Qeios

# Non-dimensionalization of the Compressible Navier-Stokes Equation by Pressure Wavelength and Period revealing its Singularity

Shisheng Wang
*Karlsruhe Institute of Technology (KIT)*
*Institute for Neutron Physics and Reactor Technology (INR)*
*Postfach 3640, D-76021 Karlsruhe, Germany*
*Shisheng.Wang@kit.edu*

## Abstract

Fluid particles oscillating in compressible fluid field will produce a pressure (density) wave. The wave propagates in the field by a finite speed – the wave speed of c. Any function where the x and t dependence is of the form (kx - ωt) (or of the form f(x-ct) and g(x+ct)) represents a traveling wave of some shape. The x and t are no longer independent variables, rather, they are inter-dependent. They are related by the wave propagating speed of c through a dimensionless scalar function – wave phase function of $\phi(x\,,t) = kx \pm \omega t$, where space and time are nondimensionalized by wavelength (wave number) and period (frequency). All the inertial observers perceive the same dimensionless wave phase function, $\phi(x\,,t)$, at a given point in space and time if they have relative motions. If the wavelength and period are picked out as a measuring unit to quantify the length and time interval, the physical values of the length and time interval must be correspondingly enlarged or shortened for different initial frames, to ensure the wave phase function to be an invariant dimensionless scalar function. The classical compressible Navier-stokes equation contains pressure and density terms, if it is non-dimensionalized by the pressure wavelength and period in an inertial frame; the resulting equation contains a Reynolds number, where the wavelength is its scaling length. Its value will change following the fluid particle motion, relative to rest frame (Lab frame); The wavelength will be shorter (frequency will be higher) if the fluid particle flows toward an observer. The flow will blow up when the flow velocity approaches wave propagation velocity. Furthermore, it shows mass-energy equivalence can be expressed as $pV = mc^2$ in the co-moving reference frame (rest frame).

https://doi.org/10.32388/CO4GWB

## 1. Introduction

The Navier–Stokes equation is very useful because they describe the physics of enormous phenomena of scientific and engineering, even of cosmology. However, our understanding of them remains minimal. The secrets hidden in the Navier-Stokes have not been unlocked. The question of whether the Navier–Stokes equations allow solutions that develop singularities in finite time remains unresolved.

In the author's previous work [1], it is shown that the classical Navier-Stokes equation is a non-relative limit of the relativistic one. The relativistic Navier-Stokes equation is derived from the conservation laws of momentum-energy and accounts for the effects of viscosity in 3+1 Minkowski space. It has shown that the classical Navier-Stokes equation neglects the "wave energy" changes. Strictly speaking, the classical Navier-Stokes equation is an approximated mathematical model for very small velocities relative to the Lab frame. It was written out in the momentarily co-moving reference frame (MCRF) - the fluid particle and observer share the same space-time coordinate point coincidently, in other words, the observer and the fluid particle have coincidently no relative motion ($v<<c$). In the flow field, the propagation speed of the pressure wave essentially is the fluid property, which is independent of the frame of reference. For different inertial reference frames with relative motion, they agree that the pressure wave has the same traveling speed, but different wavelength and period. The fundamental physical principle is that the traveling wave phase function is invariant, it is a dimensionless function and a Lorentz scalar - All inertial observers see the same phase at a given point in space-time, even though they do not agree on frequency or wavelength. Based on this consideration, if we select the pressure wavelength and period as "measuring stick" to measure a space length and time interval, we will get different physical values for space and time in different reference frames, in order to ensure the wave phase function to be invariant.

Getting inspiration from the above consideration, in this paper, we first show the traveling wave phase function is an invariant dimensionless scalar – which leads to the so-called time dilation and length contraction effect as described by the special relativity theory. Then the wavelength and period are picked out as the measuring unit (parameter) to scale the linear wave equation, it is shown that we will get a unique mathematical description of the plane wave for all inertial reference frames, provided that the time and length are covariant in the pace with the wavelength and period. In the same manner, the Navier-Stokes equation in the co-moving reference frame is then non-dimensionalized, applying pressure wavelength and period as its scale parameters; the non-dimensionalized Navier-Stokes equation includes a Reynolds number, using the wavelength as its length scale and wave speed as its characteristic velocity. When the equation is transformed into a stationary frame (Lab frame), the corresponding Reynolds number should be transformed, too. If the flow velocity approaches the wave traveling speed, the wavelength will approach to zero or infinite depending on the flow direction. The flow field will blow up or degenerate to an infinite dilute flow.

## 2. Traveling wave phase is dimensionless invariant function

Pressure wave in the flow field consists of different frequencies or wavelengths. For simplicity, in this paper, let us consider a monochromatic plane pressure wave that has one single wavelength or frequency, the wave propagates in a homogeneous media.

Suppose in the flow field there is an inertial observer, saying observer $S_A$, if observer $S_A$ and wave source $S_{A'}$ (fluid particle at position A' oscillating to produce a disturbed pressure wave) have no relative motion, or both are stationary, this is so-called the co-moving inertial reference frame. The wavelength and period (frequency) perceive by the co-moving observer $S_A$ are defined to be $\lambda$ and $T$ $(f)$, respectively, as shown in Fig. 1 (a).

Suppose observer $S_A$ stays on the coordinate origin. For $S_A$, the wave function for a plane wave is represented by a complex exponential function. The general form of a plane wave function in one dimension can be expressed as:

$$\psi(x,t) = A \cdot exp[i\phi(x\,,t)], \qquad (1)$$

where the argument of the exponential function, $\phi$, is the phase as a function of position x and time t; the expression "kx - ωt" is used to describe the phase of the wave at a given position (x) and time (t). It's often used in wave equations to express the behavior of waves and how they vary in space and time.

$$\phi(x\,,t) = kx \pm \omega t = 2\pi\left(\frac{x \pm ct}{\lambda}\right) = 2\pi\left(\frac{x}{\lambda} \pm \frac{t}{T}\right). \tag{2}$$

In view of the observer $S_A$, who sits at rest on the coordinate system origin, the negative sign represents the wave traveling in the positive x-direction. A positive sign means the wave traveling in the negative x-direction.

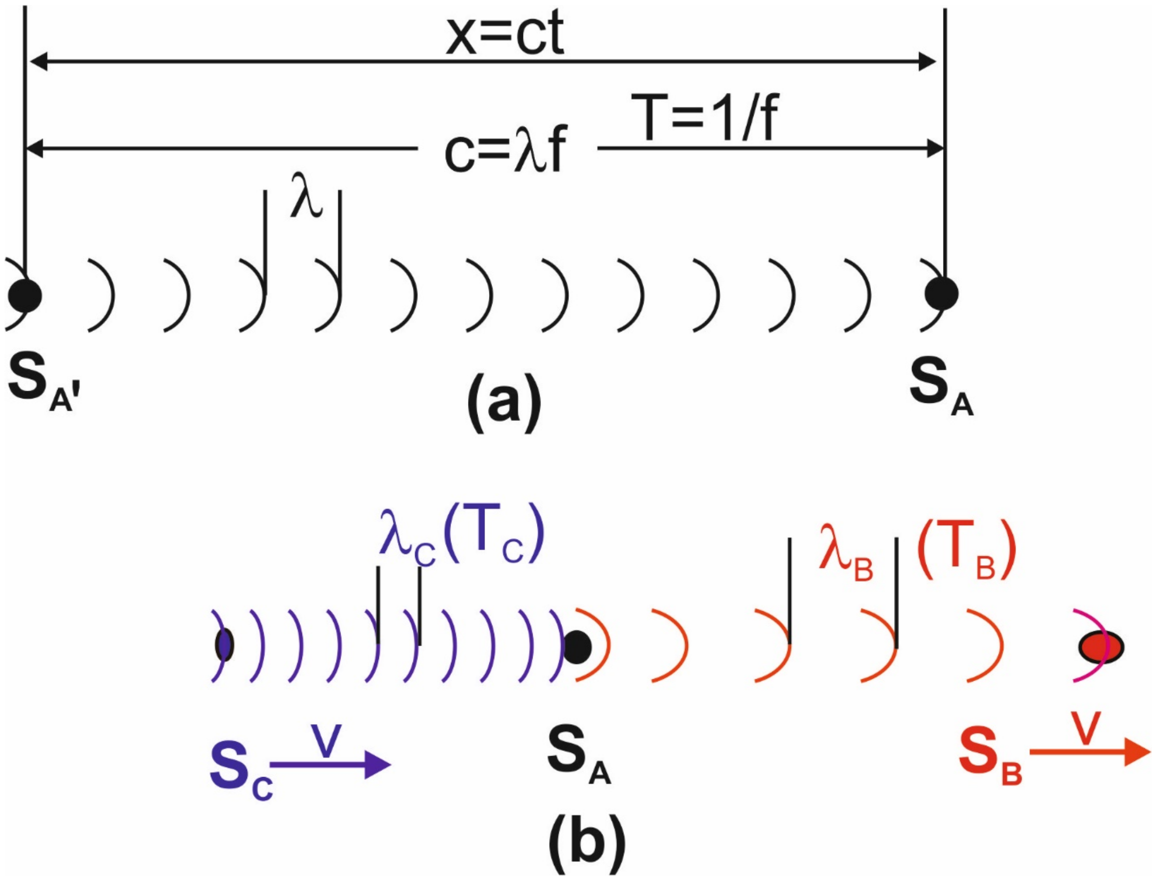


Fig. 1. (a) Observer ($S_A$) and wave source ($S_{A'}$) have no relative motion (co-moving), it can be regarded as wave source ($S_{A'}$) has only oscillation motion in Lab frame, without flow; (b) wave source of particle C flows to the observer of $S_A$ and wave source of particle B flows away from the observer of $S_A$. Observer $S_A$ can be regarded as Lab frame.

If the wave source has a relative motion to the observer, it results in a change in the wavelength (and hence, the period) of the wave as perceived by the observer.

As illustrated in Fig .1 (b), if a wave source $S_C$ (fluid particle at position c oscillating to produce wave) flows towards to observer $S_A$ with a velocity v, he will perceive a shorter wavelength $\lambda_C$ and a smaller period $T_C$ (but a higher frequency of $f_C$), this is so-called blue shift effect (Doppler effect). While if the wave source $S_B$ (fluid particle at position B oscillating to produce wave) flows away from the observer $S_A$ with a velocity v, he will feel a longer wavelength $\lambda_B$ and a larger period $T_B$ (or a lower frequency of $f_B$), the so-called red-shift effect. However, the wave phase as a function itself is invariant for observer $S_A$ – he will feel the same wave crest and trough "passing through" him, regardless of whether the wave source moves or not (particle B or C).

The wavelengths, periods, and wave phase velocity, on all circumstances, have the following dispersion relation:

$$\frac{\lambda}{T} = \frac{\lambda_B}{T_B} = \frac{\lambda_C}{T_C} = c. \tag{3}$$

The wavelength ($\lambda$) and the combination of phase velocity multiplying the period ($cT$) have the same dimension, both equal each other. If two variables are assembled as a vector

$$\vec{w} = \begin{bmatrix} cT \\ \lambda \end{bmatrix}. \tag{4}$$

It is an eigenvector of Lorentz transformation; in fact, it is a null vector in Minkowski space, to be specifically [2]:

$$\lambda^2 - (cT)^2 = 0 \quad or\ \lambda = \pm cT. \tag{5}$$

We will see that if the flow particle and the observer have relative motion, the wavelength and period values will change by a contraction or expansion factor.

Taking the wave source $S_C$, for example (Fig. 1(b)), the wavelength and period between the $S_A$ and relative motion source $S_C$ have the following relation

$$\begin{bmatrix} cT_C \\ \lambda_C \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{bmatrix} \begin{bmatrix} cT \\ \lambda \end{bmatrix}, \tag{6}$$

where $\lambda_C$ and $T_C$ are the wavelength and period emitted by $S_C$ and perceived by stationary observer $S_A$. $\gamma$ is the Lorentz factor, $\beta$ is the ratio of v to c.

Recalling the wave dispersion relation, $cT = \lambda$, we have

$$\begin{bmatrix} cT_C \\ \lambda_C \end{bmatrix} = \begin{bmatrix} \gamma(1-\beta) \\ \gamma(1-\beta) \end{bmatrix} cT = \begin{bmatrix} \gamma(1-\beta) \\ \gamma(1-\beta) \end{bmatrix} \lambda, \tag{7}$$

where the eigenvalues of Lorentz transformation are

$$\epsilon_{1,2} = \gamma(1 \mp \beta) = \sqrt{\frac{1 \mp \beta}{1 \pm \beta}}; \quad \epsilon_1 \cdot \epsilon_2 = 1. \tag{8}$$

The positive and negative sign depends on the relative velocity direction.

Using these eigenvalues, we can define two exponential functions as a contraction or expansion factor to describe the change in wavelength (and hence, period) of wave, emitted by a moving object as following,

$$e^{-u} = \epsilon_1 = \sqrt{\frac{1-\beta}{1+\beta}}; \quad e^{u} = \epsilon_2 = \sqrt{\frac{1+\beta}{1-\beta}}, \tag{9}$$

where u is the rapidity parameter or hyperbolic rotation angle [3],

$$u = \ln[\gamma(1+\beta)] = -\ln[\gamma(1-\beta)]. \tag{10}$$

In this manner, we can re-write the relationship between both wavelength and period:

$$cT_C = e^{-u}(cT); \tag{11a}$$

$$\lambda_C = e^{-u}\lambda; \tag{11b}$$

$$\lambda_C = cT_C. \tag{11c}$$

Similarly, the source $S_B$ flows away from the observer, the wavelength and period perceived by observer $S_A$ will become larger:

$$cT_B = e^{u}(cT); \tag{12a}$$

$$\lambda_B = e^{u}\lambda; \tag{12b}$$

$$\lambda_B = cT_B. \tag{12c}$$

The shortened or enlarged effect of the wavelength and period (the eigenvector or null vector of Lorentz transformation) is illustrated in Fig . 2.

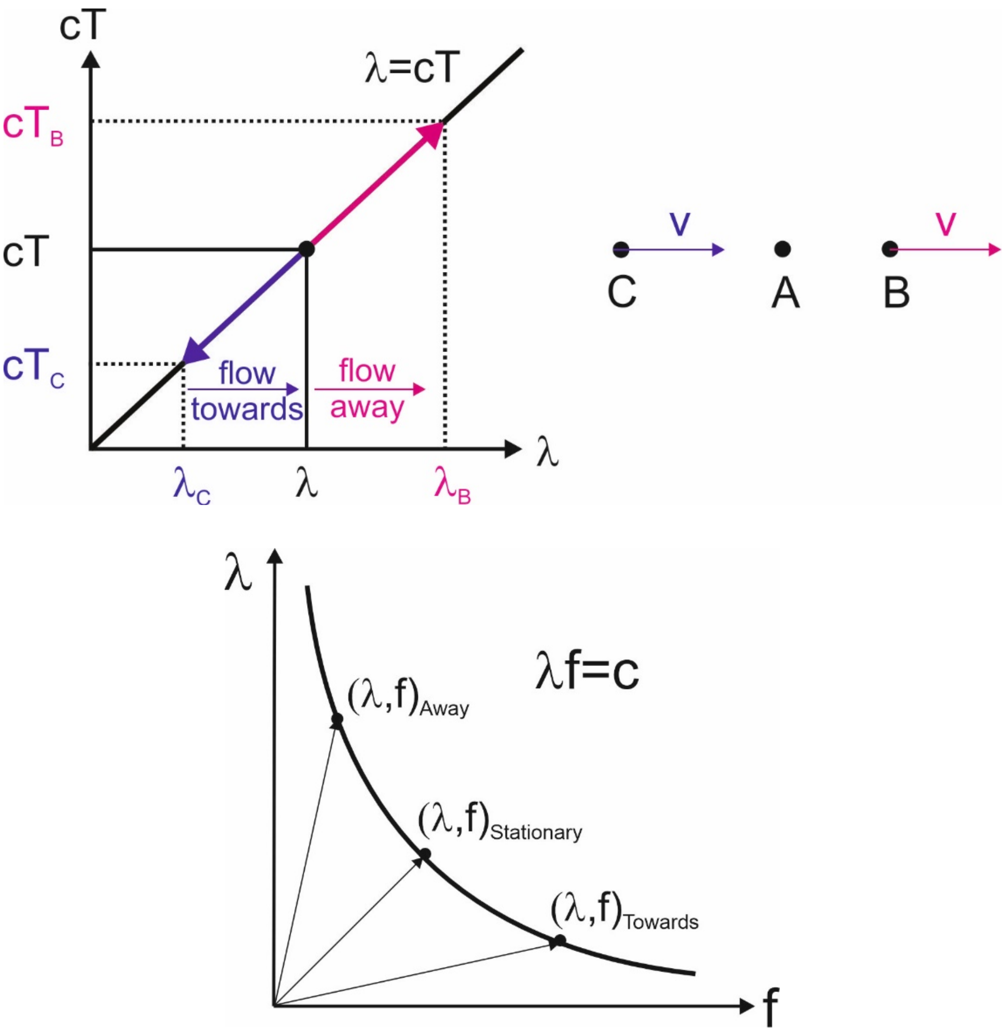


Fig. 2 Dispersion relation is a linear function: $cT = \lambda$. It is an eigenvector (null vector) of Lorentz transformation. Wave source of C flows towards to observer A, and wave source of B flows away from observer A.

As mentioned above, the wave phase function is invariant:

$$\phi(x\,,t) = \phi_B(x\,,t) = \phi_C(x\,,t). \tag{13}$$

If we write it, e.g. for wave source $S_C$, explicitly:

$$kx - \omega t = 2\pi\left(\frac{x}{\lambda} - \frac{ct}{cT}\right) = 2\pi\left(\frac{x_C}{\lambda_C} - \frac{ct_C}{cT_C}\right). \tag{14}$$

From the equation we can immediately recognize that to ensure the phase function is invariant, time duration and length must be proportionally shortened in a covariant manner, perceived by the stationary observer $S_A$,

$$ct_C = e^{-u} \cdot (ct); \tag{15a}$$

$$x_C = e^{-u} \cdot x. \tag{15b}$$

In this manner, we can ensure

$$\frac{x_C}{\lambda_C} = \frac{e^{-u} \cdot x}{e^{-u} \cdot \lambda} = \frac{x}{\lambda}; \tag{16a}$$

$$\frac{ct_C}{cT_C} = \frac{e^{-u}(ct)}{e^{-u}(cT)} = \frac{ct}{cT}. \tag{16b}$$

Finally, we get the invariant wave phase function of Eq. (14)

Similarly, for the wave source $S_B$, we have

$$2\pi\left(\frac{x_B}{\lambda_B} + \frac{ct_B}{cT_B}\right) = 2\pi\left(\frac{x}{\lambda} + \frac{ct}{cT}\right). \tag{17}$$

Where

$$ct_B = e^{u} \cdot (ct); \tag{18a}$$

$$x_B = e^{u} \cdot x. \tag{18b}$$

If the wavelength and period are selected as the measuring unit (scaling parameters) to measure the length and time, the phase function can be expressed as a no-dimensionalized scalar function, as follows:

$$\phi(x\,,t) = 2\pi(x^{*} \pm t^{*}). \tag{19}$$

where

$$x^{*} = \frac{x}{\lambda} = \frac{x_C}{\lambda_C} = \frac{x_B}{\lambda_B}; \tag{20a}$$

$$t^{*} = \frac{ct}{cT} = \frac{ct_C}{cT_C} = \frac{ct_B}{cT_B}. \tag{20b}$$

From the above discussion, we recognize that the perceived wavelength and period are different for different inertial observers, if they have relative motions, the physical length and time interval should also be enlarged or shortened by a same factor, correspondingly. This is the so-called time dilation and length contraction effect in the special relativity theory [4].

Actually, by the length and time covariant principle, the linear wave equation can be written as a non-dimensionalization form as

$$\frac{\partial^2\psi}{\partial t^2} = \left(\frac{\lambda}{T}\right)^2 \frac{\partial^2\psi}{\partial x^2} \quad \rightarrow \quad \frac{\partial^2\psi}{\partial t^{*2}} = \frac{\partial^2\psi}{\partial x^{*2}}. \tag{21}$$

It should be reminded; Eq. (21) implies the measuring unit will vary for different inertial reference frames with relative motions. When the fluid particle flows towards the observer, e.g. fluid particle C, the measuring unit will be shorter, (the eigenvalue of Lorentz transformation is smaller than one). When the fluid particle flows away from the observer, e.g. fluid particle B, the measuring unit will become bigger (the eigenvalue is larger than one). Especially, when the flow velocity approaches wave speed, the measuring unit will approach zero or infinity, depending on the relative moving direction. If $x = \pm ct$ , the wave fronts will squeeze together (wavelength equals zero) or wavelength will stretch infinitely greater. That means $\phi(x\,,t) = 0$. The wave equation is no longer valid.

With this approach, the non-dimensionalized wave equation of (21) is then independent of the coordinate frames, regardless of the relative motions. The physical meaning can be obtained from the fact that the phase of the wave is an invariant dimensionless quantity. All observers must have counted the same number of wave crests and toughs, whether they have relative motion or not [5].

## 3. Non-dimensionalization of the Navier-Stokes Equation

We begin with the Navier-Stokes equation in the momentarily co-moving reference frame (the local observer coordinates origin and the fluid particle share the same spacetime point coincidently and not considering the relative effect, namely v<<c). As mentioned in the introduction; the classical Navier-Stokes equation is the non-relativistic limit of the relativistic one and ignores the “wave energy” change.

$$\frac{\partial(\rho c u_i)}{\partial(ct)} + \frac{\partial(\rho u_i u_j)}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\sigma_{ij}; \qquad (22a)$$

$$\sigma_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \xi\,\frac{\partial u_k}{\partial x_k}\delta_{ij}. \qquad (22b)$$

By the inspiration of the above discussion, we define the following parameters as non-dimensional variables to scale the classical Navier-Stokes equation, as listed in Table 1.

Table 1. The scaling parameters

| Scale parameters | Dimensionless variable |
|---|---|
| Length $(\lambda)$ | $x^* = \frac{x}{\lambda}$ |
| Time $(\lambda/c = T)$ | $t^* = \frac{t}{T} = \frac{t}{\lambda/c}$ |
| Velocity $(v/c)$ | $v^* = \frac{v}{c} = M = \beta$ |
| Pressure $\left(\frac{p}{\rho c^2}\right)$ | $p^* = \frac{p}{\rho c^2}$ |
| Reynolds number $\left(Re_\mu = \frac{\rho c \lambda}{\mu}\right)$ | $Re_\mu = \frac{\rho c \lambda}{\mu}$ |
| Second Reynolds number $\left(Re_\xi = \frac{\rho c \lambda}{\xi}\right)$ | $Re_\xi = \frac{\rho c \lambda}{\xi}$ |

With the help of above scale parameters, we can get the following differential operator:

$$\frac{\partial}{\partial t} = \frac{1}{T}\frac{\partial}{\partial t^*}; \tag{23a}$$

$$\nabla = \frac{1}{\lambda}\nabla^*; \tag{23b}$$

$$\nabla^2 = \frac{1}{\lambda^2}\nabla^{*2}; \tag{23c}$$

$$\frac{\partial^2}{\partial x_i \partial x_j} = \frac{1}{\lambda^2}\frac{\partial^2}{\partial x_i^* \partial x_j^*} \tag{23d}$$

Substitution of the new variables of Table 1 and the differential operators, Eq. (23a-d) into Eq. (22a-b), the non-dimensionalized Navier-stokes equation is obtained as follows (see appendix for details),

$$\frac{\partial u_i^*}{\partial t^*} + \left(u_j^* \cdot \nabla^*\right)u_i^* = -\nabla^* p^* + \frac{1}{Re_\mu}\nabla^{*2}(u_i^*) + \left(\frac{1}{Re_\mu} + \frac{1}{Re_\xi}\right)\frac{\partial}{\partial x_i^*}(\nabla^* \cdot \vec{v}^*). \tag{24}$$

Here the starred variables are used to represent the dimensionless variables, it be recognized that the Reynolds number contains a wave length as its characteristic length.

### 3.1 One-Dimensional flow

For one-dimensional flow, the dimensionless form becomes

$$\frac{\partial u^*}{\partial t^*}+u^*\frac{\partial u^*}{\partial x^*}=-\frac{\partial p^*}{\partial x^*}+\frac{1}{Re_\mu}\frac{\partial^2 u^*}{\partial x^{*2}}+\left(\frac{1}{Re_\mu}+\frac{1}{Re_\xi}\right)\frac{\partial^2 u^*}{\partial x^{*2}}. \tag{25}$$

We can rearrange it into the following form

$$\frac{\partial(cu^*)}{\partial(ct^*)}=\frac{\partial}{\partial x^*}\left[-\frac{1}{2}u^{*2}-p^*+\left(\frac{2}{Re_\mu}+\frac{1}{Re_\xi}\right)\frac{\partial u^*}{\partial x^*}\right]. \tag{26}$$

If we assume the system has reached a steady state. Then, the steady-state 1-D dimensionless Navier-Stokes equation is given by

$$\frac{1}{2}u^{*2}+p^*-\left(\frac{2}{Re_\mu}+\frac{1}{Re_\xi}\right)\frac{\partial u^*}{\partial x^*}=costant. \tag{27}$$

It is possible to re-write it in dimension form:

$$\frac{1}{2}\left(\frac{u}{c}\right)^2+\frac{p}{\rho c^2}-\left(\frac{2\mu+\xi}{\rho c^2}\right)\frac{\partial u}{\partial x}=costant. \tag{28}$$

If both sides multiply the term: $\rho c^2$, then we get

$$\frac{1}{2}(\rho u)^2 + p - (2\mu + \xi)\frac{\partial u}{\partial x} = costant. \tag{29}$$

Furthermore, if we ignore the viscosity effect, the classical Bernoulli equation was recovered, with a little bit of modification.

$$\frac{1}{2}\left(\frac{u}{c}\right)^2 + \frac{p}{\rho c^2} = constant. \tag{30}$$

### 3.2 On the essence of pressure energy or mass-energy equivalence

We consider the first law of thermodynamics for a closed system, no transfer of the matter in or out the system, in the co-moving frame (the observer and fluid have no relative motion). If the heat transfer between the system and its surroundings is ignored, we consider only the mechanical energy exchange of the system with its surroundings, thus, the change in internal energy of the system is:

$$dE = dW, \tag{31}$$

where $dW$ is work done on the system by its surroundings.

$$dW = d(pV). \tag{32}$$

In a fluid, recalling the definition of the speed of sound, it depends on the bulk modulus and density:

$$c^2 = \frac{B}{\rho} \quad \rightarrow \quad B = \rho c^2. \tag{33}$$

The bulk modulus B is defined by the equation

$$B = -\frac{dp}{\left(\frac{dV}{V}\right)}. \tag{34}$$

Substituting Eq. (33) into Eq. (34), we have

$$-\frac{dp}{\left(\frac{dV}{V}\right)} = \rho c^2. \tag{35}$$

Recalling the density definition and accordingly, we have

$$dp = -\rho c^2 \left(\frac{dV}{V}\right) = -\frac{mc^2}{V^2} dV. \tag{36}$$

where m is the mass content in the researched closed system.

Integral from a reference point to present state $(p, V)$:

$$p - p_{ref} = mc^2 \left(\frac{1}{V} - \frac{1}{V_{ref}}\right). \tag{37}$$

We can choose an infinitely dilute state as the reference point (zero point):

$$zero\ point: \begin{cases} p_{ref} \rightarrow 0 \\ V_{ref} \rightarrow \infty \end{cases}. \tag{38}$$

where, the fluid volume approaches infinitely large and the pressure approaches zero.

The equation (37) becomes:

$$p = \frac{mc^2}{V} = \rho c^2. \tag{39}$$

Rearranging it, we have following expression:

$$E = pV = mc^2. \tag{40}$$

That means the mass-energy equivalence represents the "wave potential energy content" in the co-moving frame.

## 4. Discussions

In view of the fluid particle of C, the stationary observer A (the Lab frame) can be interpreted to move towards C with a relative velocity of -V. While in view of the fluid particle of B, the stationary observer A can be explained to move away from B with a relative velocity of V, as shown in Fig 3.

We want to describe the fluid motion in the Lab frame. In this circumstance, we should transform the equation from a co-moving fluid particle to the Lab frame, for instance from fluid particle C to the stationary observer frame of A. The measuring unit, namely the wavelength and period, will become shorter as perceived by the Lab observer. In another word, length and time duration should squeeze proportionally smaller by the same factor,

$$[\partial x_C \quad \partial t_C] = e^{-u}[\partial x \quad \partial t]. \tag{41}$$

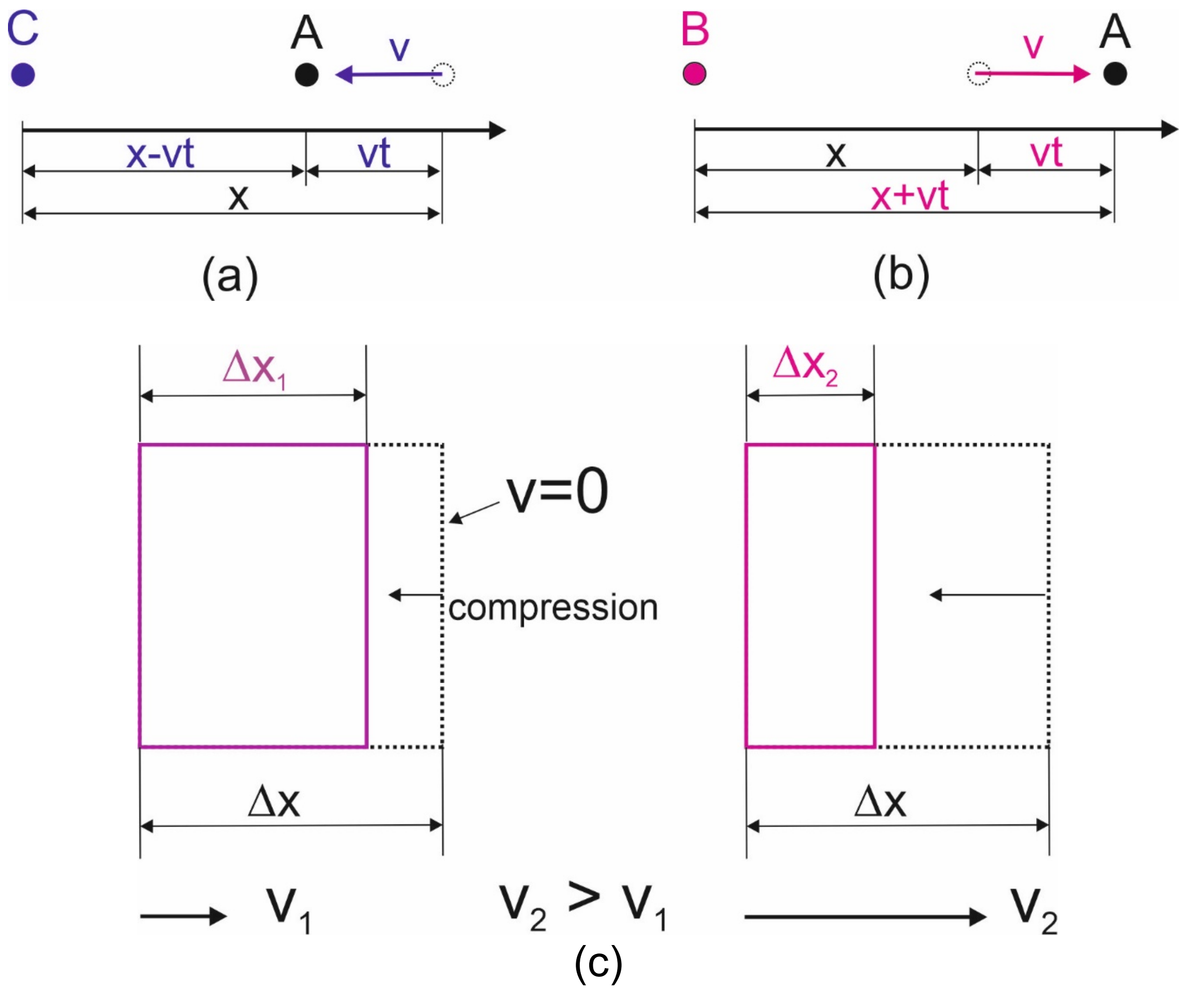


Fig. 3. (a) Observer A seems to flow towards C with a relative velocity of -v; (b) and flows away from B with a relative of velocity of +v. (c) the flow particle C flows towards observer A, the control volume will be compressed.

When the velocity of the fluid particle C approaches the speed of the wave, the wavelength (and hence, the period) will approach zero,

$$\lambda_C = e^{-u} \cdot \lambda \;\to\; 0; \tag{42a}$$

$$T_C = e^{-u} \cdot T \;\to\; 0. \tag{42b}$$

So that

$$[\partial x_C \quad \partial t_C] \;\to\; 0. \tag{43}$$

The Reynolds numbers will be transformed to

$$Re_{\mu} = \frac{\rho c\lambda}{\mu} exp\left(\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}\right) = \frac{\rho c\lambda}{\mu} exp\left(\sqrt{\frac{1-M}{1+M}}\right); \tag{44a}$$

$$Re_{\xi} = \frac{\rho c\lambda}{\xi} = \frac{\rho c\lambda}{\xi} exp\left(\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}\right) = \frac{\rho c\lambda}{\xi} exp\left(\sqrt{\frac{1-M}{1+M}}\right). \tag{44b}$$

where M is Mach number.

Hence the Reynolds number will seem to approach zero, and the resulting viscosity terms in the non-dimenalized form will approach infinity, this leads the fluid flow to become infinite or "blow up". It can be interpreted as the control volume will be compressed in the flow direction, as shown in Fig. 3 (c).

In contrast, when the fluid particle B flows away with a velocity from the stationary observer, the wavelength (and hence, the period) will appear to become larger. The “measuring unit” will become larger, in this manner, the length and time duration seems to stretch proportionally bigger by the same factor,

$$\lambda_B = e^{u} \cdot \lambda \ \rightarrow \ \infty; \tag{45a}$$

$$T_B = e^{u} \cdot T \ \rightarrow \ \infty; \tag{45b}$$

$$[\partial x_B \quad \partial t_B] = e^{u}[\partial x \quad \partial t] \ \rightarrow \ \infty. \tag{45c}$$

The Reynolds number will proceed toward infinity when the velocity of the fluid particle B approaches the speed of the wave.

$$Re_{\mu} = \frac{\rho c \lambda}{\mu} exp\left(\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}\right) = \frac{\rho c \lambda}{\mu} exp\left(\sqrt{\frac{1+M}{1-M}}\right). \quad (46)$$

If the flow velocity is very small compared with the wave speed, under this circumstance,

$$e^{-u} \approx 1\,; \quad e^{u} \approx 1. \quad (47)$$

In this case, the relativistic effect can be neglected. As an approximation, we can still use the classical Navier-stokes equation to deal with the physical phenomenon.

When the flow velocity is bigger than the wave propagation speed – supersonic flow, the argument of the exponential function (the eigenvalue of Lorentz transformation) becomes an imaginary number; however, we can decompose this function into an "even part" and an "odd part."

$$e^{iu} = \cosh(iu) + \sinh(iu) = \cos(u) + i\sin(u)\,; \quad (48a)$$

$$e^{-iu} = \cosh(-iu) + \sinh(-iu) = \cos(u) - i\sin(u). \quad (48b)$$

The contract or expansion factor for the “measuring unit” – wavelength and period – is now split into a “real part” and an “imaginary part”. Under this circumstance, if we want to solve the flow problem, the definition domain must be extended to the complex domain.

## 5. Conclusion

The compressible Navier-Stokes equation includes the pressure wave, implicitly. The wave phase function is invariant regardless of the relative motion of different inertial observers; however, the wavelength and period will change, based on the fundamental principle of the invariance of the wave phase function.

The compressible classical Navier-Stokes equation is primarily written out in the MCRF (or precisely speaking, the relative effect is ignored , namely $v \ll c$ or $M \ll 1$). In this co-moving reference frame, we can feel a wavelength (and hence, period) for the pressure wave. The equation can be non-dimensionalized in this frame, using the wavelength and period as its scaling parameters, resulting in a Reynolds number, in which the wavelength is its characteristic length scale.

When this Reynolds number is transformed to an inertial frame (e.g. to the stationary Lab. frame), which has a relative motion to the MCRF, the transformed Reynolds number will approach zero or infinity, depending on the relative motion direction. When the fluid flows toward the observer in the Lab frame, the Reynolds number will approach to zero (the Mach number approaches one), the flow will blow up. Furthermore, we show that the mass-energy equivalence represents the “wave potential energy content” or the pressure energy in the rest frame. For supersonic flow, the flow problem should be solved in a complex domain.

## Reference


1. Shisheng Wang, Extensions to the Navier–Stokes equations, Physics of Fluids **34**, 053106 (2022); https://doi.org/10.1063/5.0087550

2. Hartmann Römer, Michael Forger, Elementare Feldtheorie: Elektrodynamik, Hydrodynamik, spezielle Relativitätstheorie; VCH Verlagsgesellschaft, (1993), ISBN: 3527290656, p.110.

3. Wolfgang Rindler, *Relativity: Special, General, and Cosmological*, p. 53, Oxford University Press (2001).

4. Einstein, Zur Elektrodynamik bewegter Körper. *Annalen der Physik,* Vol. 322, 10, 891-921, (1905).

5. J.D. Jackson, *Classical Electrodynamics, p.363, John Wiley & Sons, Inc., (1962).*

## Appendix. Non-Dimensionalization of the Navier-Stokes equation

At first, to non-dimensionalize the material (total) derivative term, for the sake of simplification, let us show a 2D case. with the help of the mass conservation equation, we have:

$$\frac{\partial(\rho u)}{\partial t}+\frac{\partial(\rho uu)}{\partial x}+\frac{\partial(\rho uv)}{\partial y}=\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right). \qquad \text{(A1)}$$

Substitution of the new variables of Table (1) into each term of the Eq. (A1), we have:

$$\frac{\partial u}{\partial t}=\left(\frac{\lambda}{T^2}\right)\frac{\partial u^*}{\partial t^*}; \qquad \text{(A2)}$$

$$u\frac{\partial u}{\partial x}=\left(\frac{\lambda}{T^2}\right)u^*\frac{\partial u^*}{\partial x^*}; \qquad \text{(A3)}$$

$$v\frac{\partial u}{\partial y}=\left(\frac{\lambda}{T^2}\right)v^*\frac{\partial u^*}{\partial y^*}. \qquad \text{(A4)}$$

Thus,

$$\frac{\partial(\rho u)}{\partial t}+\frac{\partial(\rho uu)}{\partial x}+\frac{\partial(\rho uv)}{\partial y}=\rho\left(\frac{\lambda}{T^2}\right)\left[\frac{\partial u_i^*}{\partial t^*}+\left(u_j^*\cdot\nabla^*\right)u_i^*\right]. \tag{A5}$$

Dividing the pressure gradient by the coefficient of material derivative term of $\rho\left(\frac{\lambda}{T^2}\right)$, thus

$$\frac{\partial p}{\partial x}=\frac{\partial p^*}{\partial x^*}. \tag{A6}$$

Finally, we nondimensionalize the viscous forces:

$$\frac{\partial}{\partial x_j}\sigma_{ij}=\nabla\cdot[\mu(\nabla u+(\nabla u)^T)+\xi(\nabla\cdot u)\boldsymbol{I}]. \tag{A7}$$

With the help of the differential operators of Eq. (23c-d) and a bit of algebra manipulation, thus

$$\frac{\partial}{\partial x_j}\sigma_{ij}=\frac{\mu c}{\lambda^2}\nabla^{*2}(u_i^*)+\frac{\mu c}{\lambda^2}\frac{\partial}{\partial x_i^*}(\nabla^*\cdot\vec{v}^*)+\frac{\xi c}{\lambda^2}\frac{\partial}{\partial x_i^*}(\nabla^*\cdot\vec{v}^*). \tag{A8}$$

Dividing it by the coefficient of material derivative term of $\rho\left(\frac{\lambda}{T^2}\right)$, we have

$$\frac{\partial}{\partial x_j}\sigma_{ij}=\frac{1}{Re_\mu}\nabla^{*2}(u_i^*)+\frac{1}{Re_\mu}\frac{\partial}{\partial x_i^*}(\nabla^*\cdot\vec{v}^*)+\frac{1}{Re_\xi}\frac{\partial}{\partial x_i^*}(\nabla^*\cdot\vec{v}^*). \tag{A9}$$

Substitution of the new terms of (A5, A6, and A9) into the Navier-Stokes equation, finally we get the non-dimensionalized Navier-stokes equation, as described by Eq. (24).