu \underset{(A2)}{\overset{(A1)}{=}} x^{\mu'} x_{\mu'} \quad \text{with} \quad x_\mu = \eta_{\mu\nu} x^\nu \tag{A3}$$

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and the invariance of the proper time τ by the differential

$$d\tau := \frac{1}{c}\sqrt{dx_\mu dx^\mu} \overset{(2)}{=} dt\sqrt{1-\left(\frac{d\vec{x}}{dt}\right)^2}. \tag{A4}$$

The gradients transform "contragrediently" with the matrix $\bar{\Lambda}$

$$\partial_{\mu'} \overset{(A1)}{=} \bar{\Lambda}^{\nu}_{\mu'}\partial_\nu \Leftrightarrow \partial' = \bar{\Lambda}\partial \text{ with } \bar{\Lambda} := \eta\Lambda\eta \overset{(A2)}{=} \Lambda^{T^{-1}}. \tag{A5}$$

Using the non-relativistic definitions of velocity and acceleration

$$\vec{v} := \frac{d\vec{x}}{dt}, \quad \vec{a} := \frac{d\vec{v}}{dt}, \tag{A6}$$

and the abbreviations

$$\beta := \frac{\vec{v}}{c_0}, \quad \gamma := \frac{1}{\sqrt{1-\beta^2}}, \quad dt \underset{(A6)}{\overset{(A4)}{=}} \gamma d\tau, \tag{A7}$$

we obtain the four-velocity

$$u := (u^0, \vec{u}) := \frac{dx}{d\tau} = \frac{dx}{dt}\frac{dt}{d\tau} \underset{(A7)}{\overset{(2)}{=}} \gamma(c_0, \vec{v}) \overset{(A7)}{=} c_0\gamma(1, \vec{\beta}), \tag{A8}$$

and the four-acceleration[3],

$$\begin{aligned} b = (b^0, \vec{b}) := \frac{du}{d\tau} \overset{(A7)}{=} \frac{du}{dv_i}\frac{dv_i}{dt}\gamma \underset{(A6)}{\overset{(A8)}{=}} \frac{d}{dv_i}[\gamma(c_0, \vec{v})]a_i\gamma = \\ \overset{(A9)}{=} (\gamma^4\vec{a}\vec{\beta}, \gamma^2\vec{a} + \gamma^4(\vec{a}\vec{\beta})\vec{\beta}), \end{aligned} \tag{A10}$$

which for rectilinear motion is simplified to the space-like four-vector

$$b \underset{(A7)}{\overset{(A10)}{=}} \gamma^4 a(\beta, \vec{1}) \tag{A11}$$

Including the invariant rest mass m_0, we define the four-vectors for the momentum p^μ and the force vector K^μ

$$p := \left(\frac{E}{c_0}, \vec{p}\right) := m_0 u, \quad K := (K^0, \vec{K}) := \frac{dp}{d\tau} \overset{(A10)}{=} m_0 b. \tag{A12}$$

The following applies to the four-momentum

$$p \overset{(A12)}{=} m_0 u \overset{(A8)}{=} \gamma m_0(c_0, \vec{v}) =: m(\beta)(c_0, \vec{v}) \tag{A13}$$

and the four-force

$$K \overset{(A12)}{=} m_0 b \overset{(A10)}{=} m_0(\gamma^4\vec{a}\vec{\beta}, \gamma^2\vec{a} + \gamma^4(\vec{a}\vec{\beta})\vec{\beta}). \tag{A14}$$

It is noteworthy that for the derivation of the relations of the relativistic to the non-relativistic quantities of particle kinematics, in addition to the four-vector (2) of the coordinates x^μ, we have also identified the four-momentum p^μ directly with the non-relativistic quantities, energy E and momentum $\vec{p}$, see Eq. (A12) and Eq. (3). We will elaborate on this after Eq. (A27).

From the expression (A13) for the four-momentum p it follows that the mass contributing to energy and momentum increases proportionally to γ,

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$$E \overset{(A12)}{\underset{(A13)}{=}} \gamma m_0 c_0^2, \quad \vec{p} \overset{(A12)}{\underset{(A13)}{=}} \gamma m_0 \vec{v}. \tag{A15}$$

Since we retain the definitions for the power P and the force $\vec{F}$

$$P := \frac{dE}{dt}, \quad \vec{F} := \frac{d\vec{p}}{dt} \tag{A16}$$

known from non-relativistic mechanics, it follows

$$K \overset{(A12)}{=} m_0 b \overset{(A10)}{=} m_0(\gamma^4 \vec{a}\vec{\beta}, \gamma^2 \vec{a} + \gamma^4 (\vec{a}\vec{\beta})\vec{\beta}) \overset{(A12)}{\underset{(A7)}{=}} \gamma \left(\frac{1}{c_0} P, \vec{F} \right), \tag{A17}$$

i.e.

$$\vec{F} \overset{(A17)}{=} m_0 \left[\gamma \vec{a} + \gamma^3 \left(\vec{a}\vec{\beta} \right) \vec{\beta} \right] \text{ und } P = \vec{v}\vec{F}. \tag{A18}$$

For rectilinear motion, it follows that the inertial mass that must be accelerated increases with γ^3

$$\vec{F} \overset{(A18)}{=} \gamma^3 m_0 \vec{a}. \tag{A19}$$

We will now calculate invariants and draw conclusions from them. Vectors x^μ are denoted by

$$x^\mu x_\mu = \begin{cases} \rho^2 > 0 & \text{als zeitartige Vektoren,} \\ 0 & \text{als lichtartige Vektoren,} \\ -\rho^2 < 0 & \text{als raumartige Vektoren.} \end{cases} \tag{A20}$$

$x^\mu x_\mu$ is a Lorentz invariant but not an invariant of motion. In contrast, however,

$$u_\mu u^\mu \overset{(A8)}{=} c_0^2 \gamma^2 \left(1 - \vec{\beta}^2 \right) \overset{(A7)}{=} c_0^2 \tag{A21}$$

$$p_\mu p^\mu \overset{(A21)}{\underset{(A12)}{=}} m_0^2 c_0^2 \tag{A22}$$

are also invariants of motion. From the vanishing of the differential of these invariants follows the relativistic energy conservation law[4]

$$dE \overset{(A15)}{=} d\gamma m_0 c_0^2 \overset{(A23)}{=} d\vec{x} \frac{d(\gamma m_0 \vec{v})}{dt} \overset{(A13)}{=} d\vec{x} \frac{d\vec{p}}{dt} \overset{(A16)}{=} \vec{F} d\vec{x}, \tag{A24}$$

which expresses that mechanical work $\int \vec{F} d\vec{x}$ contributes to the energy. If Eq. (A24) is divided by dt, it turns out that energy conservation was already included in Eq. (A18)

$$P \overset{(16)}{=} \frac{dE}{dt} \overset{(A24)}{\underset{(A6)}{=}} \vec{v}\vec{F}. \tag{A25}$$

The two equations (A22) and (A12) show that the energy of a moving mass, in addition to the kinetic energy and its relativistic corrections, also contains an additional contribution, the rest energy $m_0 c_0^2$

$$\begin{gathered} (p^0)^2 - \vec{p}^{\,2} \overset{(A22)}{=} m_0^2 c_0^2 \Leftrightarrow E^2 \overset{(A12)}{=} m_0^2 c_0^4 + \vec{p}^{\,2} c_0^2, \\ E \overset{(A15)}{=} \gamma m_0 c_0^2 \overset{(A7)}{=} \frac{1}{\sqrt{1-\beta^2}} m_0 c_0^2 \approx m_0 c_0^2 + \frac{m_0 \vec{v}^{\,2}}{2} + \mathcal{O}(\beta^4). \end{gathered} \tag{A26}$$

Interestingly, by transferring the law of conservation of energy from classical mechanics to relativity theory, it has emerged naturally that for the two canonically conjugated quantities x and p, the spatial components coincide with the three-quantities. However, for their derivatives u, b, and K, we had to introduce separate letters to prevent confusion between their spatial components and the non-relativistic quantities $\vec{v}$, $\vec{a}$, and $\vec{F}$.

We will now show that it is sufficient to define the relationship (2) between three- and four-quantities. The relationship (3) between the momenta follows from a suitably chosen Lagrange density and the energy-momentum tensor derived from it, see Eq. (A34).

Free particles move on geodesics, on extremal paths between events. It is therefore obvious that paths with extreme (minimal) proper time

$$\int_0^1 \frac{d\tau(\lambda)}{d\lambda} d\lambda = \int_{\tau_1}^{\tau_2} d\tau \tag{A27}$$

are proportional to a suitable action function for free particles. The proper time decreases as the speed of the particles increases. The proportionality factor between extreme time and extreme action has the dimension of energy, obviously the rest energy of the free particle

$$S := -m_0 c_0^2 \int_{\tau_1}^{\tau_2} d\tau \overset{(A27)}{=} -m_0 c_0^2 \int_{t_1}^{t_2} \frac{d\tau(t)}{dt} dt \tag{A28}$$

The sign was chosen negative so that the Lagrange function

$$L := -m_0 c_0^2 \frac{d\tau(t)}{dt} \overset{(A7)}{=} -\frac{m_0 c_0^2}{\gamma} \overset{(A7)}{=} -m_0 c_0^2 \sqrt{1-\beta^2} = -m_0 \gamma c_0^2 (1-\beta^2) \tag{A29}$$

increases with increasing momentum of the particle. The components of the canonically conjugated momentum follow in the Lagrange description to

$$p_i := \frac{\partial L}{\partial v_i} \overset{(A29)}{=} m_0 \gamma v_i. \tag{A30}$$

The Hamiltonian results in

$$H := \vec{p}\vec{v} - L \underset{(A29)}{\overset{(A30)}{=}} m_0 \gamma c_0^2. \tag{A31}$$

For point-like electrons with the world line $\vec{x}_e(t)$, the Lagrange density $\mathcal{L}$, energy density $\mathcal{E}$, and momentum density $\vec{\pi}$ are

$$\begin{aligned}
\mathcal{L}(x) &\overset{(A29)}{=} -\frac{m_0 c_0^2}{\gamma} \delta^3(\vec{x} - \vec{x}_e(t)), \\
\mathcal{E}(x) &\overset{(A31)}{=} m_0 \gamma c_0^2 \delta^3(\vec{x} - \vec{x}_e(t)), \\
\vec{\pi}(x) &\overset{(A30)}{=} m_0 \gamma \vec{v} \delta^3(\vec{x} - \vec{x}_e(t))
\end{aligned} \tag{A32}$$

The energy-momentum tensor, which contains the energy density $\mathcal{E} := \Theta^{00}(x)$ and momentum density $\vec{\pi} := \frac{1}{c_0}\Theta^{0i}(x)$, is

$$\Theta^{\mu\nu}(x) \underset{(A8)}{\overset{(A32)}{=}} \frac{m_0}{\gamma} u^\mu(x) u^\nu(x) \delta^3(\vec{x} - \vec{x}_e(t)). \tag{A33}$$

The momentum of the particle consequently results as a spatial integral over the space-time components of $\Theta^{\mu 0}(x) = \Theta^{0\mu}(x)$

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$$p^\mu \overset{(A33)}{\underset{(A32)}{=}} \frac{1}{c_0}\int_\Sigma \Theta^{\mu 0}(x) d^3\sigma. \tag{A34}$$

As can be seen from the transition from the integrated quantities in Eqs. (A29-A31) to the densities (A32), the integration takes place over the three-dimensional space Σ in which the velocities $\vec{v}$ are determined, i.e., in principle, over any three-dimensional space-like volume Σ. Precisely this arbitrariness is obviously one of the characteristics of a particle.

Appendix B. Electrons in Maxwell's field model

We calculate energy and momentum for an extended classical electron of charge $e = -e_0$ and describe it from different reference systems, in a reference system $\overset{\circ}{\Sigma}$ in which the electron is at rest and in a reference system Σ in which the electron moves with a velocity $\vec{v} = c\vec{\beta}$, see Fig. 1.

Like Abraham[1], we start from the idea that the mass of the electron is purely electromagnetic in nature. We therefore use the usual Lagrangian density of electrodynamics

$$\mathcal{L} := -\frac{1}{4\mu_0} F_{\mu\nu}F^{\mu\nu} \tag{B1}$$

and the metric tensor $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. Due to the translation symmetry and after adding a four-divergence term, this leads to the symmetric energy-momentum tensor, see Ref. [6]

$$\Theta^{\mu\nu} \overset{(12.114[6])}{=} -\frac{1}{\mu_0}\eta^{\mu\kappa}F_{\kappa\lambda}F^{\nu\lambda} + \frac{1}{4\mu_0}\eta^{\mu\nu}F_{\kappa\lambda}F^{\kappa\lambda}. \tag{B2}$$

Its elements are in detail

$$\Theta^{00} =: \mathcal{E} \overset{(B2)}{=} \frac{\varepsilon_0}{2}\left(\vec{E}^2 + c_0^2\vec{B}^2\right), \tag{B3}$$

$$\Theta^{0i} = \Theta^{i0} \overset{(B2)}{=} c_0\varepsilon_0 \vec{E}\times\vec{B}, \tag{B4}$$

$$\Theta^{ij} \overset{(B2)}{=} -\varepsilon_0\left(E_iE_j + c_0^2 B_iB_j\right) + \delta_{ij}\frac{\varepsilon_0}{2}\left(\vec{E}^2 + c_0^2\vec{B}^2\right). \tag{B5}$$

The four-momentum of a field distribution generally depends on the reference system Σ in which the field strengths are determined. A reference system can be defined in a Lorentz covariant manner as the three-dimensional Euclidean space that is orthogonal to a velocity vector u^μ and is usually written as

$$d\sigma^\mu := \frac{u^\mu}{c_0}d^3\sigma \tag{B6}$$

A Lorentz covariant definition of the four-momentum is obtained by

$$P^\mu := \frac{1}{c_0}\int_\Sigma \Theta^{\mu\nu}d\sigma_\nu. \tag{B7}$$

In his 1903 paper, Abraham describes electrons on a purely electromagnetic basis with a homogeneous spherically symmetrical charge distribution $\rho(r)$ for electrons at rest. He poses the question: Can the inertia of the electron be completely described by the dynamic effect of its electromagnetic field? It turns out through the 4/3 problem that this question must ultimately be answered in the negative. We will draw conclusions from this failure.

B.1. Self-energy of the classical electron

The electric field of a point charge e at rest at the origin results in the SI according to Gauss's law to

$$\vec{E}_\infty = \frac{e}{4\pi\varepsilon_0}\frac{\vec{e}_r}{r^2}. \tag{B8}$$

As Eq. (B11) will show for the limiting case $r_0 \to 0$, such a field strength is unrealistic, since it leads to an infinite self-energy $E_e(0)$ of the charge e at rest, i.e. an energy that is infinitely greater than the electron mass requires. The more realistic assumption that the electron charge is distributed on a homogeneously charged sphere of radius r_0 or the surface charge of a conducting sphere, as used by Abraham in 1902 on page 147 of Ref. [1], leads to a finite self-energy. In the following, we prefer a regularized form of the electric field strength $\vec{E}(0)$ for a resting classical electron, which does not result in a kink or jump in the density of an extended charge distribution,

$$\vec{E}(0) := \frac{e}{4\pi\varepsilon_0}\frac{\vec{e}_r}{r^2 + r_0^2}, \tag{B9}$$

as proposed by Schwinger in Ref. [11]. In Eq. (B9) it may be irritating that the field strength at the origin has no defined direction. However, this only shows that the electric field strength is defined as the electric flux density on space-time surfaces, which can have different directions starting from the origin. As required, the expression (B9) results in a finite energy density everywhere in the system at rest

$$\mathcal{E}_0(\vec{r}) \overset{(B3)}{=} \frac{\varepsilon_0}{2}\vec{E}^2(0) \overset{(B9)}{=} \frac{\alpha_f \hbar c_0}{8\pi r_0^4}\frac{1}{(1+\rho^2)^2} \text{ with } \alpha_f := \frac{e_0^2}{4\pi\varepsilon_0\hbar c_0}, \rho := \frac{r}{r_0}. \tag{B10}$$

with a total energy, the so-called self-energy, of the charge e that results from integration over the three-dimenional space,

$$E_e(0) \overset{(B7)}{=} 4\pi\int_0^\infty r^2\mathcal{E}_0(\vec{r})dr \overset{(B10)}{=} \frac{\alpha_f\hbar c_0}{r_0}\int_0^\infty \frac{\rho^2}{2(1+\rho^2)^2}d\rho = \frac{\alpha_f\hbar c_0}{r_0}\frac{\pi}{8} =: m_s c_0^2, \tag{B11}$$

and gives the self-energy $m_s c_0^2$ of the classical electron as a function of r_0. Adjusting this self-energy to the physical value leads to $r_0 = 1.1066\,\text{fm}$.

From Eq. (B11), the known instability of the classical electron can be seen. Its mass decreases with $1/r_0$, it dissolves, its radius parameter r_0 increases indefinitely. The reason for this expansion of the core region is easy to detect. Because of the four Lorentz indices in $F_{\mu\nu}F^{\mu\nu}$, the energy density is proportional to r_0^{-4}, the spatial integral only grows with r_0^3, so overall the r_0^{-1} behavior of Eq. (B11) results.

B.2. Energy of the moving electron

To describe an electron moving in Σ, we start from the transformation of the coordinates $\overset{\circ}{x} := (c_0\overset{\circ}{t}, \overset{\circ}{\vec{r}})$ in the comoving frame $\overset{\circ}{\Sigma}$ and transform to Σ in which the electron moves with $\vec{\beta}$

$$\begin{aligned} c_0\overset{\circ}{t} &= \gamma c_0 t - \gamma\vec{\beta}\vec{r}, \\ \overset{\circ}{\vec{r}} &= \gamma\vec{r}_\parallel + \vec{r}_\perp - \gamma\vec{\beta}c_0 t, \end{aligned} \tag{B12}$$

The Lorentz transformation of the field strength tensor $F^{\mu\nu}$ is

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$$\vec{E}(\vec{\beta}) = \vec{E}_\parallel(0) + \gamma\vec{E}_\perp(0) \quad \text{mit} \quad \vec{E}_\parallel := \frac{\vec{\beta}(\vec{\beta}\vec{E})}{\beta^2}, \quad \vec{E}_\perp := \vec{E} - \vec{E}_\parallel,$$
$$c_0\vec{B}(\vec{\beta}) = \gamma\vec{\beta} \times \vec{E}(0). \tag{B13}$$

We can interpret the evaluation of the energy in Eq. (B11) in such a way that the energy of the electron at rest regularized according to Eq. (B9) takes place in the 3D space $\circ\Sigma$, which is orthogonal to the velocity vector $u = \gamma(c, \vec{\beta})$, see Fig. 1. We note for further calculations that in $\overset{\circ}{\Sigma}$, each of the three electric field components contributes to the energy density (B3) with one-third of $\mathcal{E}_0(\vec{r})$, which is due to the spherical symmetry of the field. Now, however, we consider the energy densities $\mathcal{E}_{\vec{\beta}}(\vec{r})$ in the 3D space Σ, which is orthogonal to $(1, \vec{0})$ and contains other space-time points than $\overset{\circ}{\Sigma}$, see Fig. 1. We list the contributions of the field components $\vec{E}_\parallel, \vec{E}_\perp, \vec{B}_\parallel, \vec{B}_\perp$ according to Eq. (B13) in order[5]

$$E_e(\beta) \overset{(B7)}{=} \int_\Sigma d^3\sigma\Theta^{00}(\beta) \overset{(B3)}{=} \int_\Sigma d^3\sigma\mathcal{E}_{\vec{\beta}}(\vec{r}) \underset{(B13)}{\overset{(B3)}{=}} \int_\Sigma d^3\sigma \frac{\mathcal{E}_0(\overset{\circ}{\vec{r}})}{3}(1 + 2\gamma^2 + 0 + 2\gamma^2\beta^2). \tag{B14}$$

In this calculation, we have used that $\vec{E}_\parallel$ contributes unchanged with one third of the energy density of the electron at rest. The contributions of the two orthogonal electric field components are given a factor γ^2. $\vec{B}_\parallel$ and consequently its contribution disappears. The two orthogonal magnetic field components are proportional $\beta^2\gamma^2$ according to Eq. (B13). From an expansion up to the order β^2, i.e. $\gamma^2 \approx 1 + \beta^2$, Abraham read in Eq. (15e) of Ref. [1] that the "kinetic" magnetic energy contributions proportional to β^2, $W_m \propto 2\gamma^2\beta^2 \approx 2\beta^2$, is related to the "static" electrical energy contributions of a Lorentz-contracted electron $W_e \propto 1 + 2\gamma^2 \approx 3 + 2\beta^2$ by

$$W_m \overset{(B14)}{=} \frac{2\beta^2}{3 + 2\beta^2} W_e \approx \frac{2\beta^2}{3} W_e \approx \frac{4}{3}\frac{\beta^2}{2} W_e, \tag{B15}$$

and thus the factor $4/3$ appeared for the first time.

Due to the time independence of the electric field strength in the comoving system, it was possible in Eq. (B14) to read off the field values in the comoving reference system $\vec{\circ}\Sigma$ instead of in Σ, see Fig. 1. Since we have already integrated in Eq. (B11) over the energy density in the comoving system $\vec{\circ}\Sigma$, it makes sense to carry out the integration in Eq. (B14) via $\vec{\circ}\Sigma$, whereby we take into account the Lorentz contraction of the moving electron according to Eq. (B12). The total energy of the moving classical electron thus results in

$$E_e(\beta) \underset{(B12)}{\overset{(B4)}{=}} \frac{1}{\gamma} \int_{\overset{\circ}{\Sigma}} d^3\overset{\circ}{\sigma} \frac{\mathcal{E}_0(\overset{\circ}{\vec{r}})}{3}(1 + 2\gamma^2 + 0 + 2\underbrace{\gamma^2\beta^2}_{\gamma^2-1}) \overset{(B11)}{=} \frac{E_e(0)}{3\gamma}(4\gamma^2 - 1), \tag{B16}$$

which does not have the form (A13) expected for particles. For $\gamma = 1$ the expression is correct, but for $\gamma \to \infty$ the energy increases to $4/3$ of the expected value.

B.3. Momentum of the moving electron

According to the particle interpretation, the momentum of the uniformly moving classical electron results from the Σ integration via the stress tensor or the Poynting vector

$$\vec{P}_e(\vec{\beta}) := \frac{1}{c_0}\int_{\Sigma} d^3\sigma\Theta^{0i}(\vec{\beta}) \overset{(B4)}{=} \varepsilon_0 \int_{\Sigma} d^3\sigma \vec{E}(\vec{\beta}) \times \vec{B}(\vec{\beta})$$
$$\overset{(B13)}{=} \frac{\varepsilon_0}{c_0}\gamma \int_{\Sigma} d^3\sigma[\vec{E}_{\parallel}(0) + \gamma\vec{E}_{\perp}(0)] \times [\vec{\beta} \times \vec{E}_{\perp}(0)]. \tag{B17}$$

In the transformation to $\overset{\circ}{\Sigma}$, we again take into account the Lorentz contraction

$$\vec{P}_e(\vec{\beta}) \underset{(B12)}{\overset{(B17)}{=}} \frac{\varepsilon_0}{c_0}\int_{\overset{\circ}{\Sigma}} d^3\overset{\circ}{\sigma}\left[\vec{E}_{\parallel}(0) + \gamma\vec{E}_{\perp}(0)\right] \times \left[\vec{\beta} \times \vec{E}_{\perp}(0)\right]$$
$$= \frac{\varepsilon_0}{c_0}\int_{\overset{\circ}{\Sigma}} d^3\overset{\circ}{\sigma}\left[-\vec{E}_{\perp}(0)\left(\vec{E}_{\parallel}(0)\vec{\beta}\right) + \gamma\vec{\beta}\vec{E}_{\perp}^2(0)\right]. \tag{B18}$$

The first summand in the integrand of the last expression shows momentum densities normal to the velocity and thus internal stresses in the classical electron. At points that are mirror-symmetrical to the velocity vector $\vec{\beta}$, $E_{\perp}(0)$ points in the opposite directions, so their contributions cancel each other out and do not contribute to the total momentum $\vec{P}_e(\vec{\beta})$. Since the two orthogonal field components $\vec{E}_{\perp}(0)$ in the resting electron each contribute one-third of the field energy, the second contribution provides the 4/3 factor already known from Eq. (B15)

$$\vec{P}_e(\vec{\beta}) \overset{(B18)}{=} \vec{\beta}\gamma\frac{4}{3}\frac{\varepsilon_0}{c_0}\int_{\overset{\circ}{\Sigma}} d^3\overset{\circ}{\sigma}\frac{\vec{E}^2(0)}{2} \overset{(B11)}{=} \vec{\beta}\gamma\frac{4}{3}\frac{E_e(0)}{c_0} \overset{(B11)}{=} \vec{v}\gamma\frac{4}{3}m_s. \tag{B19}$$

Both calculations, (B15) and (B19), thus show that the mass of the electron contributing to the momentum and the kinetic energy is greater by a factor of 4/3 than results from the electrical field energy $E_e(0)$ for the electron at rest in Eq. (B16). This would mean a discrepancy between inertial and gravitational mass. To eliminate this contradiction, Poincaré introduced a negative pressure[7], which has to balance the exploding tendency of the electron.

B.4. Lorentz transformed four-momentum of the electron at rest

The situation is different for the four-momentum of the electron at rest in $\overset{\circ}{\Sigma}$ when its four-momentum $P^{\mu}(\vec{\beta})$ is expressed in coordinates of Σ. As Rohrlich[4] has clearly shown, the correct expression results due to the consistency of special relativity

$$P^{\mu}(\vec{\beta}) \underset{(B6)}{\overset{(B7)}{=}} \frac{1}{c_0}\int_{\overset{\circ}{\Sigma}} d^3\overset{\circ}{\sigma}\Theta^{\mu\nu}_{\Sigma}\beta_{\nu} \underset{(B23)}{\overset{(B21)}{=}} \frac{E_e(0)}{c_0}(\gamma, \vec{\beta}\gamma), \tag{B20}$$

whereby it should be noted that the field values calculated in the laboratory system Σ are integrated over the comoving world volume $\overset{\circ}{\Sigma}$.

In detail, this results in

$$P^0(\vec{\beta}) \overset{(B20)}{=} \frac{\gamma}{c_0}\int_{\overset{\circ}{\Sigma}} d^3\overset{\circ}{\sigma}\left(\Theta^{00}_{\Sigma} - \beta_i\Theta^{0i}_{\Sigma}\right) \overset{(B12)}{=} \frac{\gamma^2}{c_0}\int_{\Sigma} d^3\sigma\left(\Theta^{00}_{\Sigma} - \beta_i\Theta^{0i}_{\Sigma}\right) \underset{(B17)}{\overset{(B14)}{=}} \gamma^2\frac{E_e(\beta)}{c_0} - \gamma^2\vec{\beta}\vec{P}_e\left(\vec{\beta}\right) =$$
$$\underset{(B19)}{\overset{(B16)}{=}} \gamma\left(4\gamma^2 - 1\right)\frac{E_e(0)}{3c_0} - \beta^2\gamma^3\frac{4E_e(0)}{3c_0} = \gamma E_e(0) \tag{B21}$$

$$P^i(\vec{\beta}) \overset{(B20)}{=} \frac{\gamma}{c_0}\int_{\overset{\circ}{\Sigma}} d^3\overset{\circ}{\sigma}\left(\Theta^{i0}_{\Sigma} - \beta_j\Theta^{ij}_{\Sigma}\right) =$$
$$\underset{(B17)}{\overset{(B5)}{=}} \gamma^2 P^i_e(\vec{\beta}) - \frac{\gamma}{c_0}\beta_j\int_{\overset{\circ}{\Sigma}} d^3\overset{\circ}{\sigma}\left\{-\varepsilon_0\left[E_i(0)E_j(0) + c_0^2 B_i(0)B_j(0)\right] + \delta_{ij}\frac{\varepsilon_0}{2}\left[E^2(0) + c_0^2 B^2(0)\right]\right\}$$
$$\overset{(B13)}{=} \gamma^2 P^i_e(\vec{\beta}) + \frac{\gamma}{c_0}\beta\varepsilon_0\int_{\overset{\circ}{\Sigma}} d^3\overset{\circ}{\sigma}\left[E_i(0)E_{\parallel}(0) + c_0^2 B_i(0)\underbrace{B_{\parallel}(0)}_{0}\right] - \frac{\gamma}{c_0}\beta_i\frac{\varepsilon_0}{2}\int_{\overset{\circ}{\Sigma}} d^3\overset{\circ}{\sigma}\left[E^2(0) + c_0^2 B^2(0)\right] \tag{B22}$$

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i.e.

$$P_{\parallel}(\vec{\beta}) \overset{(B22)}{=} \gamma^2 P_e^{\parallel}(\vec{\beta}) + \frac{\gamma}{c_0}\beta\frac{\varepsilon_0}{2}\int_{\mathring{\Sigma}} d^3\mathring{\sigma}\left[E_{\parallel}^2(0) - 2E_{\perp}^2(0) - 2c_0^2 B_{\perp}^2(0)\right] = \underset{(B13)}{\overset{(B19)}{=}} \beta\gamma^3\frac{4}{3}\frac{E_e(0)}{c_0} + \frac{\gamma}{c_0}\beta\frac{E_e(0)}{3}\left[1 - 2\gamma^2 - 2\beta^2\gamma^2\right] = \beta\gamma\frac{E_e(0)}{c_0}. \tag{B23}$$

and

$$P_{\perp}(\vec{\beta}) \overset{(B22)}{=} \frac{\gamma}{c_0}\beta\varepsilon_0\int_{\mathring{\Sigma}} d^3\mathring{\sigma}E_{\parallel}(0)E_{\perp}(0) = 0. \tag{B24}$$

The result (B20) has nothing to do with the 4/3 problem. It only shows that the four-vector $(1,\vec{0})E_e(0)/c_0$ in $\mathring{\Sigma}$ can be transformed by a Lorentz transformation to $(\gamma,\vec{\beta}\gamma)E_e(0)/c_0$ in Σ. It is important that the space-like volume that is integrated is also correctly transformed.

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Footnotes

[1] We use the metric $\eta^{\mu\nu} := \mathrm{diag}(1,-1,-1,-1)$ here, i.e. $u_\mu x^\mu := u_0 x^0 - \vec{u}\vec{x}$.

[2] With the vector arrow in $\vec{\omega}$ we denote the triple ω_k of components of $\vec{\omega}$. Since there are no tensor products of vectors in this work, we write the scalar product without multiplication point, $\vec{\omega}\vec{\sigma} := \omega_i\sigma_i$.

[3]

$$\frac{d\gamma}{dv_i} \overset{(A7)}{=} \frac{1}{c_0}\frac{d\gamma}{d\beta_i}, \quad \frac{d\gamma}{d\beta_i} \overset{(A7)}{=} \frac{\beta_i}{(1-\beta^2)^{3/2}} = \beta_i\gamma^3, \tag{A9}$$

[4]

$$u^0 du^0 \overset{(A21)}{=} \vec{u}d\vec{u} \Leftrightarrow c_0^2 d\gamma \overset{(A8)}{=} \vec{v}d(\gamma\vec{v}) \overset{(A6)}{=} \frac{d\vec{x}}{dt}d(\gamma\vec{v}) = d\vec{x}\frac{d(\gamma\vec{v})}{dt} \tag{A23}$$

[5] We point out that this calculation, which was carried out in analogy to Abraham [1], is an exact calculation according to the definition (B7) of the four-momentum and not, as Rohrlich [5] writes before his Eq. (16): "We can summarize this discussion by saying that the definition (5) is incorrect." With the definition (5), Rohrlich refers to the Abraham-Lorentz definition of the energy of a moving electron, which corresponds to Eq. (B14).

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