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Bending the Riemann critical strip to a lunula: no zeroes in $1/2 < Re(z) < 1$

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Abstract

The critical strip of the Riemann $\zeta(z)$ is transformed into a crescent-like lunula and the critical line into the unit circle by a conformal transformation. In the new extended complex plane, the argument principle is used to show that there are no zeroes outside of the unit circle, thus proving that there are no zeroes in the right half of the strip, $1/2 < Re(z) < 1$. This constitutes a truly elementary proof of the Riemann Hypothesis.

1 Introduction

The Riemann hypothesis states that all non-trivial zeroes of the $\zeta(z)$ function lie on the $Re(z) = 1/2$ critical line. This conjecture is an unsolved problem that has enormous consequences in many different fields, from number theory to cryptography[1, 2]. In particular, it is of great significance for the distribution of prime numbers and the prime number theorem. It is already well-known that most zeroes lie on the critical line, but their presence or absence from the region $0 < Re(z) < 1$ is still under scrutiny. In this paper it is shown that there are no zeroes in the right half of the strip, i.e. for $1/2 < Re(z) < 1$. This fact, combined with the symmetry of the zeroes about the critical line, insures that, if any nontrivial zero exists, it must lie on the vertical critical line.

2 The *lunula*

The complex plane $z = x + iy$ is transformed into the w-plane, with $w = u + iv$, see Fig.1, by the following invertible conformal bilinear fractional transformation, that is also a Möbius transformation, $\mathcal{T} : z \to w/$

$$w = \frac{z+1}{2-z}\,, \qquad z = \frac{2w-1}{w+1} \tag{1}$$

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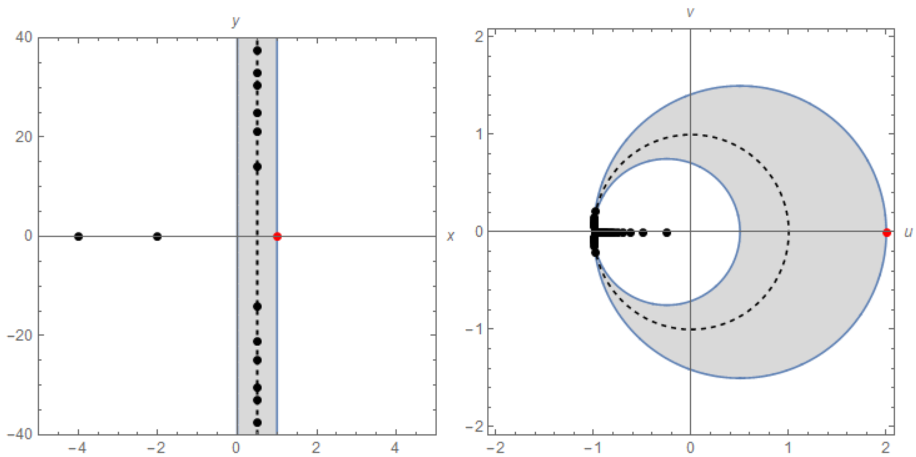

Figure 1: Left: z-plane, $z = x + iy$ with critical strip in gray, dashed black critical line $x = 1/2$, simple pole at $z = \{1, 0\}$ in red, trivial and critical zeroes as black dots. Right: transformed w-plane, $w = u + iv$ with the critical lunula in gray, dashed black critical line, simple pole at $w = \{2, 0\}$ in red, trivial and critical zeroes as black dots.

that maps lines to circles. The Riemann ζ function, in the new plane, is mapped onto a new function $\mathcal{T}(\zeta(z)) \to \vartheta(w)$ such that:

$$\vartheta(w) = \zeta(z) = \zeta\left(\frac{2w-1}{1+w}\right) . \tag{2}$$

The simple pole at $z = \{1, 0\}$ is transformed into $w = \{2, 0\}$, the critical strip is bent into a *lunula*, a concave-convex crescent-like figure. The critical line $x = 1/2$ is mapped into the circumference of the unit circle. The trivial zeroes, that are found at $z = \{-2n, 0\}$ are mapped into $w = \{\frac{1-2n}{2+2n}, 0\}$, while the critical zeroes on the critical line are squeezed onto the corresponding points on the circumference, the first being located at an angle $\pm 167^\circ.885$ with respect to the positive u-axis. As Hardy proved [3], there are infinitely many of them. The black dots in the right panel of Fig. (1) are disconnected, but they are so close to form a hammer-shaped black figure.

The point $w = \{-1, 0\}$ is very remarkable as it is at the same time an accumulation point for trivial zeroes and an essential singularity. A theorem states [4] that, for an analytic function on a simply connected domain, when we have a sequence of zeroes converging to a limiting (or accumulation) point, then the function in that point is either vanishing identically or it is an essential singularity. It is obviously an accumulation point for trivial zeroes, because the formula above insures that these zeroes become denser and denser as n grows, approaching the point. If we now consider the Riemann sphere, i.e. the $\{x, y\}$ complex plane augmented with complex infinity, we have that the Riemann ζ

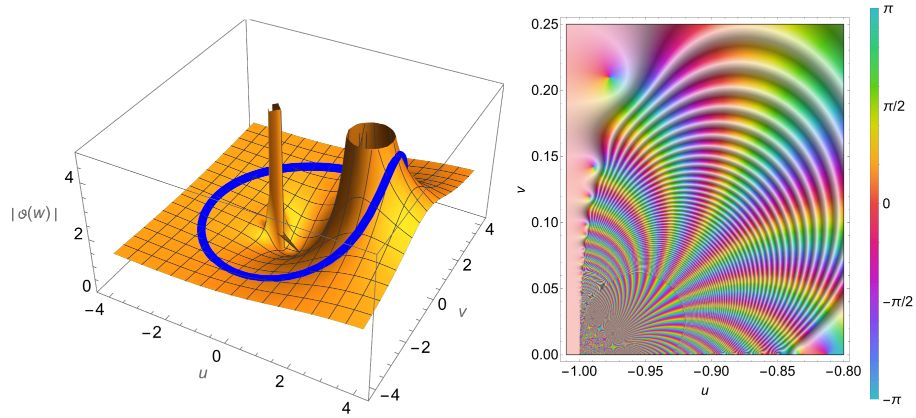

Figure 2: Left: Modulus of the theta function, $|\vartheta(w)|$, in the w-plane, showing the pole in $\{2, 0\}$ and the essential singularity in $\{-1, 0\}$. A circular path with radius $r = 3$ is shown in blue. Right: close up of the complex plot of the ϑ function near $\{-1, 0\}$. The trivial zeroes are visible as black dots on the arc, surrounded by a $2\pi i$ change in argument (rainbow). The internal region reaches all possible values and the argument winds more and more often as one approaches the essential singularity.

has an essential singularity at $\infty_{\mathbb{C}}$ that is mapped onto the point $\{-1, 0\}$. On the w-plane the limits $\lim_{w \to -1} \theta(w)$ and $\lim_{w \to -1} 1/\theta(w)$ are both indeterminate, therefore the point $\{-1, 0\}$ is also an essential singularity. To see the behaviour of the function at these points, we can plot the modulus of the ζ function in the w-plane as in Fig. 2. It is cut at some finite height, and, on the left side, one sees a hint of the fact that $\lim_{w->\infty_{\mathbb{C}}} \theta(w) \to \frac{\pi^2}{6} \simeq 1.645$. One recovers the same constant on all sides, also on the right side, after the pole. The other part of the figure shows the complex plot, i.e. the plot of the argument of θ in the corner close to $\{-1, 0\}$. The rainbow colors appear whenever the argument winds by $2\pi i$, around zeroes. According to the Great Picard's theorem, any punctured neighborhood of an essential singularity attains all possible complex values infinitely often, with at most one exception. That's why the complex plot shows an intricate pattern close to the singularity.

3 Choudury's formula and the argument principle

B.K. Choudury gave a formula (to be found in Ref. [5], unnumbered, just before Eq. 8) for the logarithmic derivative of the Riemann zeta-function :

$$\frac{\zeta'(z)}{\zeta(z)} = \frac{1}{1-z} + \gamma - \sum_{n=1}^{\infty} \bar{A}_n (z-1)^n \tag{3}$$

where γ is the Euler-Mascheroni constant and the coefficients A_n and $\bar{A}_n$ are connected to the Stieltjes' constants γ_n by:

$$\bar{A}_n = -(n+1)A_n - \sum_{k=0}^{n-1} A_k \bar{A}_{n-k-1} \; ; \qquad \bar{A}_0 = -\gamma; \qquad A_n = (-1)^n \gamma_n / n! \tag{4}$$

This is exact, but slowly converging. Since the formula is given in [5] without proof, we give one in the Appendix. The corresponding formula in the w-plane reads

$$f(w) = \gamma + \frac{1+w}{2-w} - \sum_{n=1}^{\infty} \bar{A}_n \left(\frac{w-2}{1+w} \right)^n \tag{5}$$

In order to apply Cauchy's argument principle (See Ref. [6], Ch. 11 or Ref. [7], Ch.5), we will need to evaluate the logarithmic derivative:

$$\frac{\vartheta'(w)}{\vartheta(w)} = \frac{\zeta'(\frac{2w-1}{w+1})}{\zeta(\frac{2w-1}{w+1})} \frac{3}{(w+1)^2} \tag{6}$$

where the last term is the derivative of the argument, or dz/dw. This approach works, and indeed this can be shown either numerically or by using the argument principle on small circular paths around the isolated zeroes or the pole.

Now we want to apply the argument principle **in the w-plane** to the ϑ function, anticlockwise along circles C of radius R centered around the origin, i.e. along $Re^{i\omega}$:

$$\frac{1}{2\pi i} \oint_C \frac{\vartheta'(w)}{\vartheta(w)} dw = \frac{1}{2\pi i} \oint_C f(w) \frac{3}{(1+w)^2} dw = N - P \tag{7}$$

that is connected to the number of zeroes (N) and the number of poles (P) inside the path in a anticlockwise manner, or, that is the same because ϑ is analytic on circles far away from the origin, on the number of zeroes and poles outside of it, if run across clockwisely. This is illustrated in Fig. 3. On the surface of the Riemann sphere, inside and outside loose their meaning and the argument principle is valid on the simply connected portion of the sphere, where circles are contractible to a point. Our function is analytic on the "outside" of the path C, with the exception of a finite number of points, actually only the pole at $w = 2$ in this case.

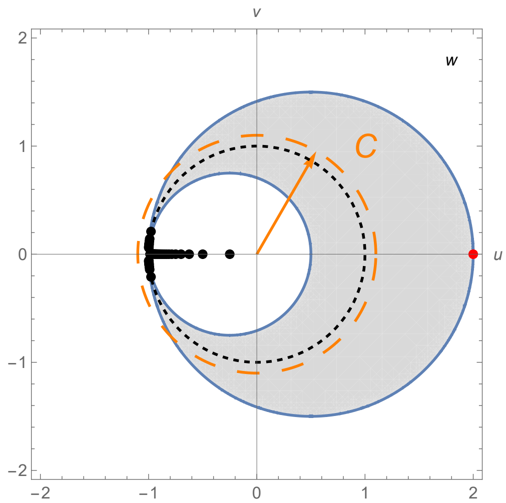

Figure 3: Integration path C (circumference with $R > 1$) in the w-plane in orange.

Now, by plugging in the definition of f from Eq.(5) into Eq.(7), it is easy to see that the first term (γ) evaluates to zero by the residue theorem,

$$\frac{1}{2\pi i}\oint_C \underbrace{\gamma\frac{3}{(1+w)^2}}_{f_\gamma} dw = Res(f_\gamma, -1) = 0 \tag{8}$$

The second term gives a residue of 1 if the circle does not encompass the simple pole at $w = 2$ and goes to zero when the circle is larger, because it takes in the residue at $w = 2$:

$$\frac{1}{2\pi i}\oint_C \underbrace{\frac{(1+w)}{(2-w)}\frac{3}{(1+w)^2}}_{f_f} dw = \begin{cases} Res(f_f, -1) = 1 & \text{if } 1 < R < 2 \\ Res(f_f, -1) + Res(f_f, 2) = 0 & \text{if } R > 2 \end{cases} \tag{9}$$

The third term gives a null residue, $\forall n > 0$, because the Laurent series expansion around $w = -1$ of each term of the type

$$3\frac{(2-w)^n}{(1+w)^{n+2}} \tag{10}$$

always starts from the term $(1+w)^{-2}$, therefore the residue, i.e. the coefficient of the term $(1+w)^{-1}$ is always 0. Thus, since the sum

$$N - P = \begin{cases} 1 & \text{if } 1 < R < 2 \\ 0 & \text{if } R > 2 \end{cases} \tag{11}$$

the difference is a constant on all circles of radius R larger then 1. Here the value $N - P = 1$ is not counting the essential singularity and the infinity of trivial and nontrivial zeroes of the hammer shape, but rather is counting only the single pole outside of the circle, changed in sign because of the equivalence between the inside counted anticlockwise and the outside counted clockwise. There cannot be other zeroes in this region or the argument principle would count them when $R \to 1$. We have just proven that there are no zeroes in the annular region that goes from the black dashed circle in Fig. 1 to the circle touching $w = 2$, see Fig. 3. But that region comprises the whole outer part of the lunula that maps back to the right half of the critical strip, therefore we have just proven that there are no zeroes of the Riemann ζ function for $1/2 < Re(z) < 1$. This is a big step forward with respect to any estimate found so far (See [8, 9]).

Now, if a zero cannot exist in this region, because of the symmetry established by the functional relation between $\zeta(z)$ and $\zeta(1 - z)$, this implies that there cannot be zeroes also on the left part of the strip, i.e. for $0 < Re(z) < 1/2$. The only place left is the critical line itself, as Riemann conjectured back in 1859. The present approach constitutes a truly elementary proof of the Riemann Hypothesis.

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A Appendix: Derivation of Choudury's formula

Since the original reference [5] does not give a proof for the Choudury formula, and since there was a typo in the Eq.(4) of the previous version of this paper, we give a simple derivation here, starting from the Laurent series expansion of the Riemann ζ:

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \underbrace{(-1)^n \frac{\gamma_n}{n!}}_{A_n} (s-1)^n \tag{12}$$

where γ_n are the Steltjes constants, the first $\gamma_0 = \gamma$ being the Euler-Mascheroni constant. We have also indicated the A_n costants, cfr. eq. (4).

The first derivative of the ζ is thus:

$$\zeta'(s) = \frac{-1}{(s-1)^2} + \sum_{n=0}^{\infty} A_n \; n(s-1)^{n-1} \tag{13}$$

The inverse of Eq.(12) can be written as

$$\frac{1}{\zeta(s)} = \frac{(s-1)}{1 + \sum_{n=0}^{\infty} A_n (s-1)^{n+1}} = \tag{14}$$

$$= (s-1)\Big(1 - \sum_{n=0}^{\infty} A_n (s-1)^{n+1} + \Big(\sum_{n=0}^{\infty} A_n (s-1)^{n+1}\Big)^2 - \cdots\Big) \tag{15}$$

where the formal expansion $(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$ with $\alpha = -1$ has been used.

Now, the logarithmic derivative is the product of Eqs. (13) and Eqs. (15) above:

$$\frac{\zeta'(s)}{\zeta(s)} = \left(\frac{1}{(1-s)} + \sum_{n=0}^{\infty} A_n n (s-1)^n\right)\left(1 - \sum_{n=0}^{\infty} A_n (s-1)^{n+1} + \Big(\sum_{n=0}^{\infty} A_n (s-1)^{n+1}\Big)^2 - \cdots\right) \tag{16}$$

Note that the first sum, might start from 1. It is quite complicated to account for all powers of the infinite sums appearing in the second parenthesis, but the first few powers of the expansion are:

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{(1-s)} + A_0 + (2A_1 - A_0^2)(s-1) + (A_0^3 - 3A_0A_1 + 3A_2)(s-1)^2$$
$$+(-A_0^4 + 4A_0^2A_1 - 2A_1^2 - 4A_0A_2 + 4A_3)(s-1)^3 + \cdots \tag{17}$$

that coincide with coefficients given in the text, mind the signs.

Differences with respect to Ver.1

- a line appended to abstract and conclusions.
- sentence mentioning the prime number theorem added in intro
- a couple of references added in the intro
- typo in Eq.(4) corrected: in the sum $A_k \bar{A}_{n-k-1}$ as in Ref. 4
- the sentence "from Eq.(5) into Eq.(7)" has been added just before Eq.(8)
- Appendix with derivation of Choudury's formula added
- reference to standard textbooks added (Arfken, Ahlfors)
- figure 3 added, with caption and description in Sect. 3
- this list of changes added