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RESEARCH ARTICLE

Longitudinal Doppler for Observers in Uniform Acceleration

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Abstract

This study derives the exact frequency observed for electromagnetic waves emitted during uniform acceleration. In the classic case the frequency observed is typically calculated using the formula $f = f_0 (1+gH/c^2)$ where 'f_0' is relevant to the source and g represents the constant acceleration. At least in Feynman's Lectures, such formula is derived directly as a first order approximation from the relativistic Doppler effect. Relativistic relations are used to determine the exact light-time between the source and observer. It is found that, if g represents the proper acceleration, the frequency ratio remains the same as in the classic case $(1+gH/c^2)$. Being that value exact, no higher order relation exists where it can be derived from. The absence of higher order terms depends on a peculiar compensation of two relativistic effects.

1. Introduction

Feynman[1], following Einstein's reasoning[2], considered the EM waves emitted from an oscillator A situated at the front of a rocket towards B at its rear, while measuring an acceleration g.

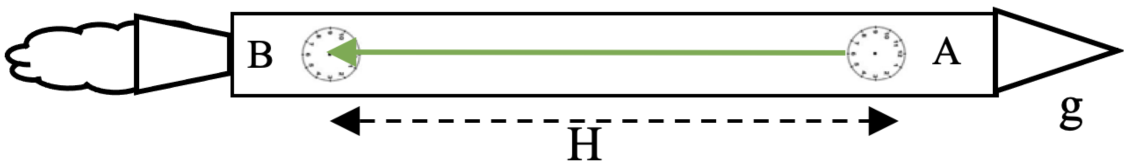

Fig.1. Rocket acceleration scenario with the observer in the rear, receiving EM waves.

In the scenario illustrated in Fig. 1, at constant acceleration g, B gets closer to the position of emission of the light emitted by A. The light-time is approximated as H/c[1][2], which is the same as having a source and an observer at rest.

An observer in motion within the reference frame of the emitter experiences a longitudinal Doppler. If f_A represents the frequency of the emitted light at rest with the source, and the velocity v is much less than the speed of light (v<<c)[1][2], then the approximation $f_B/f_A = 1 + v/c$ holds. By substituting $v=g*H/c$, into the previous equation, we obtain

$f_B/f_A \approx (1+gH/c^2)$, a result derived by Feynman[1], as a first order approximation of the relativistic Doppler effect:

$$\frac{\left(1+\frac{v}{c}\right)}{\sqrt{1-\frac{v^2}{c^2}}} \approx (1+gH/c^2).$$

By considering the simplest case of a stationary source and observer in uniform acceleration, the light-time connecting source and observer will be calculated using the relativistic formulas. This calculation will lead to the derivation of a general result for the frequency shift of light for accelerated observers. Contrary to Feynman's assertion, the formula maintains the same form as the classic one, with g representing the proper acceleration. The derivation of the relevant results will follow, including the case of the accelerated rocket. A short discussion is presented afterwards.

2. Longitudinal Doppler in Uniform Acceleration

Two clocks, C' and C'', are stationary at distance H; they are synchronized using Einstein's synchronization procedure. At time t_0, observer B departs in the position B (t_0) with a constant acceleration g, while simultaneously A emits a pulse of radiation. The instantaneous speed v_1 of B, in the inertial reference frame of the source, is calculated at the time of absorption t_1, in position B(t_1) as shown in the figure below, across the displacement S. This calculation allows for the application of the tested formula of the Relativistic Doppler effect.

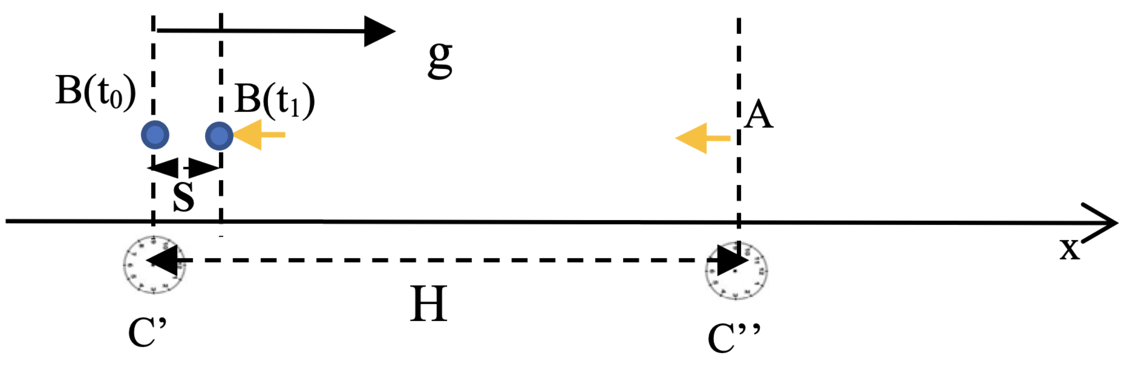

Fig.2. Emission by a stationary source and absorption of a pulse by an accelerated observer

According to relativistic dynamics[3] the displacement S of an accelerated body with proper acceleration g as a function of the elapsed time t, starting with t_0=0 is given by the formula

$$S = \frac{c^2}{g}\sqrt{1+\left(\frac{gt}{c}\right)^2} - \frac{c^2}{g} \quad (\approx 1/2gt^2 \text{ in the classic case})$$

The light-time to reach B is $ct_1 = H - S$. (2)

The speed[3]

$$v_1 = \frac{gt_1}{\sqrt{1+\left(\frac{gt_1}{c}\right)^2}} \quad (\approx gt_1 \text{ for } v_B << c)$$

is the value of the speed of B in the IRF of the source at the instant t_1 of absorption.

From (1) and (2) the following equation is obtained $\left(\frac{H}{c} + \frac{c}{g} - t_1\right) = \frac{c}{g}\sqrt{1 + \left(\frac{gt_1}{c}\right)^2}$ *(see appendix part 1)*

the elapsed time is found by solving in t_1.

$$t_1 = \frac{\frac{H}{c}\left(1+\frac{gH}{2c^2}\right)}{1+\frac{gH}{c^2}} = \frac{H}{c} - \frac{gH^2}{2c^3} + \frac{g^2H^3}{2c^5} + \ldots \left(\approx \frac{H}{c} \text{ for } gH << c^2\right)$$ *(see Appendix Part 2)* (4)

This is the light-time between the emission by the stationary source and the absorption by the accelerated observer, as measured by a stationary clock located at that point. In the limit of small gH, the interval is well approximated by H/c.

The difference of light-times compared to the classical case is the following, from Eq. (4): $\Delta t_{var} = t_1 - \frac{H}{c} \approx -\frac{gH^2}{2c^3}$

That represents the variation of the light-time to connect the source with an observer experiencing constant acceleration g. As expected, when the observer is moving towards the source (approaching the source) the exact light-time is shorter than H/c, $\Delta t_{var} < 0$. Conversely, if the acceleration has a negative value (observer departing from the source), the time interval becomes longer, with $\Delta t_{var} > 0$.

By replacing t_1 from Eq(4), in the Eq(3) we have *(see Appendix Part 3)*

$$v_1 = \frac{\frac{gH}{c}\left(1+\frac{gH}{2c^2}\right)}{\frac{1}{2}\left(\frac{gH}{c^2}\right)^2 + \frac{gH}{c^2} + 1} = \frac{gH}{c} - \frac{(gH)^2}{2c^3} + \frac{(gH)^4}{4c^7} + \ldots \left(\approx \frac{gH}{c} \text{ for } gH << c^2\right)$$ (5)

The final speed, approximated to the second order in gH, is $v_1 \approx \frac{gH}{c} - \frac{(gH)^2}{2c^3} = \frac{gH}{c}\left(1 - \frac{gH}{2c^2}\right)$

If the direction of the speed v is towards the emitter (when g > 0), the classic speed $V_{classic}$ is greater than v (all positive). Consequently, the observer's speed in the event of the absorption is a slightly lower than the calculated classic value.

However, if the observer departs from the source (illustrated as v<0) the magnitude of the speed increases more than in the classic case. This results in light taking more time to reach the observer compared to the classic scenario.

The Relativistic Doppler Effect formula gives $f_B/f_A = \frac{1+\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}}$, considering $v=v_1$

The numerator $(1+ v_1/c)$ can be expressed also as $\frac{\left(1+\frac{gH}{c^2}\right)^2}{\frac{1}{2}\left(\frac{gH}{c^2}\right)^2+\frac{gH}{c^2}+1}$ *(see appendix part 4)* (6)

The denominator $1/\sqrt{(1-v_1^2/c^2)}$ can be also expressed as $\frac{\frac{1}{2}\left(\frac{gH}{c^2}\right)^2+\frac{gH}{c^2}+1}{\left(1+\frac{gH}{c^2}\right)}$, (7)

By multiplying Eq. (6) and Eq. (7) to find the Doppler effect, the final result is

$f_B/f_A = (1+gH/c^2)$ (8)

To some surprise the frequency ratio given by of Eq. (8) is identical to the classic formula and is exact, with g representing the proper acceleration of the observer. Additionally the frequency shift $\Delta f/f = gH/c^2$, often found in electrodynamics, is exact.

When heading towards the light wave (g>0), the light-time gets shorter than H/c, hence the speed reached is reduced in comparison to v=gH/c: $v_1 \approx \frac{gH}{c}\left(1-\frac{gH}{2c^2}\right) < gH/c$.

In the Eq. (6) the value $(1+gH/c^2)$ is multiplied by the quantity $(1+gH/c^2) / [1+gH/c^2+1/2(gH/c^2)^2] \approx 1 - (gH/c^2)^2/2$, representing the higher order contribution. This effect causes a decrease in the blueshift detected by the observer due to the lower final speed at the moment of detection.

The decrease mentioned above is exactly counterbalanced by the relativistic effects in Eq. (7), which are the reciprocal of the previous term (a factor not considered in the classic configuration). This occurs because the period of the absorber becomes longer due to its velocity in the frame of the source. As a result, the incoming radiation is observed with a greater blueshift compared to the classic case. This leads to the conclusion that distinct relativistic effects precisely cancelled each other out. The original formula of the Doppler ratio in accelerated motion, derived classically, turns out to be exact, at least for a stationary emitter, when considering the proper acceleration of the observer.

The obtained formulas, rely exclusively on the relativity of time and the experimentally tested relativistic Doppler effect.

Now let's explore how to apply these formulas to the configuration of the accelerated rocket.

3. The Case of the Accelerated Rocket

In the context of the accelerated rocket, the emission also takes place within a non-inertial frame. It's important to highlight that the choice of an initial speed $v_0=0$ is a deliberate decision to prevent v_0 from appearing in the formula for the frequency shift. This maintains the independence of the formula from the initial speed v_0. If the frequency shift measured internally were to depend on the initial speed, it could potentially allow one to deduce their speed from within the system solely by measuring the frequency shift. This scenario requires only the knowledge of the acceleration g and distance H, without referring to external observations or internal clock readings. From a relativistic standpoint, as discussed in[4], if the observer at the rear detects an acceleration g, the front clock within the rocket will register a slightly different acceleration. According to Special Relativity, the two clocks will exhibit different proper accelerations due to these factors.

It was reported by Feynman[1], starting from the formula $(1+ v/c) /\sqrt{(1- v^2/c^2)}$ "*Assuming that the acceleration and the length of the ship are small enough that this velocity is much smaller than c, we can neglect the term in v^2/c^2*". He derived the result as already mentioned considering that the first order approximation is $(1+ v/c)$ where $v<<c$, hence his first order approximation of the Relativistic Doppler effect $(1+ v/c) /\sqrt{(1- v^2/c^2)} \approx (1+gH/c^2)$.

However, the outcome of the detailed calculations results in $(1+ v/c) /\sqrt{(1- v^2/c^2)} = (1+gH/c^2)$. This implies that the classic formula is exact when g represents the proper acceleration. Consequently, it does not have any higher-order formula.

The longitudinal Doppler effect in accelerated motion, as derived classically, is indeed exact, and cannot be derived as a first order approximation from a higher order formula. While the difference between the light-times in inertial motion and acceleration experience a slight adjustment, its first order approximation is $\Delta t_{var} \approx - ½ gH^2/c^3 = - ½ vH/c^2$ where $v=gH/c$.

4. Conclusions

The light-time required to connect an observer undergoing uniform accelerated motion with constant acceleration g and an emitter, initially separated by H, is $t_1 = \frac{H}{c}^{\frac{\left(1+\frac{gH}{2c^2}\right)}{1+\frac{gH}{c^2}}}$. Being the light-time at rest H/c, the difference with t_1 is approximately $|\Delta t_{var}| \approx |½ H/c\ gH/c^2|$, a tiny fraction of the light-time at rest, $|\Delta t_{var}|/(H/c) \approx |½ gH/c^2|$.

To some surprise, the Doppler effect, with a source at frequency f_0, set at adistance H from an observer measuring acceleration g, is exactly the ratio $f/f_0 = (1+gH/c^2)$ or $\Delta f /f_0 = gH/c^2$. It is the same as in the classic case, with g the proper acceleration. That formula is valid also in an accelerated rocket. Feynman incorrectly considered it derivable as a first-order approximation from the Relativistic Doppler formula.

In the scenario of blueshift, when the observer is moving towards the source, the observer's oscillator would register, due solely to its speed, the incoming radiation exhibits a reduced blueshift, compared to the classic case. Conversely, relativistic effects lead to an increase in the proper period of the observer's oscillator, resulting in a more pronounced blueshift in the detected incoming radiation. What's particularly intriguing is the observation that these two effects are reciprocals of each other and thus cancel each other out, resulting in no net effect on the observed blueshift.

These findings have implications for understanding wave propagation in accelerating frames, with potential applications in astrophysics and relativistic mechanics

Appendix

1. **From the first equation** $c\,t = H – S$, replace S with the Eq. (1)

   $ct = H – (c^2/g \sqrt{(1+(g\,t/c)^2)} - c^2/g)$;

   $c\,t - H - c^2/g = – (c^2/g \sqrt{(1+(g\,t/c)^2)})$,

   $(H/c + c/g - t) = c/g \sqrt{(1+(gt/c)^2)}$

2. **Find t_1 by solving the previous equation**,

   by setting $k=c/g$,

   $(H/c + k - t_1) = k \sqrt{(1+(t_1/k)^2)}$ by solving in t_1 (with Wolfram Alpha)

   $t_1 = H/c\,(H/c+2\,c/g) / 2(H/c + c/g)$, by multiplying and dividing by (g/c)

   $t_1 = [H/c\,(g/c)\,(H/c+2\,c/g)] / [2(H/c + c/g)(g/c)] = [H/c\,(gH/c^2 +2)]/[2(gH/c^2+1)]$

   $t_1 = H/c\,(1+gH/2c^2)/(1+ gH/c^2)$

3. **Calculate $v = gt/\sqrt{(1+(gt/c)^2)}$ replacing t_1 (as function g, H, c)**

   By replacing $t = t_1$ as found in the previous, set $k= gH/c^2$; (with Wolfram Alpha)

   $\mathbf{(1+(g\,t/c)^2)} = 1+ g^2/c^2\,(H/c\,(1+k/2)/(1+ k))^2 =1+ g^2/c^2\,H^2/c^2\,(1+k/2)^2/(1+ k)^2 = 1+ [(gH/c^2)^2\,(1+k/2)^2/(1+ k)^2]$; $1+ [k^2\,(1+k/2)^2/(1+ k)^2] =[(1+ k)^2 +k^2\,(1+k/2)^2]/(1+ k)^2 =1/2\,[(k^2 +2k+2) / (k+1)]^2$;

   $\mathbf{\sqrt{(1+(g\,t/c)^2)}} = (k^2 +2k+2) /2(k+1)$; $1/\sqrt{(1+(g\,t/c)^2)} = 2(k+1)/ (k^2 +2k+2)$

   $v = g\,t\,[2(k+1)/ (k^2 +2k+2)] = gH/c\,[(1+k/2)/(1+ k)]\,[2(k+1)/ (k^2 +2k+2)] = 2\,gH/c\,(1+k/2)/ (k^2 +2k+2)$

   replacing back $k= gH/c^2$,

   $v= gH/c\,(1+gH/2c^2)/(1/2(gH/c^2)^2 +gH/c^2+1)$

4. **Find $f_B/f_A= (1+ v/c) /\sqrt{(1-v^2/c^2)}$, replacing v (as function g, H, c)**

   Setting $k= gH/c^2$, from the previous calculations considering v (instead of v1)

   $\mathbf{v/c} = 2\,gH/c^2\,(1+k/2)/ (k^2 +2k+2) = 2\,k\,(1+k/2)/ (k^2 +2k+2)$

   from the previous

   $\mathbf{(1+ v/c)} = 1 + 2\,k\,(1+k/2)/ (k^2 +2k+2) = [(k^2 +2k+2) +2\,k\,(1+k/2)]/ (k^2 +2k+2) = (2k^2 +4k+2)/ (k^2 +2k+2) =$

   $= 2(k^2 +2k+1)/(k^2 + 2k+2)$

   And

   $\mathbf{v^2/c^2)} = 1- [2k(1+k/2)/ (k^2 +2k+2)]^2 = [(k^2 +2k+2)^2- (2k(1+k/2))^2]/ (k^2 +2k+2)^2 =$

$[(k^2 + 2k+2)^2 - (2k+k^2)^2] / (k^2 + 2k+2)^2 = 4(k+1)^2 / (k^2 + 2k+2)^2$

hence

$\sqrt{(1 - v^2/c^2)} = 2(k+1) / (k^2 + 2k+2)$

Final result

$(1 + v/c) / \sqrt{(1 - v^2/c^2)} = 2(k^2 + 2k+1) / [2(k+1)] = (k+1) = (1 + gH/c^2)$

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