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Conclusions not yet drawn from the unsolved 4/3-problem. How to get a stable classical electron.

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It has been known for over 100 years that there is a discrepancy between Maxwell's electrodynamics and the idea of a classical electron as the "atom" of electricity. This incompatibility is known under the terms 4/3 problem and radiation reaction force and has been circumvented in the currently most successful theories, the quantum field theories, by limit value considerations, by the mutual subtraction of infinities, i.e. by purely mathematical methods that eliminate obvious contradictions but are not really based on an intuitive understanding and can therefore never really be understood by the physically interested public. The actual cause of the classical problem lies in the instability of the classical electron. Stabilization cannot be achieved within the framework of Maxwell's electrodynamics. This raises the question of what a minimal change in the foundations of electrodynamics should look like that contains Maxwell's theory as a limiting case. A detailed analysis of the 4/3 problem points to models that fulfill these requirements.

Keywords: classical electron, Maxwell equation, special relativity, energy-momentum tensor

I. INTRODUCTION

The discussion about the origin of the 4/3 problem is basically about the concept of particles. The question is whether the idea of assuming atoms of electricity to describe electrodynamic phenomena, as Helmholtz had already suggested, is expedient. Stoney suggested the name "electron" for these "atoms". In formulating this description, classical electrodynamics encountered two unsolvable problems, the 4/3 problem and the problem of radiation reaction [1, 2]. This article focuses in particular on the cause of the 4/3 problem and examines what conclusions can be drawn from the form of the discrepancy and what kind of models can solve both problems of classical electrodynamics.

Early on in the formulation of the dynamics of electrons, an idea emerged that is still generally accepted today: a distinction is made between the dynamics of electromagnetic fields, the dynamics of electrons and the interaction between particles and fields. According to the special theory of relativity, the mass of particles is expected to increase with velocity according to the well-known equation (4) with the γ factor. It is a characteristic of moving particles that are described in a "stationary" three-dimensional Euclidean reference system Σ, see section II, that the velocity vector $u^\mu = \gamma(c, \vec{v})$ does not have to be orthogonal to the position vectors $x^\mu := (0, \vec{x})$

$$u_\mu x^\mu := -\gamma \vec{v}\vec{x} \leq 0. \tag{1}$$

Orthogonality for all $\vec{x}$ only applies to particles at rest in Σ.

For electrons, which are described by electromagnetic fields, such a particle behavior was already expected by Lorentz and Abraham [2, 3]. Rohrlich clearly demonstrated in [4, 5] that the formalism of special relativity only guarantees that the four-momentum of a field distribution behaves Lorentz covariantly. However, the 4/3 problem already established by Abraham [1] shows that it was not possible to consistently represent electrons moving in Σ by fields. The reason lies in the instability of the classical electron. To discuss the 4/3 problem, it is sufficient to consider electrons moving at constant speed. The result of such a field description of the classical electron is given in section II in order to be able to discuss in detail which minimal changes lead to a stable model of classical electrons, so that the energy-momentum relations also apply in reference systems that are not orthogonal to u^μ. In such a model, which allows a stable classical electron to be formulated, no divergences occur. Maxwell's electrodynamics would thus no longer contradict Millikan's famous experiment, which proved a quantization of the electric charge before quantum mechanics moved quantization to the center of scientific interest, as explained in section III.

II. PARTICLE AND FIELD DESCRIPTION OF THE CLASSICAL ELECTRON

Particles are lumps of matter that remain undestroyed when scattered with sufficiently low energies. Such particles can be assigned an invariant mass m_0 and the concepts of kinematics can be applied without contradiction. To understand what this means, it is helpful to look at the definitions and relationships of relativistic kinematics and their relationship to the non-relativistic terms, see Appendix A.

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From the assignment of the space-time coordinates x^μ to time t and the position vector $\vec{x}$

$$x := (ct, \vec{x}) \tag{2}$$

and the four-vector p^μ of the momenta to the energy E and the spatial momentum $\vec{p}$

$$p := (\frac{E}{c}, \vec{p}) \tag{3}$$

it follows that the mass m of the particles depends on the ratio of their velocity v to the speed of light c

$$m(\beta) = \gamma m_0 \quad \text{mit} \quad \gamma := \frac{1}{\sqrt{1-\beta^2}} \quad \text{und} \quad \beta := \frac{v}{c}, \tag{4}$$

so the four-momentum results in

$$p = \gamma m_0 (c, \vec{v}). \tag{5}$$

A closer look shows that the definition (3) follows from the definition (2) if a suitable action function for a free particle is defined and the momentum is derived from it as the temporal component of the energy-momentum tensor

$$p^\mu \overset{(A35)}{=} \int_\Sigma \Theta^{\mu 0}(x)\, \mathrm{d}^3\sigma. \tag{6}$$

The integration here takes place over the three-dimensional space Σ in which the velocities $\vec{v}$ are determined, i.e. in principle over any three-dimensional spacelike plane Σ. Precisely this arbitrariness of Σ is obviously one of the characteristics of a particle, see Fig. 1.

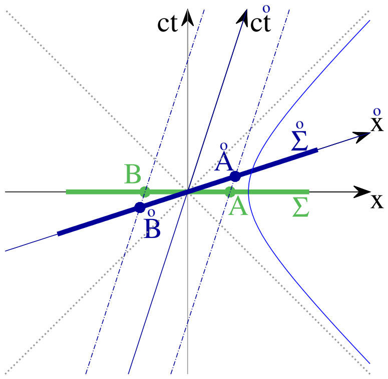

FIG. 1: We describe an electron in the laboratory system Σ and in the co-moving system $\mathring{\Sigma}$. Due to the temporal translation invariance, the electric and magnetic field values determined at different times $\mathring{t}$ in $\mathring{\Sigma}$ are the same in A and $\mathring{A}$, as well as in B and $\mathring{B}$.

In the field description, we assume, like Abraham [1], that the mass of the electron is purely electromagnetic in nature, see Appendix B, and calculate energy and momentum according to Eq. (6) for the field of a charge $e = -e_0$ from the symmetric energy-momentum tensor [6]

$$\Theta^{\mu\nu} := -\frac{1}{\mu_0}\eta^{\mu\kappa}F_{\kappa\lambda}F^{\nu\lambda} + \frac{1}{4\mu_0}\eta^{\mu\nu}F_{\kappa\lambda}F^{\kappa\lambda}. \tag{7}$$

In a reference frame Σ, in which the electron moves at a speed $\vec{v} = c\vec{\beta}$, see Fig. 1, the field strengths transform according to the Lorentz transformation of the field strength tensor $F^{\mu\nu}$ to

$$\begin{aligned} &\vec{E}(\vec{\beta}) = \vec{E}_\parallel(0) + \gamma \vec{E}_\perp(0) \quad \text{mit} \quad \vec{E}_\parallel := \frac{\vec{\beta}(\vec{\beta}\vec{E})}{\beta^2}, \quad \vec{E}_\perp = \vec{E} - \vec{E}_\parallel, \\ &c\vec{B}(\vec{\beta}) = \gamma\vec{\beta} \times \vec{E}(0). \end{aligned} \tag{8}$$

For a charge at rest, each of the three electric field components contributes to the energy density (B3) with one third of the rest energy density $\mathcal{E}_0(\vec{r})$. This leads to an energy of the moving charge

$$E_e(\beta) \overset{(B16)}{=} E_e(0)\,\frac{4\gamma^2-1}{3\gamma}, \tag{9}$$

that does not have the form expected for particles (5), $E_e(\beta) = \gamma E_e(0)$. The momentum

$$\vec{P}_e(\vec{\beta}) \overset{(B18)}{=} \frac{\varepsilon_0}{c}\int_{\mathring{\Sigma}} \mathrm{d}^3\,\mathring{\sigma}\,\left[-\vec{E}_\perp(0)\left(\vec{E}_\parallel(0)\vec{\beta}\right) + \gamma\vec{\beta}\vec{E}_\perp^2(0)\right] \overset{(B19)}{=} \vec{\beta}\gamma\frac{4}{3}\frac{E_e(0)}{c} \tag{10}$$

shows momentum densities normal to the velocity and thus internal stresses in the classical electron, which cancel each other out and therefore do not contribute to the total momentum. The factor 4/3 in $\vec{P}_e(\vec{\beta})$ means a discrepancy between inertial and gravitational mass of the classical electron. However, if the momentum of an electron at rest in $\mathring{\Sigma}$ is described in the reference frame Σ, in which the electron moves with a velocity $\vec{v} = c\vec{\beta}$, by integration over $\mathring{\Sigma}$, the expected expression is obtained, see Appendix B,

$$P^\mu(\vec{\beta}) \overset{(B20)}{=} \frac{E_e(0)}{c}(\gamma, \vec{\beta}\gamma), \tag{11}$$

which only shows the consistency of the relativistic description, as Rohrlich [5] has clearly shown.

III. CONCLUSIONS

An interpretation of the frustrating result (9) for the energy of the moving electron is facilitated by a comparison with the Sine-Gordon model, which is illustrated very clearly in Ref. [7]. The rest energy $E_{\rm SG}(0)$ of a Sine-Gordon soliton increases for a moving soliton to

$$E_{\rm SG}(\beta) = \frac{E_{\rm SG}(0)}{2\gamma}(\gamma^2 + \underbrace{1+\gamma^2\beta^2}_{\gamma^2}) = \gamma E_{\rm SG}(0). \tag{12}$$

Here, the stress energy $\propto \gamma$, the potential energy $\propto 1/\gamma$ and the kinetic energy $\propto \gamma\beta^2$ are listed in order. The energy of the Sine-Gordon soliton meets the expectations of a particle subject to the laws of relativistic kinematics. A soliton at rest is stable because the broadening stress term and the compressing potential energy balance each other. These two energy contributions must be equal in order for stability to occur according to the Hobart-Derrick theorem [8, 9]. This results from the one-dimensional integration over the real axis and the number of derivatives – the stress term contains two derivatives and the potential term contains no derivative.

When comparing the energy expressions (9) and (12), it is noticeable that adding an energy contribution with a $1/\gamma$ behaviour to the energy $E_e(\beta)$ in Eq. (9) leads to the behaviour expected for a stabilized classical electron

$$E_e(\beta) \to E_{\rm stab}(\beta) = \frac{4}{3}\gamma E_e(0), \tag{13}$$

i.e. the energy value required by the momentum calculation (10) of the moving classical electron. The added energy $E_{\rm stab}(\beta) - E_e(\beta) = \frac{E_e(\beta)}{3\gamma}$ is obviously the energy contribution required for stabilization. After taking it into account, the energy of the electric field $E_e(0)$ is only 3/4th of the rest energy of a stable classical electron. The size of the added contribution, one third of the electromagnetic field energy for a particle at rest, shows that this contribution must not have any Lorentz indices, i.e. it must be a potential energy. This is because only such a contribution

$$E_{\rm pot}(0) := \int \mathrm{d}^3r\,\mathcal{E}_{\rm pot}(\vec{r}) \tag{14}$$

to the energy of a particle at rest scales under the substitution $r \to \lambda r$ as one third of the electric field energy of a particle at rest

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\int \mathrm{d}^3(\lambda r)\,\mathcal{E}_0(\lambda r)\right]_{\lambda=1} + \frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\int \mathrm{d}^3(\lambda r)\,\mathcal{E}_{\rm pot}(\lambda r)\right]_{\lambda=1} = -E_e(0) + 3E_{\rm pot}(0) = 0, \tag{15}$$

if the total energy $E_{\text{stab}}(\beta) = E_e(\beta) + E_{\text{pot}}(\beta)$ has a minimum at $\lambda = 1$. Such an energy contribution $E_{\text{pot}}(\beta)$ also does not contribute to the momentum $\vec{P}_e(\vec{\beta})$ of a moving electron.

The solution to the 4/3 problem requires a formulation of the degrees of freedom of an electron that allows to attribute a high potential energy density for the field values occurring in the center of the electron and thus prevent an unlimited increase in the core region. This is only possible with a formulation of electrodynamics in which the vector fields A_μ are not the fundamental fields, but rather with vector fields based on a scalar (Higgs) field $Q(x)$, with which a suitable potential energy density $\mathcal{E}_{\text{pot}}(0)$ can be formulated, which disappears sufficiently fast at infinity. It is necessary to find a formulation in which the dynamics of the scalar field is formulated by the vector field A_μ in the usual form with four Lorentz indices.

The potential energy density just demanded has the dimension of Einstein's cosmological constant Λ, but would be a cosmological function that, when integrated over space, would amount to a quarter of the electron mass. If one assumes such a relationship also for the proton and neutron, then one can derive an average matter density in the universe from the measured value of the cosmological constant. In its measurement results from 2017, the Planck Collaboration [10] states a cosmological energy density of $0.69\rho_{\text{crit}}$, whereby the critical energy density is $4.9\,\text{GeV/m}^3$. This would correspond to a density of around 15 nucleons/m^3.

In connection with the value of the momentum in Eq. (10), we pointed out the internal stresses in the electron reflect the instability of the classical electron, but are canceled out in the total momentum. These stresses, revealed by the first term in the second expression in Eq. (10) only disappear if the field strength components $\vec{E}_\parallel(0)$ and $\vec{E}_\perp(0)$ are orthogonal to each other, which is only possible in non-Abelian formulations of the field strength tensor, as they occur in quantum chromodynamics or in the su(2) algebra, in which such field components can belong to orthogonal directions in the algebra.

This simplest solution of the 4/3 problem addressed here implies that there is no separation of the field degrees of freedom between degrees of freedom for electrons and degrees of freedom for electromagnetic fields, no division of the Lagrangian function into a dynamics of free fields, a dynamics of free particles and an interaction term between these free fields, as exemplified by the Sine-Gordon model. The Lagrangian function of the scalar field should consist of a dynamic term with four Lorentz indices and a potential term and generate the interactions through its non-linearity. In such a description, electrons are a concentrated electromagnetic field. One cannot separate these classical electrons from the electric and magnetic fields they generate. Moreover, this inseparability is a clear conclusion from the experiments in which electrons never appear without their electromagnetic fields. The field itself is uncharged, so that the problem of the instability of the classical electron, which has remained unsolved for 100 years, cannot be attributed to the repulsion of charged regions inside the electron, as has often been assumed [11]. It is the structure of the field and the Hamiltonian function that lead to stability and to attractive or repulsive forces between different charges. For the formulation of electrodynamics proposed here, the second problem of classical electrodynamics, the radiation reaction problem, is irrelevant. A single fundamental non-Abelian field with a suitable Lagrangian function as proposed above cannot have a reaction on itself, but can only follow its dynamics as formulated in the Lorentz invariant Lagrangian. As far as the Lagrangian is a Lorentz scalar, no contradiction with special relativity can occur due to the consistency of the theory.

This paper shows what minimal modifications could be made to Maxwell's electrodynamics in order to eliminate these more than hundred-year-old inconsistencies in Maxwell's formulation of electrodynamics. Maxwellian electrodynamics should then turn out to be a clever linear approximation to the nonlinear theory. It should be interesting to study models with such properties [12].

Appendix A: Kinematics of point particles

A relativistic description of the motion of point particles requires that from the knowledge of the kinematic quantities in one inertial system, their values can be calculated in other inertial systems. It is therefore useful to formulate kinematic quantities as scalars, vectors, and tensors. This allows for a simple transformation of these quantities between the reference systems and an easy check of whether the principle of relativity is fulfilled, i.e., whether the same physical laws apply in all reference systems.

We already know the transformation behavior for the four-vector

$$x^\mu :\overset{(2)}{=} (ct, \vec{r}), \tag{A1}$$

which contains the time t and the position $\vec{x}$ as components. Its transformation

$$x^{\mu\prime} := \Lambda^{\mu\prime}{}_\nu x^\nu \quad \Leftrightarrow \quad x' := \Lambda x, \tag{A2}$$

is defined by the condition

$$\Lambda^T \eta \Lambda \overset{!}{=} \eta \quad \text{mit} \quad \eta = \mathrm{diag}(1, -1, -1, -1). \tag{A3}$$

The gradients transform "contragrediently" with the matrix $\bar{\Lambda}$

$$\partial_{\mu'} \overset{(A2)}{=} \bar{\Lambda}_{\mu'}{}^{\nu} \partial_\nu \quad \Leftrightarrow \quad \partial' = \bar{\Lambda}\partial \quad \text{mit} \quad \bar{\Lambda} \overset{(A2)}{=} \eta \Lambda \eta \overset{(A3)}{=} \Lambda^{T^{-1}} \tag{A4}$$

The Lorentz transformation (A3) guarantees the invariant length

$$x^\mu x_\mu \underset{(A3)}{\overset{(A2)}{=}} x^{\mu'} x_{\mu'} \tag{A5}$$

and the invariance of the proper time τ with the differential

$$\mathrm{d}\tau := \frac{1}{c}\sqrt{\mathrm{d}x_\mu \mathrm{d}x^\mu} \overset{(A1)}{=} \mathrm{d}t \sqrt{1 - \left(\frac{\mathrm{d}\vec{x}}{\mathrm{d}t}\right)^2} \tag{A6}$$

Using the non-relativistic definitions of velocity and acceleration

$$\vec{v} := \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}, \quad \vec{a} := \frac{\mathrm{d}\vec{v}}{\mathrm{d}t}, \tag{A7}$$

and the abbreviations

$$\vec{\beta} := \frac{\vec{v}}{c}, \quad \gamma := \frac{1}{\sqrt{1-\beta^2}}, \quad \mathrm{d}t \underset{(A7)}{\overset{(A6)}{=}} \gamma \,\mathrm{d}\tau, \tag{A8}$$

we obtain the four-velocity

$$u = (u^0, \vec{u}) := \frac{\mathrm{d}x}{\mathrm{d}\tau} = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} \underset{(A8)}{\overset{(A1)}{=}} \gamma(c, \vec{v}) \overset{(A8)}{=} c\gamma(1, \vec{\beta}), \tag{A9}$$

and the four-acceleration,[1]

$$\begin{aligned} b = (b^0, \vec{b}) := \frac{\mathrm{d}u}{\mathrm{d}\tau} &\overset{(A8)}{=} \frac{\mathrm{d}u}{\mathrm{d}v_i}\frac{\mathrm{d}v_i}{\mathrm{d}t}\gamma \underset{(A7)}{\overset{(A9)}{=}} \frac{\mathrm{d}}{\mathrm{d}v_i}\left[\gamma(c, \vec{v})\right] a_i \gamma = \\ &\overset{(A10)}{=} \left(\gamma^4 \vec{a}\vec{\beta}, \gamma^2 \vec{a} + \gamma^4 (\vec{a}\vec{\beta})\vec{\beta}\right), \end{aligned} \tag{A11}$$

which for rectilinear motion is simplified to the space-like four-vector

$$b \overset{(A11)}{=} \gamma^4 a\,(\beta, 1)\,. \tag{A12}$$

Including the invariant rest mass m_0, we define the four-vectors for the momentum p^μ and the force vector K^μ

$$p = (\frac{E}{c}, \vec{p}) := m_0 u, \quad K = (K^0, \vec{K}) := \frac{\mathrm{d}p}{\mathrm{d}\tau} \overset{(A11)}{=} m_0 b. \tag{A13}$$

The following applies to the four-momentum

$$p \overset{(A13)}{=} m_0 u \overset{(A9)}{=} \gamma m_0 (c, \vec{v}) \tag{A14}$$

and the four-force

$$K \overset{(A13)}{=} m_0 b \overset{(A11)}{=} m_0 \left(\gamma^4 \vec{a}\vec{\beta}, \gamma^2 \vec{a} + \gamma^4 (\vec{a}\vec{\beta})\vec{\beta}\right) \tag{A15}$$

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[1]

$$\frac{\mathrm{d}\gamma}{\mathrm{d}v_i} := \frac{1}{c}\frac{\mathrm{d}\gamma}{\mathrm{d}\beta_i}, \quad \frac{\mathrm{d}\gamma}{\mathrm{d}\beta_i} \overset{(A8)}{=} \frac{\beta_i}{(1-\beta^2)^{3/2}} = \beta_i \gamma^3, \tag{A10}$$

It is noteworthy that for the derivation of the relationships of the relativistic to the non-relativistic quantities of particle kinematics, in addition to the four-vector (A1) of the coordinates x^μ in Eq. (A13), we also directly identified the four-momentum p^μ with the non-relativistic quantities, energy E and momentum $\vec{p}$. We will elaborate on this after Eq. (A28).

From the expression (A14) for the four-momentum p it follows that the mass contributing to energy and momentum increases proportionally to γ,

$$E \underset{(A14)}{\overset{(A13)}{=}} \gamma m_0 c_0^2, \quad \vec{p} \underset{(A14)}{\overset{(A13)}{=}} \gamma m_0 \vec{v}. \tag{A16}$$

Since we retain the definitions for the power P and the force $\vec{F}$

$$P := \frac{\mathrm{d}E}{\mathrm{d}t}, \quad \vec{F} := \frac{\mathrm{d}\vec{p}}{\mathrm{d}t}, \tag{A17}$$

known from non-relativistic mechanics, it follows

$$K \overset{(A13)}{=} m_0 b \overset{(A11)}{=} m_0 \left(\gamma^4 \vec{a}\vec{\beta}, \gamma^2 \vec{a} + \gamma^4 (\vec{a}\vec{\beta})\vec{\beta}\right) \underset{(A8)}{\overset{(A13)}{=}} \gamma \left(\frac{1}{c_0} P, \vec{F}\right), \tag{A18}$$

i.e.

$$\vec{F} \overset{(A18)}{=} m_0[\gamma \vec{a} + \gamma^3 (\vec{a}\vec{\beta})\vec{\beta}] \quad \text{and} \quad P = \vec{v}\vec{F}. \tag{A19}$$

If velocity and acceleration are parallel, it follows that the inertial mass increases with γ^3

$$\vec{F} \overset{(A19)}{=} \gamma^3 m_0 \vec{a}. \tag{A20}$$

We will now calculate invariants and draw conclusions from them. Vectors x^μ are denoted by

$$x^\mu x_\mu = \begin{cases} \rho^2 > 0 & \text{timelike vectors,} \\ 0 & \text{lightlike vectors,} \\ -\rho^2 < 0 & \text{spacelike vectors.} \end{cases} \tag{A21}$$

$x^\mu x_\mu$ is a Lorentz invariant but not an invariant of motion. In contrast, however,

$$u_\mu u^\mu \overset{(A9)}{=} c^2 \gamma^2 (1 - \vec{\beta}^2) \overset{(A8)}{=} c^2, \tag{A22}$$

$$p_\mu p^\mu \underset{(A22)}{\overset{(A13)}{=}} m_0^2 c^2, \tag{A23}$$

are also invariants of motion. From the vanishing of the differential of these invariants follows the relativistic energy conservation law [2]

$$\mathrm{d}E \overset{(A16)}{=} \mathrm{d}\gamma m_0 c^2 \overset{(A24)}{=} \mathrm{d}\vec{x} \frac{\mathrm{d}(\gamma m_0 \vec{v})}{\mathrm{d}t} \overset{(A14)}{=} \mathrm{d}\vec{x} \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} \overset{(A17)}{=} \vec{F}\mathrm{d}\vec{x}, \tag{A25}$$

which expresses that mechanical work $\int \vec{F}\mathrm{d}\vec{x}$ contributes to the energy. If Eq. (A25) is divided by $\mathrm{d}t$, it turns out that energy conservation was already included in Eq. (A19)

$$P \overset{(A17)}{=} \frac{\mathrm{d}E}{\mathrm{d}t} \underset{(A7)}{\overset{(A25)}{=}} \vec{v}\vec{F}. \tag{A26}$$

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[2]

$$u^0 \mathrm{d}u^0 \overset{(A22)}{=} \vec{u}\,\mathrm{d}\vec{u} \quad \overset{(A9)}{\Leftrightarrow} \quad c^2 \mathrm{d}\gamma = \vec{v}\,\mathrm{d}(\gamma \vec{v}) \overset{(A7)}{=} \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}\,\mathrm{d}(\gamma \vec{v}) = \mathrm{d}\vec{x}\,\frac{\mathrm{d}(\gamma \vec{v})}{\mathrm{d}t} \tag{A24}$$

The two equations (A23) and (A13) show that the energy of a moving mass, in addition to the kinetic energy and its relativistic corrections, also contains an additional contribution, the rest energy m_0c^2

$$
\begin{aligned}
(p^0)^2 - \vec{p}^{\,2} &\overset{(A23)}{=} m_0^2c^2 \quad \Leftrightarrow \quad E^2 \overset{(A13)}{=} m_0^2c^4 + \vec{p}^{\,2}c^2, \\
E &\overset{(A16)}{=} \gamma m_0 c^2 \overset{(A8)}{=} \frac{1}{\sqrt{1-\beta^2}} m_0c^2 \approx m_0c^2 + \frac{m_0\vec{v}^{\,2}}{2} + \mathcal{O}(\beta^4).
\end{aligned}
\tag{A27}
$$

Interestingly, by transferring the law of conservation of energy from classical mechanics to relativity theory, it has emerged unforced that for the two canonically conjugated quantities x and p, the spatial components coincide with the three-quantities. However, for their derivatives u, b, and K, we had to introduce separate letters to prevent confusion between their spatial components and the non-relativistic quantities $\vec{v}$, $\vec{a}$, and $\vec{F}$.

We will now show that it is sufficient to define the relationship (A1) between three- and four-quantities. The relationship (A13) between the momenta follows from a suitably chosen Lagrange density and the energy-momentum tensor derived from it.

Free particles move on geodesics, on extremal paths between events. It is therefore obvious that paths with extreme (minimal) proper time

$$
\int_0^1 \frac{\mathrm{d}\tau(\lambda)}{\mathrm{d}\lambda}\,\mathrm{d}\lambda = \int_{\tau_1}^{\tau_2} \mathrm{d}\tau \tag{A28}
$$

are proportional to a suitable action function for free particles. The proper time decreases as the speed of the particles increases. The proportionality factor between extreme time and extreme action has the dimension of energy, obviously the rest energy of the free particle

$$
S := -m_0c^2 \int_{\tau_1}^{\tau_2} \mathrm{d}\tau \overset{(A28)}{=} -m_0c^2 \int_{t_1}^{t_2} \frac{\mathrm{d}\tau(t)}{\mathrm{d}t}\,\mathrm{d}t \tag{A29}
$$

The sign was chosen negative so that the Lagrange function

$$
L := -m_0c^2 \frac{\mathrm{d}\tau(t)}{\mathrm{d}t} \overset{(A8)}{=} -\frac{m_0c^2}{\gamma} \overset{(A8)}{=} -m_0c^2\sqrt{1-\beta^2} = m_0\gamma c^2(1-\beta^2) \tag{A30}
$$

increases with increasing momentum of the particle. The components of the canonically conjugated momentum follow in the Lagrange description to

$$
p_i := \frac{\partial L}{\partial v_i} \overset{(A30)}{=} m_0\gamma\, v_i. \tag{A31}
$$

The Hamiltonian results in

$$
H := \vec{p}\,\vec{v} - L \underset{(A30)}{\overset{(A31)}{=}} m_0\gamma c^2. \tag{A32}
$$

For point-like electrons with the world line $\vec{x}_e(t)$, the Lagrange density $\mathcal{L}$, energy density $\mathcal{E}$, and momentum density $\vec{\pi}$ are

$$
\begin{aligned}
\mathcal{L}(x) &\overset{(A30)}{=} \frac{m_0c^2}{\gamma}\delta^3(\vec{x}-\vec{x}_e(t)), \\
\mathcal{E}(x) &\overset{(A32)}{=} m_0\gamma\, c^2\delta^3(\vec{x}-\vec{x}_e(t)), \\
\vec{\pi}(x) &\overset{(A31)}{=} m_0\gamma\,\vec{v}\,\delta^3(\vec{x}-\vec{x}_e(t))
\end{aligned}
\tag{A33}
$$

The energy-momentum tensor, which contains the energy density $\mathcal{E}$ and momentum density $\vec{\pi}$, is

$$
\Theta^{\mu\nu}(x) \underset{(A9)}{\overset{(A33)}{=}} \frac{m_0}{\gamma}\, u^\mu(x)u^\nu(x)\,\delta^3(\vec{x}-\vec{x}_e(t)). \tag{A34}
$$

The momentum of the particle consequently results as a spatial integral over the space-time components of $\Theta^{\mu 0}(x) = \Theta^{0\mu}(x)$

$$
p^\mu \underset{(A33)}{\overset{(A34)}{=}} \int_\Sigma \Theta^{\mu 0}(x)\,\mathrm{d}^3\sigma. \tag{A35}
$$

As can be seen from the transition from the integrated quantities in Eqs. (A30-A32) to the densities (A33), the integration takes place over the three-dimensional space Σ in which the velocities $\vec{v}$ are determined, i.e., in principle, over any three-dimensional spatial world surface Σ. Precisely this arbitrariness is obviously one of the characteristics of a particle.

Appendix B: Electrons in Maxwell's field model

We calculate energy and momentum for an extended classical electron of charge $e = -e_0$ and describe it from different reference systems, in a reference system $\mathring{\Sigma}$ in which the electron is at rest and in a reference system Σ in which the electron moves with a velocity $\vec{v} = c\vec{\beta}$, see Fig. 1.

Like Abraham [1], we start from the idea that the mass of the electron is purely electromagnetic in nature. We therefore use the usual Lagrangian density of electrodynamics

$$\mathcal{L} := -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} \tag{B1}$$

and the metric tensor $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. Due to the translation symmetry and after adding a four-divergence term, this leads to the symmetric energy-momentum tensor [6]

$$\Theta^{\mu\nu} := -\frac{1}{\mu_0}\eta^{\mu\kappa} F_{\kappa\lambda} F^{\nu\lambda} + \frac{1}{4\mu_0}\eta^{\mu\nu} F_{\kappa\lambda} F^{\kappa\lambda}. \tag{B2}$$

Its elements are in detail

$$\Theta^{00} =: \mathcal{E} \overset{(B2)}{=} \frac{\varepsilon_0}{2}(\vec{E}^2 + c^2\vec{B}^2), \tag{B3}$$

$$\Theta^{0i} = \Theta^{i0} \overset{(B2)}{=} c\varepsilon_0\, \vec{E} \times \vec{B}, \tag{B4}$$

$$\Theta^{ij} \overset{(B2)}{=} -\varepsilon_0\,(E_i E_j + c^2 B_i B_j) + \delta_{ij}\frac{\varepsilon_0}{2}(\vec{E}^2 + c^2\vec{B}^2). \tag{B5}$$

The four-momentum of a field distribution generally depends on the reference system Σ in which the field strengths are determined. A reference system can be defined in a Lorentz covariant manner as the three-dimensional Euclidean space that is orthogonal to a velocity vector u^μ and is usually written as

$$\mathrm{d}\sigma^\mu := \frac{u^\mu}{c}\,\mathrm{d}^3\sigma. \tag{B6}$$

A Lorentz covariant definition of the four-momentum is obtained by

$$P^\mu := \frac{1}{c}\int_\Sigma \Theta^{\mu\nu}\,\mathrm{d}\sigma_\nu. \tag{B7}$$

In his 1903 paper, Abraham describes electrons on a purely electromagnetic basis with a homogeneous spherically symmetrical charge distribution $\rho(r)$ for electrons at rest. He poses the question: Can the inertia of the electron be completely described by the dynamic effect of its electromagnetic field? It turns out through the 4/3 problem that this question must ultimately be answered in the negative. We will draw conclusions from this failure.

The electric field of a point charge e at rest at the origin results in the SI according to Gauss' law to

$$\vec{E}_\infty = \frac{e}{4\pi\varepsilon_0}\frac{\vec{e}_r}{r^2}. \tag{B8}$$

To obtain a finite field energy of a classical electron with $\vec{\beta} = \vec{v}/c = 0$ in these considerations, we must regularize the field. Instead of a surface charge or a homogeneous charge distribution, it is more realistic to use a regularized field strength of the form

$$\vec{E}(0) := \frac{e}{4\pi\varepsilon_0}\frac{\vec{e}_r}{r^2 + r_0^2}. \tag{B9}$$

In Eq. (B9) it may be irritating that the field strength at the origin has no defined direction. However, this only shows that the electric field strength is defined as the electric flux density on space-time surfaces, which can have different directions starting from the origin. As required, the expression (B9) results in a finite energy density everywhere in the system at rest

$$\mathcal{E}_0(\vec{r}) \overset{(B3)}{=} \frac{\varepsilon_0}{2}\vec{E}^2(0) \overset{(B9)}{=} \frac{\alpha_f \hbar c}{8\pi r_0^4}\frac{1}{(1+\rho^2)^2} \quad \text{mit} \quad \alpha_f := \frac{e_0^2}{4\pi\varepsilon_0\hbar c}, \quad \rho := \frac{r}{r_0}. \tag{B10}$$

with a total energy, the so-called self-energy

$$E_e(0) \overset{(B7)}{=} 4\pi \int_0^\infty r^2 \mathcal{E}_0(\vec{r})\mathrm{d}r \overset{(B10)}{=} \frac{\alpha_f \hbar c}{r_0} \int_0^\infty \frac{\rho^2}{2(1+\rho^2)^2}\mathrm{d}\rho = \frac{\alpha_f \hbar c}{r_0}\frac{\pi}{8} = m_s c^2, \tag{B11}$$

which leads to a radius parameter $r_0 = 1.1066\,\mathrm{fm}$.

From Eq. (B11), the known instability of the classical electron can be seen. Its mass decreases with $1/r_0$, it dissolves. The reason for this expansion of the core region is easy to determine. The energy density is proportional to r_0^{-4} because of the four Lorentz indices in $F_{\mu\nu}F^{\mu\nu}$, the spatial integral only grows with r_0^3, so overall an r_0^{-1} behavior results.

To describe an electron moving in Σ, we start from the transformation of the coordinates $\overset{\circ}{x} = (c\overset{\circ}{t}, \overset{\circ}{\vec{r}})$ and fields $\vec{E}(0)$ and $\vec{B}(0)$ in the comoving frame $\overset{\circ}{\Sigma}$ and transform to Σ in which the electron moves with $\vec{\beta}$

$$\begin{aligned} c\overset{\circ}{t} &= \gamma ct - \gamma\vec{\beta}\vec{r}, \\ \overset{\circ}{\vec{r}} &= \gamma\vec{r}_\parallel + \vec{r}_\perp - \gamma\vec{\beta}ct. \end{aligned} \tag{B12}$$

The Lorentz transformation of the field strength tensor $F^{\mu\nu}$ is

$$\begin{aligned} \vec{E}(\vec{\beta}) &= \vec{E}_\parallel(0) + \gamma\vec{E}_\perp(0) \quad \text{mit} \quad \vec{E}_\parallel := \frac{\vec{\beta}(\vec{\beta}\vec{E})}{\beta^2}, \quad \vec{E}_\perp = \vec{E} - \vec{E}_\parallel, \\ c\vec{B}(\vec{\beta}) &= \gamma\vec{\beta}\times\vec{E}(0). \end{aligned} \tag{B13}$$

We can interpret the evaluation of the energy in Eq. (B11) in such a way that the energy of the electron at rest regularized according to Eq. (B9) takes place in the 3D space $\overset{\circ}{\Sigma}$, which is orthogonal to the velocity vector $u = \gamma(c, \vec{\beta})$, see Fig. 1. We note for further calculations that in $\overset{\circ}{\Sigma}$, each of the three electric field components contributes to the energy density (B3) with one-third of $\mathcal{E}_0(\vec{r})$, which is due to the spherical symmetry of the field. Now, however, we consider the energy densities $\mathcal{E}_{\vec{\beta}}(\vec{r})$ in the 3D space Σ, which lies orthogonal to $(1, \vec{0})$ and contains other space-time points than $\overset{\circ}{\Sigma}$, see Fig. 1. We list the contributions of the field components $\vec{E}_\parallel, \vec{E}_\perp, \vec{B}_\parallel, \vec{B}_\perp$ according to Eq. (B13) in order [3].

$$E_e(\beta) \overset{(B7)}{=} \int_\Sigma \mathrm{d}^3\sigma\Theta^{00}(\beta) \overset{(B3)}{=} \int_\Sigma \mathrm{d}^3\sigma\mathcal{E}_{\vec{\beta}}(\vec{r}) \underset{(B13)}{\overset{(B3)}{=}} \int_\Sigma \mathrm{d}^3\sigma \frac{\mathcal{E}_0(\overset{\circ}{\vec{r}})}{3}(1 + 2\gamma^2 + 0 + 2\gamma^2\beta^2). \tag{B14}$$

From approximation expressions in the non-relativistic limit in an expansion up to the order β^2, i.e., $\gamma^2 \approx 1 + \beta^2$, Abraham read off in Eq. (15e) of Ref. [1] that the "kinetic" magnetic energy contributions $W_\mathrm{m} \propto 2\gamma^2\beta^2 \approx 2\beta^2$ proportional to β^2 are related to the "static" electric energy contributions $W_\mathrm{e} \propto 1 + 2\gamma^2 \approx 3 + 2\beta^2$, to the self-energy, in the relationship

$$W_\mathrm{m} \overset{(B14)}{=} \frac{4}{3}\frac{\beta^2}{2} W_\mathrm{e}. \tag{B15}$$

Due to the time independence of the electric field strength in the comoving system, it was possible in Eq. (B14) to read off the field values in the comoving reference system $\overset{\circ}{\Sigma}$ instead of in Σ, see Fig. 1. It now makes sense to also perform the integration over $\overset{\circ}{\Sigma}$, whereby we take into account the Lorentz contraction of the moving electron according to Eq. (B12). The total energy of the moving classical electron thus results in

$$E_e(\beta) \underset{(B12)}{\overset{(B14)}{=}} \frac{1}{\gamma} \int_{\overset{\circ}{\Sigma}} \mathrm{d}^3\,\overset{\circ}{\sigma}\,\frac{\mathcal{E}_0(\overset{\circ}{\vec{r}})}{3}(1 + 2\gamma^2 + 0 + 2\underbrace{\gamma^2\beta^2}_{\gamma^2-1}) \overset{(B11)}{=} \frac{E_e(0)}{3\gamma}(4\gamma^2 - 1), \tag{B16}$$

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[3] We point out that this calculation, which was carried out in analogy to Abraham [1], is an exact calculation according to the definition (B7) of the four-momentum and not, as Rohrlich [5] writes before his Eq. (16): "We can summarize this discussion by saying that the definition (5) is incorrect." With the definition (5), Rohrlich refers to the Abraham-Lorentz definition of the energy of a moving electron, which corresponds to Eq. (B14)

which does not have the form (A14) expected for particles.

According to the particle interpretation, the momentum of the uniformly moving classical electron results from the Σ integration via the stress tensor or the Poynting vector

$$\vec{P}_e(\vec{\beta}) := \frac{1}{c}\int_\Sigma \mathrm{d}^3\sigma\,\Theta^{0i}(\vec{\beta}) \overset{(B4)}{=} \varepsilon_0 \int_\Sigma \mathrm{d}^3\sigma \vec{E}(\vec{\beta}) \times \vec{B}(\vec{\beta}) \overset{(B13)}{=} \frac{\varepsilon_0}{c}\gamma \int_\Sigma \mathrm{d}^3\sigma \left[\vec{E}_\parallel(0) + \gamma \vec{E}_\perp(0)\right] \times \left[\vec{\beta} \times \vec{E}_\perp(0)\right]. \quad \text{(B17)}$$

In the transformation to $\overset{\circ}{\Sigma}$, we again take into account the Lorentz contraction

$$\vec{P}_e(\vec{\beta}) \underset{(B12)}{\overset{(B17)}{=}} \frac{\varepsilon_0}{c}\int_{\overset{\circ}{\Sigma}} \mathrm{d}^3\,\overset{\circ}{\sigma} \left[\vec{E}_\parallel(0) + \gamma\vec{E}_\perp(0)\right] \times \left[\vec{\beta} \times \vec{E}_\perp(0)\right] = \frac{\varepsilon_0}{c}\int_{\overset{\circ}{\Sigma}} \mathrm{d}^3\,\overset{\circ}{\sigma}\left[-\vec{E}_\perp(0)\left(\vec{E}_\parallel(0)\vec{\beta}\right) + \gamma\vec{\beta}\vec{E}_\perp^2(0)\right]. \quad \text{(B18)}$$

The first summand in the integrand of the last expression shows momentum densities normal to the velocity and thus internal stresses in the classical electron. At points that are mirror-symmetrical to the velocity vector $\vec{\beta}$, $E_\perp(0)$ points in the opposite directions, so their contributions cancel each other out and do not contribute to the total momentum $\vec{P}_e(\vec{\beta})$. Since the two orthogonal field components $\vec{E}_\perp(0)$ in the resting electron each contribute one-third of the field energy, the second contribution provides the 4/3 factor already known from Eq. (B15)

$$\vec{P}_e(\vec{\beta}) \overset{(B18)}{=} \vec{\beta}\gamma\frac{4}{3}\frac{\varepsilon_0}{c}\int_{\overset{\circ}{\Sigma}} \mathrm{d}^3\,\overset{\circ}{\sigma}\,\frac{\vec{E}^2(0)}{2} \overset{(B11)}{=} \vec{\beta}\gamma\frac{4}{3}\frac{E_e(0)}{c}. \quad \text{(B19)}$$

Both calculations, (B15) and (B19), thus show that the mass of the electron contributing to the momentum and the kinetic energy is greater by a factor of 4/3 than results from the electrical field energy $E_e(0)$ for the electron at rest in Eq. (B16). This would mean a discrepancy between inertial and gravitational mass. To eliminate this contradiction, Poincaré introduced a negative pressure [13], which has to balance the exploding tendency of the electron.

The situation is different for the four-momentum of the electron at rest in $\overset{\circ}{\Sigma}$ when its four-momentum $P^\mu(\vec{\beta})$ is expressed in coordinates of Σ. As Rohrlich [4] has clearly shown, the correct expression results due to the consistency of special relativity

$$P^\mu(\vec{\beta}) \overset{(B7)}{=} \frac{1}{c}\int_{\overset{\circ}{\Sigma}} \mathrm{d}^3\,\overset{\circ}{\sigma}\,\Theta^{\mu\nu}_{\Sigma}\beta_\nu \underset{(B23)}{\overset{(B21)}{=}} \frac{E_e(0)}{c}(\gamma, \vec{\beta}\gamma). \quad \text{(B20)}$$

In detail, this results in

$$P^0(\vec{\beta}) \overset{(B20)}{=} \frac{1}{c}\int_{\overset{\circ}{\Sigma}} \mathrm{d}^3\,\overset{\circ}{\sigma}\,\gamma(\Theta^{00}_\Sigma - \beta_i\Theta^{0i}_\Sigma) \underset{(B17)}{\overset{(B14)}{=}} \frac{\gamma^2}{c}E_e(\beta) - \gamma^2\vec{\beta}\vec{P}_e(\vec{\beta}) \underset{(B19)}{\overset{(B16)}{=}} \frac{E_e(0)}{3c}\gamma(4\gamma^2 - 1) - \frac{E_e(0)}{3c}4\beta^2\gamma^3 = \gamma E_e(0) \quad \text{(B21)}$$

$$\begin{aligned} P^i(\vec{\beta}) &\overset{(B20)}{=} \frac{1}{c}\int_{\overset{\circ}{\Sigma}} \mathrm{d}^3\,\overset{\circ}{\sigma}\,\gamma(\Theta^{i0}_\Sigma - \beta_j\Theta^{ij}_\Sigma) = \\ &\underset{(B5)}{\overset{(B17)}{=}} \gamma^2 P^i_e(\vec{\beta}) - \frac{\gamma}{c}\beta_j\int_{\overset{\circ}{\Sigma}} \mathrm{d}^3\,\overset{\circ}{\sigma}\,\{-\varepsilon_0\left[E_i(0)E_j(0) + c^2B_i(0)B_j(0)\right] + \delta_{ij}\frac{\varepsilon_0}{2}[\vec{E}^2(0) + c^2\vec{B}^2(0)]\} \\ &\overset{(B13)}{=} \gamma^2 P^i_e(\vec{\beta}) + \frac{\gamma}{c}\beta\varepsilon_0\int_{\overset{\circ}{\Sigma}} \mathrm{d}^3\,\overset{\circ}{\sigma}\,[E_i(0)E_\parallel(0) + c^2B_i(0)\underbrace{B_\parallel(0)}_{0}] - \frac{\gamma}{c}\beta_i\frac{\varepsilon_0}{2}\int_{\overset{\circ}{\Sigma}} \mathrm{d}^3\,\overset{\circ}{\sigma}\,[\vec{E}^2(0) + c^2\vec{B}^2(0)], \end{aligned} \quad \text{(B22)}$$

i.e.

$$\begin{aligned} P_\parallel(\vec{\beta}) &\overset{(B22)}{=} \gamma^2 P^\parallel_e(\vec{\beta}) + \frac{\gamma}{c}\beta\frac{\varepsilon_0}{2}\int_{\overset{\circ}{\Sigma}} \mathrm{d}^3\,\overset{\circ}{\sigma}\,[E^2_\parallel(0) - 2E^2_\perp(0) - 2c^2\vec{B}^2_\perp(0)] = \\ &\underset{(B13)}{\overset{(B19)}{=}} \beta\gamma^3\frac{4}{3}\frac{E_e(0)}{c} + \frac{\gamma}{c}\beta\frac{E_e(0)}{3}[1 - 2\gamma^2 - 2\beta^2\gamma^2] = \beta\gamma\frac{E_e(0)}{c}. \end{aligned} \quad \text{(B23)}$$

and

$$P_\perp(\vec{\beta}) \overset{(B22)}{=} \frac{\gamma}{c}\beta\varepsilon_0\int_{\overset{\circ}{\Sigma}} \mathrm{d}^3\,\overset{\circ}{\sigma}\,E_\parallel(0)E_\perp(0) = 0. \quad \text{(B24)}$$

The result (B20) has nothing to do with the 4/3 problem. It only shows that the four-vector $(1,\vec{0})\,E_e(0)/c$ in $\overset{\circ}{\Sigma}$ can be transformed by a Lorentz transformation to $(\gamma,\vec{\beta}\gamma)\,E_e(0)/c$ in Σ.

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