Fundamentals of classical and supersymmetric quantum stochastic filtering theory

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This paper is dedicated to my beloved father Late Prof. K R Parthasarathy

1 abstract

In this paper, we first derive the discrete time recursive stochastic filtering equations for a classical Markov process when noisy measurements of the process are made with the measurement noise at each discrete sample forms an iid Gaussian or non-Gaussian process. The filter involves deriving general formulae for the conditional mean process given measurements upto that time and also simultaneously recursive formulae for the conditional moments of the estimation error process. In the next section, we build upon the method of John Gough etal, to construct a supersymmetric quantum stochastic filter, ie, the supersymmetric Belavkin filter. The construction of supersymmetric noise in the Hudson Parthasarathy noisy Schrodinger equation is based on constructing Fermionic noise by applying a twist to Bosonic noise, ie, $dJ = (-1)^{\Lambda} dA, dJ^* = (-1)^{\Lambda} dA^*$. Such a Fermionic noise has memory unlike the Bosonic noise and satisfies the CAR. We prove that when a superposition of Bosonic quantum Brownian motion and Bosonic counting process along with Fermionic counting process is measured through the Hudson-Parthasarathy noisy Schrodinger system with both Bosonic and supersymmetric noise, then it satisfies the non-demolition property and hence conditional expectations can be defined. The construction of the quantum filter here involves taking into consideration all the positive integer powers of the output measurement noise differential and we give an algorithm for computing these powers based on the quantum Ito formula. The method for deriving a countably infinite number of linear equations for the filter coefficients is based on the reference probability method of Gough et.al. The method of constructing supersymmetric noise for driving the HPS equation is based on the work of Timothy Eyre. The main result of this paper is

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to derive an infinite system of linear equations for the quantum filter coefficients thereby yielding a real time implementable filter for estimating system observables or equivalently, the system mixed state from the most general form of non-demolition measurements comprising Bosonic continuous and counting noise plus Fermionic counting noise.

keywords:Belavkin quantum filter; Fermionic noise using Poisson twist; Refrence probability approach; non-demolition measurement; supersymmetric quantum noise

2 Introduction

The primary aim of this work is to develop a Belavkin quantum real time filter for the case when the non-demolition measurement Y_t^o consist of a mixture of quantum Brownian motions and quantum Poisson processes passed through the noisy H-P dynamics taking into account the presence of Fermionic noise having memory in the H-P quantum stochastic differential equation (QSDE). The resulting filter is again a classical Schrödinger equation but contains all powers of the measurement differential i.e. $(dY_t^o)^k, k \geq 1$. This is the characteristic feature of any Levy process driven sde beacuse if Y_t is a Levy process then in general, all powers of dY_t are of $O(dt)$ or larger. In the quantum case however, in addition, the Brownian and Poisson component of the measurement do not commute because of the quantum Ito table so things are more complex. The filter coefficients are determined using the reference probability approach.

Gough and Kostler provided a comprehensive analysis of the Belavkin filter assuming that the bath is in a coherent state and that the measurements are either quadrature only, or $(A + A^\dagger)$, or photon counts alone, or Λ. The derivation of the filter for mixtures of these two processes have been covered in Naman et al. When measurement noise is Gaussian white noise but process noise is non-Gaussian white noise, i.e., the state process is Markov, classical nonlinear filtering theory is applicable and. Not much research on non-linear filtering in the case of white non-Gaussian measurement noise, or a differential of a Levy process has been carried out. The emergence of all powers of the output noise differential in the stochastic partial differential equation for the filter characterizes these kinds of problems.

In this paper, we present the Kushner-Kallianpur Belavkin quantum filtering equation for the scenario where the noisy Schrodinger equation in the H-P sense contains both Bosonic and Fermionic Creation, Annihilation, and Conservation noise. Additionally, the input non-demolition measurement includes the previous three Bosonic components and also the conservative/counting part of quantum Fermionic noise. Quantum Fermionic noise was constructed by KRP and Hudson using the Poisson twist $(-1)^\Lambda dA, (-1)^\Lambda dA^\dagger$. Accordingly, the stochastic differential structure of the Belavkin dynamical quantum filter for an evolving system observable X based on non-demolition measurement Y^0

will be as follows:

$$d\pi_t(X) = F_t(X)dt + \sum_{k\geq 1} G_{kt}(X)(dY_t^0)^k$$

where the linear functionals $F_t(X)$ and $G_{kt}(X)$ are measurable with respect to the non-demolition output noise of X. Filtration of Abelian type $\sigma(Y_s^0 : s \leq t)$. This Belavkin filter can be thought of as a classical sde [13] as the operators $F_t(X)$ and $G_{kt}(X)$ commute. It is simple to derive a series of linear equations for $F_t(X)$ and $G_{kt}(X)$ in terms of the Hamiltonian and Lindblad operators of the H-P equation and the coefficients of $A_t, A_t^\dagger, \Lambda_t$ in the input measurement process Y_t^i. This is done by applying the orthogonality principle for conditional expectation $\pi_t(X) = \mathbb{E}(j_t(X)|Y_s^0 : s \leq t)$. It is not possible to solve these equations in closed form. On the other hand, using system operators and the homomorphism j_t that transfers system observables to their Heisenberg evolution after time t in the tensor product of the system and bath space, we develop recursive equations for computing output measurements.

The primary new feature of this paper is to derive the quantum filter when the HP QSDE has both Bosonic and Fermionic noise with the measurement noise being a superposition of Bosonic quantum Brownian motions, Bosonic counting processes and Fermionic counting processes. That such processes yield non-demolition measurements is established via two theorems. The new idea here involves computing the evolution of not only $\mathbb{E}(j_t(X)|\eta_0(t)) = \pi_{0,t}(X)$ but also $\mathbb{E}(j_t(X(-1)^{\Lambda_t})|\eta_0(t)) = \pi_{1,t}(X)$ where X is a system observable while Λ_t is the quantum Poisson process used as a twist to define the Fermionic noise differentials.These computations are carried out in the supersymmetric notaion of T. Eyre[14].

The two major novel features of this paper are therefore deriving an infinite sequence of linear equations for the quantum filter coefficients in the case where noise in the H-P QSDE is supersymmetric in the language of Timothy Eyre [14] and in addition, when the measurement process is a mixture of quantum Bosonic Gaussian process, quantum Bosonic counting process and quantum Fermionic counting process passed through the HP QSDE.

3 Fundamentals of stochastic filtering theory

Let $x[n], n \geq 0$ be a Markov process in discrete time having transition probability density $\pi_n(y|x)$, ie,

$$P(x[n+1] \in E|x[n] = x) = \int_E \pi_n(y|x)dy$$

The measurement process is

$$z[n] = h_n(x[n]) + v[n]$$

where $v[n]$ has independent samples with pdf $q_n(v)$ v and x are assumed to be independent processes. Let

$$Z_n = \{z[k] : k \le n\}$$

The aim is to determine an integro-difference equation satisfied by $p_n(x|Z_n)$, the density of $x[n]$ given Z_n and hence calculate approximate recursions for $\hat{x}[n|n] = \mathbb{E}(x[n]|Z_n), \hat{x}[n+1|n] = \mathbb{E}(x[n+1]|Z_n)$ and $\mathbb{E}(x[n]-\hat{x}[n|n])^{\otimes k}|Z_n), k = 2, 3, ...$ We have using the Markov property

$$p(x[n+1]|Z_{n+1}) = \frac{p(x[n+1], z[n+1], Z_n)}{p(z[n+1], Z_n)}$$

$$= \frac{\int p(z[n+1]|x[n+1])p(x[n+1]|x[n])p_n(x[n]|Z_n)\,dx[n]}{p(z[n+1]|Z_n)}$$

$$= \frac{\int q_{n+1}(z[n+1] - h_{n+1}(x[n+1]))\pi_n(x[n+1]|x[n])p_n(x[n]|Z_n).dx[n]}{\int q_{n+1}(z[n+1] - h_{n+1}(x[n+1]))\pi_n(x[n+1]|x[n])p_n(x[n]|Z_n).dx[n+1]}$$

In case we have the model

$$x[n+1] = f_n(x[n], w[n+1])$$

with $w[n]$ independent having pdf $r_n(w)$, we can write

$$p(x[n+1]|Z_{n+1}) = \frac{\int q_{n+1}(z[n+1] - h_{n+1}(f(x[n], w))r_{n+1}(w)p_n(x[n]|Z_n)\,dw\,dx[n]}{\int q_{n+1}(z[n+1] - h_{n+1}(f(x[n], w))r_{n+1}(w)p_n(x[n]|Z_n)\,dw\,dx[n+1]}$$

In the general case, we write

$$x[n] = \hat{x}[n|n] + e[n|n], x[n+1] = \hat{x}[n+1|n] + e[n+1|n]$$

If $\phi(x)$ is any function of the state, we define

$$\Pi_n(\phi) = \mathbb{E}(\phi(x[n])|Z_n)$$

and sometimes write this as $\Pi_n(\phi(x))$, the argument x being emphasized to make sure that it is to be replaced by $x[n]$ and then followed by calculating the conditional expectation. Then the above gives

$$\Pi_{n+1}(\phi) = \frac{\sigma_{n+1}(\phi)}{\sigma_{n+1}(1)}$$

where

$$\sigma_{n+1}(\phi) = \Pi_n\left(\int q_{n+1}(z[n+1] - h_{n+1}(y))\phi(y)\pi_n(y|x)dy\right)$$

We also have the following relations:

$$\mathbb{E}(\phi(x[n+1])|Z_n) = \Pi_n\left(\int \phi(y)\pi_n(y|x)dy\right) p(x[n+1]|Z_n) = \int \pi_n(x[n+1]|x[n])p(x[n]|Z_n)dx[n]$$

so that

$$\hat{x}[n+1|n] = \mathbb{E}(x[n+1]|Z_n) = \Pi_n\left(\int y\pi_n(y|x)dy\right)$$

For example, suppose

$$x[n+1] = f_n(x[n]) + w[n+1]$$

where w has independent samples with pdf $r_n(w)$. Then,

$$\pi_n(y|x) = r_{n+1}(y - f_n(x))$$

and

$$\hat{x}[n+1|n] = \mathbb{E}(f_n(x[n])|Z_n) = \sum_m \frac{f_n^{(m)}(\hat{x}[n|n])}{m!}\mathbb{E}(e[n|n]^{\otimes m}|Z_n)$$

where

$$e[n|n] = x[n] - \hat{x}[n|n]$$

More generally, if

$$x[n+1] = f_n(x[n], w[n+1])$$

where w has independent samples with density $r_n(w)$, then

$$\pi_n(y|x)dy = P(f_n(x, w[n+1]) \in dy)$$

or equivalently,

$$\pi_n\phi(x) = \int \phi(y)\pi_n(y|x)dy = \mathbb{E}\phi(f_n(x, w[n+1])) = \int \phi(f_n(x,w))r_n(w)dw$$

then defining

$$f_n^{(m)}(x,w) = \partial_x^m f_n(x,w)$$

we get

$$\hat{x}[n+1|n] = \sum_m (m!)^{-1}\mathbb{E}(f_n^{(m)}(0, w[n+1]))\mathbb{E}(x[n]^{\otimes m}|Z_n)$$

or equivalently, in terms of central moments,

$$\hat{x}[n+1|n] = \sum_m (m!)^{-1}\mathbb{E}(f_n^{(m)}(0, w[n+1]))\mathbb{E}(x[n|n] + e[n|n])^{\otimes m}|Z_n)$$

We can expand

$$(x[n|n] + e[n|n])^{\otimes m} = \sum_{|k|\le|m|} P_k(\hat{x}[n|n])e[n|n]^{\otimes k}$$

and then observe that

$$\mathbb{E}(x[n|n] + e[n|n])^{\otimes m}|Z_n) = \sum_{|k|\le|m|} P_k(\hat{x}[n|n])\mathbb{E}(e[n|n]^{\otimes k}|Z_n)$$

Here, P_k's are multivariate polynomials. To complete the recursion, we must compute $\hat{x}[n+1|n+1]$ and $\mathbb{E}(e[n+1|n+1]^{\otimes m}|Z_{n+1})$ in terms of $\hat{x}[n+1|n]$.

To this end, we recall that

$$p(x[n+1]|Z_{n+1}) = \frac{\int q_{n+1}(z[n+1]-h_{n+1}(x[n+1]))\pi_n(x[n+1]|x[n])p_n(x[n]|Z_n)dx[n]}{\int q_{n+1}(z[n+1]-h_{n+1}(x[n+1]))\pi_n(x[n+1]|x[n])p_n(x[n]|Z_n)dx[n+1]}$$

$$= \frac{A(z[n+1],x[n+1])}{B(z[n+1])}$$

where $A(z[n+1],x[n+1])$ is the numerator and $B(z[n+1]) = \int A(z[n+1],x[n+1])dx[n+1]$ is the denominator. We then get with $e = e[n+1|n]$,

$$A(x[n+1],z[n+1]) =$$

$$= A(\hat{x}[n+1|n]+e,z[n+1]) = \int q_{n+1}(z[n+1]-h_{n+1}(\hat{x}[n+1|n]+e))\pi_n(\hat{x}[n+1|n]+e|x)p_n(x|Z_n)dx$$

Thus, the filtering recursion is completed by using the fact that

$$\mathbb{E}(x[n+1]^{\otimes m}|Z_{n+1}) = \int y^{\otimes m}A(z[n+1],y)dy/B(z[n+1])$$

or equivalently, in terms of central moments,

$$\mathbb{E}(e[n+1|n+1]^{\otimes m}|Z_{n+1}) = \int e^{\otimes m}A(\hat{x}[n+1|n]+e,z[n+1])de/B(z[n+1])$$

Having classical general non-Gaussian stochastic filtering theory in discrete time, we now proceed to a discussion of its quantum continuous time counterpart.

4 Quantum filtering in the presence of both Bosonic and supersymmetric generalized noise processes

Here, we present a complete derivation of the Boson-Fermion quantum filter introduced first by Belavkin and developed further upto perfection by John Gough etal in [3]-[4]. The Hudson-Parthasarathy-noisy Schrodinger equation with both Bosonic and Fermionic creation, annihilation and conservation noise processes is first introduced based on the graded tensor product between the system and noise Hilbert spaces and then Bosonic non-demolition measurements like the Bosonic Gaussian quadrature measurement and the Fermionic counting measurement processes are introduced. Then, using the reference probability approach developed by J.Gough et.al, we derive the quantum filtering equations, namely a non-commutative supersymmetric generalization of the classical Kushner-Kallianpur stochastic nonlinear filter. In our formalism, we assume that the Fermionic quantum Brownian motion processes are derived from the Bosonic quantum Brownian motion processes by applying the number operator twist to the latter and more generally, supersymmetric quantum noise processes are derived by applying graded number operator twists to the ungraded noise processes as discussed in detail in [14].

In order to introduce Mixed Bosonic and Fermionic or equivalently, supersymmetric quantum filtering [12], we start with the Hilbert space

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$$

where $\mathcal{H}_k, k = 1, 2$ are copies of $L^2(\mathbb{R}_+) \otimes \mathbb{C}^N$ and identify the Boson Fock space $\Gamma_s(\mathcal{H})$ with $\Gamma_s(\mathcal{H}_1) \otimes \Gamma_s(\mathcal{H}_2)$ via the canonical isomorphism $e(u_1 \oplus u_2) \to e(u_1) \otimes e(u_2)$ where $e(u)$ is the standard exponential vector [1]. Let $A_j(t), 1 \leq j \leq N$ denote the canonical annihilation processes in the Fock space $\Gamma_s(\mathcal{H}_1)$. They are defined in terms of canonical annihilation field a as $a(\chi_{[0,t]} e_j), j = 1, 2, ..., N$ where $e_j, j = 1, 2, ..., N$ is the standard basis for $\mathbb{C}^N$. We define the generalized quantum noise processes by

$$d\Lambda^a_b(t) = dA_b(t)^* dA_a(t)/dt \oplus 0,$$

$$d\tilde{\Lambda}^a_b(t) = 0 \oplus dA_b(t)^* dA_a(t)/dt$$

where $a, b = 0, 1, ..., N$ with $A_0(t) = A_0(t)^* = t$. The quantum Ito formula is then

$$d\Lambda^a_b(t).d\Lambda^c_d(t) = \epsilon^a_d d\Lambda^c_b(t), d\tilde{\Lambda}^a_b(t).d\tilde{\Lambda}^c_d(t) = \epsilon^a_d d\tilde{\Lambda}^c_b(t)$$

where $\epsilon^a_b = 1$ if $a = b \geq 1$ and is zero otherwise. Note that a, b assume values $0, 1, ..., N$.

Note that we can write $A_a(t) = a(\chi_{[0,t]}(e_a \oplus 0))$ and $\tilde{A}_a(t) = a(\chi_{[0,t]}(0 \oplus e_a))$ in the Boson Fock space $\Gamma_s(\mathcal{H}) = \Gamma_s(\mathcal{H}_1 \oplus \mathcal{H}_2) \approx \Gamma_s(\mathcal{H}_1) \otimes \Gamma_s(\mathcal{H}_2)$ and $d\Lambda^a_b(t) = dA^*_b(t) dA_a(t)/dt$ and $d\tilde{\Lambda}^a_b(t) = d\tilde{A}^*_b(t) d\tilde{A}_a(t)/dt$ and of course

$$[\Lambda^a_b(t), \tilde{\Lambda}^c_d(s)] = 0, \forall t, s$$

These generalized quantum Ito formulae can be seen easily to be consequences of the fundamental quantum Ito formula

$$dA_i(t) dA_j(t)^* = \delta_{ij} dt, i, j = 1, 2, ..., N$$

The generalized Bosonic noise processes are $\Lambda^a_b(t), a, b \geq 0$ and the supersymmetric noise processes are

$$\tilde{\xi}^a_b(t) = \int_0^t \tilde{G}(s)^{\sigma(a,b)} d\tilde{\Lambda}^a_b(s), a, b \geq 0$$

where

$$\tilde{G}(t) = (-1)^{\tilde{\Lambda}_t(H)} = exp(i\pi \tilde{\Lambda}_t(H))$$

with

$$H = diag[0_r, I_{N-r}]$$

and $\sigma(a,b) = 0$ if either $1 \leq a, b \leq r$ or $r+1 \leq a, b, \leq N$ and $\sigma(a,b) = 1$ otherwise. Note that on setting $\tilde{\Lambda}(t) = \tilde{\Lambda}_t(H) = \tilde{\lambda}(H_t)$,

$$\tilde{G}(t) = \tilde{\Gamma}(exp(i\pi H_t)) = (-1)^{\tilde{\Lambda}(t)}, H_t = H.\chi_{[0,t]}$$

so that

$$\tilde{\Lambda}(t) = \sum_{k=r+1}^{N} \tilde{\Lambda}_k^k(t)$$

and hence

$$\tilde{G}(t)|e(u)>=|e(exp(i\pi H)u\chi_{[0,t]} + u\chi_{(t,\infty)}>=|Ku\chi_{[0,t]} + u\chi_{(t,\infty)}>$$

Note that

$$exp(i\pi H) = K = (-1)^H = diag[I_r, -I_{N-r}]$$

It is easily verified that for $s \leq t$,

$$<e(v)|d\tilde{\Lambda}_b^a(s)\tilde{G}(t)|e(u)>=(Ku)_a(s)\bar{v}_b(s)ds<e(v)|\tilde{G}(t)|e(u)>$$

and

$$<e(v)|\tilde{G}(t)d\tilde{\Lambda}_b^a(s)|e(u)>=<\tilde{G}(t)e(v)|d\tilde{\Lambda}_b^a(s)|e(u)>u_a(s)\bar{(}Kv)_b(s)ds<e(v)|\tilde{G}(t)|e(u)>$$

Note that here, we are assuming that the Boson Fock space is

$$\Gamma_s(\mathcal{H}_1 \oplus \mathcal{H}_2) = \Gamma_s(\mathcal{H}_1) \otimes \Gamma_s(\mathcal{H}_2)$$

and that the noise processes Λ_b^a act in $\Gamma_s(\mathcal{H}_1)$ while the noise processes $\tilde{\Lambda}_b^a$ act in $\Gamma_s(\mathcal{H}_2)$. Thus, these two noise processes mutually commute. Noting that

$$(Ku)_a(s) = (-1)^{\sigma(a)}u_a(s), (\bar{K}v)_b(s) = (-1)^{\sigma(b)}\bar{v}(s) \qquad (1)$$

where $(-1)^{\sigma(a)} = K_{aa}$ is one for $1 \leq a \leq r$ and -1 for $r+1 \leq\leq N$, we get that

$$\tilde{G}(t)d\tilde{\Lambda}_b^a(s) = (-1)^{\sigma(a,b)}d\tilde{\Lambda}_b^a(s)\tilde{G}(t)$$

since

$$\sigma(a,b) = \sigma(a) + \sigma(b) mod 2$$

Thus,

$$\tilde{G}(t)^{\sigma(c,d)}d\tilde{\Lambda}_b^a(s) = (-1)^{\sigma(a,b)\sigma(c,d)}d\tilde{\Lambda}_b^a(s)\tilde{G}(t)^{\sigma(c,d)}, s \leq t$$

From the commutativity of $\tilde{G}(t), t \geq 0$, and of $\tilde{G}(s)$ with $d\tilde{\Lambda}_b^a(t), t \geq s$, we thus easily deduce that for $s \neq t$, we have

$$d\tilde{\xi}_b^a(s).d\tilde{\xi}_d^c(t) - (-1)^{\sigma(a,b)\sigma(c,d)}d\tilde{\xi}_d^c(t).d\tilde{\xi}_b^a(s) = 0$$

and hence, using quantum Ito's formula in the form

$$d\tilde{\xi}_b^a(t).d\tilde{\xi}_d^c(t) = \tilde{G}(t)^{\sigma(a,b)+\sigma(c,d)}d\tilde{\Lambda}_b^a(t).d\tilde{\Lambda}_d^c(t)$$

$$= \epsilon^a_d . \tilde{G}(t)^{\sigma(a,b)+\sigma(c,d)} d\tilde{\Lambda}^c_b(t) = \epsilon^a_d d\tilde{\xi}^c_b(t)$$

(because when $a = d$, we get $\sigma(a,b) + \sigma(c,d) = \sigma(b,c) mod 2$), we deduce from these results, the following Lie super-algebra representation property of the super-symmetric noise processes:

$$\tilde{\xi}^a_b(t).\tilde{\xi}^c_d(s) - (-1)^{\sigma(a,b)\sigma(c,d)} \tilde{\xi}^c_d(s).\tilde{\xi}^a_b(t) =$$

$$\epsilon^a_d.\tilde{\xi}^c_b(min(t,s)) - (-1)^{\sigma(a,b)\sigma(c,d)} \epsilon^c_b \tilde{\xi}^a_d(min(t,s))$$

This equation was first derived by Timothy Eyre [14] wherein he used this formula to illustrate how representations of Lie-Super algebras can be constructed using the Hudson-Parthasarathy quantum stochastic calculus [2].

Now consider the super-symmetric Hudson-Parthasarathy-noisy Schrodinger equation

$$dU(t) = (L^b_a d\Lambda^a_b(t) + M^b_a d\tilde{\xi}^a_b(t))U(t)$$

where L^b_a, M^b_a are system operators chosen to ensure unitarity of $U(t)$, ie,

$$0 = d(U(t)^* U(t)) = dU(t)^*.U(t) + U(t)^*.dU(t) + dU(t)^*.dU(t)$$

This happens iff

$$(L^a_b)^* + L^b_a + (L^d_b)^* L^c_a \epsilon^d_c = 0$$

or and likewise,

$$(M^a_b)^* + M^b_a + (M^d_b)^* M^c_a \epsilon^d_c = 0$$

Note that the $\tilde{\xi}^a_b$ processes have memory while the Λ^a_b processes do not have memory. By this, we mean that $\Lambda^a_b(t_2) - \Lambda^a_b(t_1)$ commutes with $\Lambda^a_b(t_4) - \Lambda^a_b(t_3)$ when $t_1 < t_2 < t_3 < t_4$ but this property is not shared by the process $\tilde{\xi}^a_b$.

A suitable candidate for the input non-demolition measurement process is therefore a linear combination of Bosonic noise $\Lambda^a_b(t), a, b = 0, 1, ..., N$ and thje supersymmetric counting processes $\tilde{\xi}^a_a(t) = \tilde{\Lambda}^a_a(t), a = 1, 2, ..., N$. Thus, we take as our input measurement process

$$Y_i(t) = c^b_a \Lambda^a_b(t) + d(a)\tilde{\Lambda}^a_a(t)$$

where summation over the repeated indices a, b is implicitly being assumed. Here, to maintain self-adjointedness of $Y_i(t)$, we require to assume that $c^a_b = \bar{c}^b_a$ and $d(a) = \bar{d}(a)$. This form of the measurement process is a generalization of that given in John Gough et.al [Fermionic filter] wherein the input measurement consisted of Bosonic noise (either quantum Brownian motion or Bosonic counting process) plus Fermionic counting noise. In the present situation, the input measurement consists of a mixture of Bosonic quantum Brownian motion, Bosonic counting processes and Fermionic counting processes in the different channels.

Therefore, the measured output non-demolition process obtained by passing the input measurement process through the HP system is given by

$$Y_o(t) = U(t)^* Y_i(t) U(t)$$

We require to show that $Y_o(.)$ satisfies the non-demolition property, ie, firstly, that it is an a Abelian family of observables, ie, $[Y_o(t), Y_o(s)] = 0 \forall t, s$ and secondly, that for any $T > t$, $Y_o(t)$ commutes with $j_T(X.\tilde{G}(T)^m) = U(T)^* X G(T)^m U(T), m = 0, 1$ where X is any system observable. To prove the non-demolition property, we first show that

Theorem 1:

$$d_T(U(T)^* Y_i(t) U(T)) = 0, T \geq t$$

where d_T denotes differential w.r.t T.

Proof: Let $T \geq t$. Owing to the conditions on the system operators L_a^b, M_a^b that ensure unitarity of $U(T)$ and the fact that $Y_i(t)$ commutes with all the the system operators and that $Y_i(t)$ also commutes with the Bosonic noise differentials $d\Lambda_b^a(T)$, with $\tilde{G}(T)$ and with $d\tilde{\Lambda}_b^a(T)$, we can easily deduce that

$$d_T(U(T)^* Y_i(t) U(T)) = dU(T)^* Y_i(t) U(T) + U(T)^* Y_i(t) dU(T) + dU(T)^* Y_i(t).dU(T) = 0, T \geq t$$

Note that

$$dU(T)^* = U(T)^* (L_a^b)^* d\Lambda_a^b(T) + (M_a^b)^* \tilde{G}(T)^{\sigma(a,b)} d\tilde{\Lambda}_a^b(T))$$

and

$$dU(T) = (L_a^b d\Lambda_b^a(T) + M_a^b \tilde{G}(T)^{\sigma(a,b)} d\Lambda_b^a(T)) U(T)$$

and $Y_i(t)$ commutes with all the terms $(L_a^b)^*, L_a^b, (M_a^b)^*, M_a^b, d\Lambda_a^b(T), d\tilde{\Lambda}_a^b(T), \tilde{G}(T)$. The only difficult part here is the proof that $Y_i(t)$ commutes with $\tilde{G}(T)$. To prove this we require to show that $Y_i(t)$ commutes with $\tilde{\Lambda}(T) = \sum_{c=r+1}^n \tilde{\Lambda}_c^c(T)$. But that is immediate since $\Lambda_b^a(t)$ commutes with $\tilde{\Lambda}_c^c(T)$ (The two processes operate in different components of the tensor product of two Boson Fock spaces) and also that $\tilde{\Lambda}_a^a(t)$ commutes with $\tilde{\Lambda}_c^c(T)$ because of the fact that for $t \neq s$, $[d\tilde{\Lambda}_a^a(t), d\tilde{\Lambda}_c^c(s)] = 0$ and

$$d\tilde{\Lambda}_a^a(t) d\tilde{\Lambda}_c^c(t) = d\tilde{\Lambda}_c^c(t).d\tilde{\Lambda}_a^a(t)$$

$$= \epsilon_c^a.d\tilde{\Lambda}_c^c(t)$$

Corollary 1: $Y_o(t) = U(T)^* Y_i(t) U(T) \forall T \geq t$.

Proof: Immediate.

This corollary immediately implies that $[Y_o(t), Y_o(s)] = 0 \forall t, s$. In fact, choosing $T > t, s$ and applying this theorem gives

$$[Y_o(t), Y_o(s)] = [U(T)^* Y_i(t) U(T), U(T)^* Y_i(s) U(T)] = U(T)^* [Y_i(t), Y_i(s)] U(T) = 0$$

since obviously $Y_i(.)$ forms an Abelian family.

Theorem 2: $[Y_o(t), j_T(X\tilde{G}(T)^m)] = 0, m = 0, 1, T \geq t$ for all system operators X.

Proof: For $T \geq t$, we write $Y_o(t) = U(T)^* Y_i(t) U(T)$ and then proving the result amounts to proving that $[Y_i(t), X.\tilde{G}(T)^m] = 0$ for $T \geq t$. But $Y_i(t)$ commutes with X and also with $\tilde{G}(T)$ as observed in Theorem 1. This completes the proof.

Note that by non-demolition, we mean that $Y_o(t)$ commutes with $Y_o(s)$ for all $s \neq t$ and secondly that $j_T(XG(T)^m)$ commutes with $Y_o(t)$ for all $T \geq t$ and for $m = 0, 1$.

Now let $\eta_i(t)$ be the Abelian algebra generated by $Y_i(s), s \leq t$ and let $\eta_o(t)$ be the Abelian algebra generated by $Y_o(s), s \leq t$. Note that this latter is Abelian because in view of what we have just noted, $\eta_o(t) = U(t)^* \eta_i(t) U(t)$ (ie, because $Y_o(s) = U(t)^* Y_i(s) U(t), t \geq s$). Since we have just proved that $j_t(X.G(t)^m)$ commutes with $\eta_o(t) = \sigma(Y_o(s) : s \leq t)$, it follows that we can talk of the conditional expectations

$$\pi_{m,t}(X) = \mathbb{E}(j_t(X.\tilde{G}(t)^m)|\eta_o(t)), m \geq 0$$

Now,

$$dY_o(t) = d(U(t)^* Y_i(t) U(t)) = dY_i(t) + dU(t)^* dY_i(t) U(t) + U(t)^* dY_i(t) dU(t)$$

because of the unitarity of $U(t)$ and the fact that

$$dU(t)^* Y_i(t) U(t) + U(t)^* Y_i(t) dU(t) + dU(t)^* Y_i(t) dU(t) = 0$$

since $Y_i(t)$ commutes with all the system operators and also with $d\Lambda^a_b(t), d\tilde{\Lambda}^a_b(t)$ and $\tilde{G}(t)$. This latter statement again would not be true if $Y_i(t)$ contained a term $\tilde{\xi}^a_b(t)$ because then $\tilde{\xi}^a_b(t) = \int_0^t \tilde{G}(s)^{\sigma(a,b)} d\tilde{\Lambda}^a_b(s)$ does not commute with $\tilde{G}(t)$ since $d\tilde{\Lambda}^a_b(s)$ does not commute with $\tilde{G}(t)$ for $s \leq t$. Now we can assume that

$$d\pi_{m,t}(X) = F_{m,t}(X)dt + \sum_{k \geq 1} G_{m,k,t}(X)(dY_o(t))^k$$

where $F_{m,t}(X), G_{m,k,t}(X)$ are all $\eta_0(t)$ measurable. Obviously, this filter is commutative because the $\eta_0(.)$ is a commuting family of algebras. Let $C(t)$ satisfy

$$dC(t) = \sum_{k \geq 1} f_k(t) C(t) dY_o(t)^k, t \geq 0, C(0) = 1$$

where the f'_k are complex valued functions. Then, consider the orthogonality principle

$$\mathbb{E}[(j_t(X.\tilde{G}(t)^m) - \pi_{m,t}(X))C(t)] = 0-$$

Application of Ito's formula and use of the arbitrariness of $f_k(t)'s$ then gives us

$$\mathbb{E}[(dj_t(X.\tilde{G}(t)^m) - d\pi_{m,t}(X))|\eta_o(t)) = 0 - - - (a)$$

$$\mathbb{E}[(j_t(X.\tilde{G}(t)^m) - \pi_{m,t}(X)) dY_o(t)^k | \eta_o(t)]$$

$$+\mathbb{E}[(dj_t(X.\tilde{G}(t)^m) - d\pi_{m,t}(X)) dY_o(t)^k | \eta_o(t)] = 0 - - - (b)$$

Now, we have

$$dY_o(t) = dY_i(t) + dU(t)^* dY_i(t) U(t) + U(t)^* dY_i(t) dU(t)$$

$$= dY_i(t) + j_t((L_a^b)^*)d\Lambda_a^b(t)dY_i(t) + j_t(L_b^a)dY_i(t)d\Lambda_a^b(t)$$

$$+j_t((M_a^b)^*\tilde{G}(t)^{\sigma(a,b)})d\tilde{\Lambda}_a^b(t)dY_i(t) + j_t(M_b^a\tilde{G}(t)^{\sigma(a,b)})dY_i(t)d\tilde{\Lambda}_a^b(t)$$

Now,

$$d\Lambda_a^b(t)dY_i(t) = c_\nu^\mu d\Lambda_a^b(t).d\Lambda_\mu^\nu(t)$$

$$= c_\nu^\mu \epsilon_\mu^b d\Lambda_a^\nu(t)$$

$$dY_i(t)d\Lambda_a^b(t) = c_\nu^\mu d\Lambda_\mu^\nu(t)d\Lambda_a^b(t)$$

$$= c_\nu^\mu \epsilon_a^\nu d\Lambda_\mu^b(t)$$

$$d\tilde{\Lambda}_a^b(t)dY_i(t) = d(c)d\tilde{\Lambda}_a^b(t)d\tilde{\Lambda}_c^c(t) = d(c)\epsilon_c^b d\tilde{\Lambda}_a^c(t)$$

$$dY_i(t)d\tilde{\Lambda}_a^b(t) = d(c)d\tilde{\Lambda}_c^c(t)d\tilde{\Lambda}_a^b(t) = d(c)\epsilon_a^c d\tilde{\Lambda}_c^b(t)$$

with summation over the repeated index c being implied.

Thus, we can write

$$dY_o(t) = j_t(K_b^a)d\Lambda_a^b(t) + j_t(Q_b^a\tilde{G}(t)^{\sigma(a,b)})d\tilde{\Lambda}_a^b(t)$$

where K_b^a are system operators expressible in terms of c_b^a, L_b^a while Q_b^a are system operators expressible in terms of $d(a), M_b^a$.

Now writing

$$dY_o(t)^m = j_t(K_b^a(m))d\Lambda_a^b(t) + j_t(Q_b^a(m)\tilde{G}(t)^{\sigma(a,b)})d\tilde{\Lambda}_a^b(t), m \geq 1,$$

we derive the recursions:

$$j_t(K_b^a(m+1))d\Lambda_a^b(t) + j_t(Q_b^a(m+1)\tilde{G}(t)^{\sigma(a,b)})d\tilde{\Lambda}_a^b(t)$$

$$= j_t(K_d^c K_b^a(m))d\Lambda_c^d.d\Lambda_a^b +$$

$$+j_t(Q_d^c Q_b^a(m)\tilde{G}(t)^{\sigma(c,d)+\sigma(a,b)})d\tilde{\Lambda}_c^d.d\tilde{\Lambda}_a^b$$

$$= j_t(K_d^c K_b^a(m))\epsilon_a^d d\Lambda_c^b + j_t(Q_d^c Q_b^a(m))\tilde{G}(t)^{\sigma(b,c)})\epsilon_a^d d\tilde{\Lambda}_c^b$$

and hence,

$$K_b^c(m+1) = \epsilon_a^d K_d^c K_b^a(m), Q_b^c(m+1) = \epsilon_a^d Q_d^c Q_b^a(m)$$

Equivalently, in matrix notation,

$$K(m+1) = K\epsilon K(m), Q(m+1) = Q\epsilon.Q(m), K_b^a(1) = K_b^a, Q_b^a(1) = Q_b^a$$

yielding the solution

$$K(m) = (K\epsilon)^{m-1}K, Q(m) = (Q\epsilon)^{m-1}Q, m \geq 1, K(1) = K, Q(1) = Q, m \geq 1$$

Thus, we also get

$$\mathbb{E}(j_t(X.\tilde{G}(t)^m)(dY_o(t))^k|\eta_o(t))$$

$$= \mathbb{E}(j_t(X.\tilde{G}(t)^m)(j_t(K_b^a(k))d\Lambda_a^b(t) + j_t(Q_a^b(k)\tilde{G}(t)^{\sigma(a,b)})d\tilde{\Lambda}_a^b(t))|\eta_o(t))$$
$$= \mathbb{E}(j_t(X.K_b^a(k)\tilde{G}(t)^m)d\Lambda_a^b(t) + j_t(XQ_b^a(k)\tilde{G}(t)^{m+\sigma(a,b)})d\tilde{\Lambda}_a^b(t))|\eta_o(t))$$
$$= \pi_{m,t}(XK_b^a(k))u_b(t)\bar{u}_a(t)dt + \pi_{m+\sigma(a,b),t}(XQ_b^a(k))\tilde{u}_b(t)\bar{\tilde{u}}_a(t)^*dt$$

Now, we compute

$$d\tilde{G}(t) = ((-1)^{\tilde{\Lambda}(t)+1)} - (-1)^{\tilde{\Lambda}(t)})d\tilde{\Lambda}(t)$$

$$= -2\tilde{G}(t)d\tilde{\Lambda}(t), \tilde{\Lambda}(t) = \sum_{k=r+1}^{N} \tilde{\Lambda}_k^k(t)$$

and more generally,

$$d\tilde{G}(t)^m = d(-1)^{m\tilde{\Lambda}(t)}$$
$$= [(-1)^{m(\tilde{\Lambda}(t)+1)} - (-1)^{m\tilde{\Lambda}(t)}]d\tilde{\Lambda}(t)$$
$$= ((-1)^m - 1)\tilde{G}(t)^m d\tilde{\Lambda}(t)$$
$$= -2\rho_m\tilde{G}(t)^m d\tilde{\Lambda}(t)$$

where $\rho_m = 1$ if m is odd and $= 0$ if m is even. Note that $\tilde{G}(t)^m = 1$ if m is even because the eigenvalues of the observable $\tilde{\Lambda}(t)$ are all zero or one since $\tilde{\Lambda}(t)^2 = \tilde{\Lambda}_t(H)^2 = \tilde{\Lambda}_t(H^2) = \tilde{\Lambda}_t(H) = \tilde{\Lambda}(t)$, or equivalently since $(d\tilde{\Lambda}(t))^2 = d\tilde{\Lambda}(t)$ and $d\tilde{\Lambda}(t).d\tilde{\Lambda}(s) = 0$ for $t \neq s$. Now,

$$dj_t(X.\tilde{G}(t)^m) = d(U(t)^*X.\tilde{G}(t)^m U(t))$$

$$= j_t(X.d\tilde{G}(t)^m) + dU(t)^*X\tilde{G}(t)^m U(t) + U(t)^*X\tilde{G}(t)^m dU(t)$$
$$+dU(t)^*X.\tilde{G}(t)^m dU(t) + dU(t)^*X.d\tilde{G}(t)^m)U(t)$$
$$+U(t)^*X.d\tilde{G}(t)^m dU(t)$$
$$= j_t(X.d\tilde{G}(t)^m) + dU(t)^*X.d\tilde{G}(t)^m)U(t) + U(t)^*X.d\tilde{G}(t)^m dU(t)$$
$$+j_t((L_b^a)^*X\tilde{G}(t)^m)d\Lambda_b^a(t) + j_t((M_b^a)^*X\tilde{G}(t)^{\sigma(a,b)+m})d\tilde{\Lambda}_b^a(t)$$
$$+j_t(X.L_a^b\tilde{G}(t)^m)d\Lambda_b^a(t) + j_t(X.M_a^b\tilde{G}(t)^{m+\sigma(a,b)})d\tilde{\Lambda}_b^a(t)$$
$$+j_t((L_b^a)^*XL_d^c\tilde{G}(t)^m)d\Lambda_b^a.d\Lambda_c^d$$
$$+j_t((M_b^a)^*XM_d^c\tilde{G}(t)^{m+\sigma(a,b)+\sigma(c,d)})d\tilde{\Lambda}_b^a.d\tilde{\Lambda}_c^d$$

Note that $d\tilde{\Lambda}_b^a(t), d\Lambda_b^a(t)$ and system operators, all commute with $\tilde{G}(t)$. Now using the commutativity of system operators with noise operators and quantum Ito's formula (note that $d\tilde{\Lambda}_b^a(t).d\Lambda_d^c(t) = 0$), we get

$$dU(t)^*X.d\tilde{G}(t)^m U(t)$$

$$= -2\rho_m U(t)^*(M_b^a)^*X\tilde{G}(t)^{m+\sigma(a,b)}d\tilde{\Lambda}_b^a(t)d\tilde{\Lambda}(t)U(t)$$

$$= -2\rho_m j_t((M_b^a)^* X.\tilde{G}(t)^{m+\sigma(a,b)}) \sum_{k=r+1}^{M} d\tilde{\Lambda}_b^a(t) d\tilde{\Lambda}_k^k(t)$$

$$= -2\rho_m j_t((M_b^a)^* X.\tilde{G}(t)^{m+\sigma(a,b)}) \sum_{k=r+1}^{M} \epsilon_k^a d\tilde{\Lambda}_b^k(t)$$

$$= -2\rho_m \theta_a . j_t((M_b^a)^* X \tilde{G}(t)^{m+\sigma(a,b)}) d\tilde{\Lambda}_b^a(t)$$

where $\theta_a = \theta(a-r-1)$ which equals unity for $N \geq a \geq r+1$ and zero otherwise. Likewise,

$$U(t)^* X.d\tilde{G}(t)^m dU(t) = -2\rho_m \theta_a . j_t(X.M_a^b \tilde{G}(t)^{m+\sigma(a,b)}) d\tilde{\Lambda}_b^a(t)$$

Thus,

$$dj_t(X.\tilde{G}(t)^m) = A + B$$

where

$$A = -2\rho_m . j_t(X\tilde{G}(t)^m) d\tilde{\Lambda}(t) - 2\rho_m \theta_a . j_t((XM_a^b + (M_b^a)^* X)\tilde{G}(t)^{m+\sigma(a,b)}) d\tilde{\Lambda}_b^a(t)$$

$$= j_t(\phi(1,b,a,m,X)\tilde{G}(t)^m) d\tilde{\Lambda}_b^a(t) + j_t(\phi(2,b,a,X)\tilde{G}(t)^{m+\sigma(a,b)}) d\tilde{\Lambda}_b^a(t)$$

where

$$\phi(1,b,a,m,X) = -2\rho_m \theta_a \delta(a-b) X, \phi(2,b,a,X) = -2\rho_m \theta_a (XM_a^b + (M_b^a)^* X),$$

and

$$B = j_t((L_b^a)^* X \tilde{G}(t)^m) d\Lambda_b^a(t) + j_t((M_b^a)^* X \tilde{G}(t)^{\sigma(a,b)+m}) d\tilde{\Lambda}_b^a(t)$$
$$+ j_t(X.L_a^b \tilde{G}(t)^m) d\Lambda_b^a(t) + j_t(X.M_a^b \tilde{G}(t)^{m+\sigma(a,b)}) d\tilde{\Lambda}_b^a(t)$$
$$+ j_t((L_b^d)^* X L_a^c \tilde{G}(t)^m) \epsilon_c^d . d\Lambda_b^a + j_t((M_b^d)^* X M_a^c \tilde{G}(t)^{m+\sigma(b,a)}) \epsilon_c^d d\tilde{\Lambda}_b^a(t)$$
$$= j_t(\phi(3,b,a,X)\tilde{G}(t)^m) d\Lambda_b^a(t) + j_t(\phi(4,b,a,X)\tilde{G}(t)^{m+\sigma(b,a)}) d\tilde{\Lambda}_b^a(t)$$

where

$$\phi(3,b,a,X) = (L_b^a)^* X + X L_a^b + \epsilon_c^d (L_b^d)^* X L_a^c \phi(4,b,a,X) = (M_b^a)^* X + X.M_a^b + \epsilon_c^d (M_b^d)^* X M_a^c$$

Thus, combining all of the above equations, we get

$$dj_t(X.\tilde{G}(t)^m)$$
$$= j_t(\phi(1,b,a,m,X)\tilde{G}(t)^m) d\tilde{\Lambda}_b^a(t)$$
$$+ j_t(\phi(2,b,a,X)\tilde{G}(t)^{m+\sigma(a,b)}) d\tilde{\Lambda}_b^a(t)$$
$$+ j_t(\phi(3,b,a,X)\tilde{G}(t)^m) d\Lambda_b^a(t)$$
$$+ j_t(\phi(4,b,a,X)\tilde{G}(t)^{m+\sigma(b,a)}) d\tilde{\Lambda}_b^a(t)$$
$$= j_t(\psi(1,b,a,X)\tilde{G}(t)^m) d\Lambda_b^a(t) + j_t(\psi(2,b,a,X)\tilde{G}(t)^m) d\tilde{\Lambda}_b^a(t)$$
$$+ j_t(\psi(3,b,a,X)\tilde{G}(t)^{m+\sigma(b,a)}) d\tilde{\Lambda}_b^a(t)$$

where

$\psi(1,b,a,X)=\phi(3,b,a,X),\psi(2,b,a,X)=\phi(1,b,a,X),\psi(3,b,a,X)=\phi(2,b,a,X)+\phi(4,b,a,X)$

Then,

$$\mathbb{E}(dj_t(X.\tilde{G}(t)^m)|\eta_o(t))=[\pi_{m,t}(\psi(1,b,a,X))u_a(t)\bar{u}_b(t)$$
$$+\pi_{m,t}(\psi(2,b,a,X))\tilde{u}_a(t)^*\tilde{u}_b(t)$$
$$+\pi_{m+\sigma(b,a),t}(\psi(3,b,a,X))\tilde{u}_a(t)^*\tilde{u}_b(t)]dt$$

because the expectation is being computed in the state $|f>\otimes|\phi(u)\otimes\phi(\tilde{u})>$ where $|f>$ is a pure system state and $|\phi(u)>=exp(-|u|^2/2)|e(u)>$ is a bath coherent state for the first kind of Bosons and likewise $|\phi(\tilde{u})>$ for the second kind. Note that the total coherent state of the bath is $|\phi(u)>\otimes|\phi(\tilde{u})>=|\phi(u\oplus\tilde{u})>$. Note that the noise operators $\Lambda^a_b(t)$ act on the first kind of Bosons while $\tilde{\Lambda}^a_b(t)$ act on the second kind. Equivalently, the annihilation and creation fields are $a(u\oplus\tilde{u}),a(u\oplus\tilde{u})^*$ and the creation and annihilation operators of the first kind are $A_a(t)^*=a(e_a\chi_{[0,t]}\oplus 0)^*,A_a(t)=a(e_a\chi_{[0,t]}\oplus 0)$ while those of the second kind are $\tilde{A}_a(t)^*=a(0\oplus e_a\chi_{[0,t]})^*,\tilde{A}_a(t)=a(0\oplus e_a\chi_{[0,t]})$. Here, the Hilbert space for the first kind of Bosons is $\mathcal{H}_1=\mathbb{C}^d\otimes L^2(\mathbb{R}_+)$ and the Hilbert space for the second kind of Bosons $\mathcal{H}_2$ is a copy of $\mathcal{H}_1$. The two spaces are orthogonal to each other and the Boson Fock space for the entire system is

$$\Gamma_s(\mathcal{H})=\Gamma_s(\mathcal{H}_1\oplus\mathcal{H}_2)=\Gamma_s(\mathcal{H}_1)\otimes\Gamma_s(\mathcal{H}_2),\mathcal{H}=\mathcal{H}_1\oplus\mathcal{H}_2$$

Thus, we have

$$d\Lambda^a_b(t)=dA_b(t)^*dA_a(t)/dt,d\tilde{\Lambda}^a_b(t)=d\tilde{A}_b(t)^*d\tilde{A}_a(t)/dt$$

Note that the general annihilation field can be expressed as

$$a(u\oplus\tilde{u})=a(u\oplus 0)+a(0\oplus\tilde{u})a(u\oplus 0)=a_1(u)\oplus 0,a(0\oplus\tilde{u})=0\oplus a_2(\tilde{u})$$

where a_1,a_2 are copies of each other acting respectively in the component Boson Fock spaces $\Gamma_s(\mathcal{H}_1)$ and $\Gamma_s(\mathcal{H}_2)$ respectively. $a(u\oplus\tilde{u})$ acts in the total Boson Fock space $\Gamma_s(\mathcal{H})=\Gamma_s(\mathcal{H}_1)\otimes\Gamma_s(\mathcal{H}_2)$

Further,

$$\mathbb{E}(j_t(X.\tilde{G}(t)^m)(dY_o(t))^k|\eta_o(t))$$
$$=\mathbb{E}(j_t(X.\tilde{G}(t)^m)(j_t(K^a_b(k))d\Lambda^b_a(t)+j_t(Q^a_b(k)\tilde{G}(t)^{\sigma(a,b)}d\tilde{\Lambda}^b_a(t))|\eta_o(t))$$
$$=\mathbb{E}(j_t(XK^a_b(k)\tilde{G}(t)^m)d\Lambda^b_a(t)+j_t(X.Q^a_b(k)\tilde{G}(t)^{\sigma(a,b)})d\tilde{\Lambda}^b_a(t)|\eta_o(t))$$
$$=\pi_{m,t}(XK^a_b(k))u_b(t)\bar{u}_a(t)dt+\pi_{m+\sigma(a,b),t}(XQ^a_b(k))\tilde{u}_b(t)\tilde{u}_a(t)^*dt$$

In particular, we note that taking $X=I,m=0$

$$\mathbb{E}(dY_o(t)^k|\eta_o(t))=\pi_{0,t}(K^a_b(k))u_b(t)\bar{u}_a(t)dt+\pi_{\sigma(a,b),t}(Q^a_b(k))\tilde{u}_b(t)\tilde{u}_a(t)^*dt$$

Also,

$$\mathbb{E}(\pi_{m,t}(X)dY_o(t)^k|\eta_o(t)) = \pi_{m,t}(X)\pi_{0,t}(K^a_b(k))u_b(t)\bar{u}_a(t)dt+\pi_{m,t}(X)\pi_{\sigma(a,b),t}(Q^a_b(k))\tilde{u}_b(t)\tilde{u}_b(t)^*dt,$$

Further,

$$\mathbb{E}(dj_t(X.\tilde{G}(t)^m).dY_o(t)^k|\eta_o(t)) = \mathbb{E}[[j_t(\psi(1,b,a,X)\tilde{G}(t)^m)d\Lambda^a_b(t)$$
$$+j_t(\psi(2,b,a,X)\tilde{G}(t)^m)d\tilde{\Lambda}^a_b(t)$$
$$+j_t(\psi(3,b,a,X)\tilde{G}(t)^{m+\sigma(b,a)})d\tilde{\Lambda}^a_b(t)][j_t(K^c_d(k))d\Lambda^d_c(t)+j_t(Q^c_d(k)\tilde{G}(t)^{\sigma(c,d)})d\tilde{\Lambda}^d_c(t)]|\eta_o(t)]$$
$$= \mathbb{E}[[j_t(\psi(1,b,a,X)K^c_d(k)\tilde{G}(t)^m).d\Lambda^a_b(t)d\Lambda^d_c(t)|\eta_o(t)]$$
$$+\mathbb{E}[j_t(\psi(2,b,a,X)Q^c_d(k)\tilde{G}(t)^{m+\sigma(c,d)})d\tilde{\Lambda}^a_b(t).d\tilde{\Lambda}^d_c(t)|\eta_o(t)]$$
$$+\mathbb{E}[j_t(\psi(3,b,a,X)Q^c_d(k)\tilde{G}(t)^{m+\sigma(b,a)+\sigma(c,d)})d\tilde{\Lambda}^a_b(t)d\tilde{\Lambda}^d_c(t)|\eta_o(t)]$$
$$= \mathbb{E}[[j_t(\psi(1,b,a,X)K^c_d(k)\tilde{G}(t)^m)\epsilon^a_c d\Lambda^d_b(t)|\eta_o(t)]$$
$$+\mathbb{E}[j_t(\psi(2,b,a,X)Q^c_d(k)\tilde{G}(t)^{m+\sigma(c,d)})\epsilon^a_c d\tilde{\Lambda}^d_b(t)|\eta_o(t)]$$
$$+\mathbb{E}[j_t(\psi(3,b,a,X)Q^c_d(k)\tilde{G}(t)^{m+\sigma(b,a)+\sigma(c,d)})\epsilon^a_c d\tilde{\Lambda}^d_b(t)|\eta_o(t)]$$
$$= \pi_{m,t}(\psi(1,b,a,X)K^c_d(k))\epsilon^a_c u_d(t)\bar{u}_b(t)dt$$
$$+\pi_{m+\sigma(c,d),t}(\psi(2,b,a,X)Q^c_d(k))\epsilon^a_c\tilde{u}_d(t)\tilde{u}_b(t)^*dt$$
$$+\pi_{m+\sigma(b,d),t}(\psi(3,b,a,X)Q^c_d(k))\epsilon^a_c\tilde{u}_d(t)\tilde{u}_b(t)^*dt$$

Finally, for $k \geq 1$

$$\mathbb{E}(d\pi_{m,t}(X).dY_o(t)^k|\eta_o(t)) = \sum_{r\geq 1}\mathbb{E}(G_{m,r,t}(X)dY_o(t)^{k+r}|\eta_o(t))$$
$$= \sum_{r\geq 1}G_{m,r,t}(X)\mathbb{E}(dY_o(t)^{k+r}|\eta_o(t))$$
$$= \sum_{r\geq 1}G_{m,r,t}(X)(\pi_{0,t}(K^c_d(k+r))u_d(t)\bar{u}_c(t)dt+\pi_{\sigma(c,d),t}(Q^c_d(k+r))\tilde{u}_d(t)u_c(t)^*dt)$$

and

$$\mathbb{E}(d\pi_{m,t}(X)|\eta_o(t)) = F_{m,t}(X)dt+\sum_{r\geq 1}G_{m,r,t}(X)(\pi_{0,t}(K^c_d(r))u_d(t)\bar{u}_c(t)dt+\pi_{\sigma(c,d),t}(Q^c_d(r))\tilde{u}_d(t)\tilde{u}_c(t)^*dt$$

In this way, all the components required for iteratively determining the filter on a real time basis have been obtained. Specifically, from (a) and the subsequent equations,

$$\pi_{m,t}(\psi(1,b,a,X))u_a(t)\bar{u}_b(t)+\pi_{m,t}(\psi(2,b,a,X))\tilde{u}_a(t)\tilde{u}_b(t)^*+\pi_{m+\sigma(b,a),t}(\psi(3,b,a,X))$$
$$-F_{m,t}(X)-\sum_{k\geq 1}G_{m,k,t}(X)(\pi_{0,t}(K^a_b(k))u_b(t)\bar{u}_a(t)+\pi_{\sigma(a,b),t}(Q^a_b(k))\tilde{u}_b(t)\tilde{u}_a(t)^*) = 0, m = 0,1---(c)$$

where summation over the repeated indices $a, b = 0, 1, ..., N$ is implied. From (b) and the subsequent equations,

$$\pi_{m,t}(XK_b^a(k))u_b(t)\bar{u}_a(t) + \pi_{m+\sigma(a,b),t}(XQ_b^a(k))\tilde{u}_b(t)\tilde{u}_a(t)^*$$

$$-\pi_{m,t}(X)\pi_{0,t}(K_b^a(k))u_b(t)\bar{u}_a(t) - \pi_{m,t}(X)\pi_{\sigma(a,b),t}(Q_b^a(k))\tilde{u}_b(t)\tilde{u}_a(t)^*$$

$$+\pi_{m,t}(\psi(1, b, a, X)K_d^c(k))\epsilon_c^a u_d(t)\bar{u}_b(t)$$

$$+\pi_{m+\sigma(c,d),t}(\psi(2, b, a, X)Q_d^c(k))\epsilon_c^a \tilde{u}_d(t)\tilde{u}_b(t)^* dt$$

$$+\pi_{m+\sigma(b,d),t}(\psi(3, b, a, X)Q_d^c(k))\epsilon_c^a \tilde{u}_d(t)\tilde{u}_b(t)^* dt$$

$$-\sum_{r\geq 1} G_{m,r,t}(X)(\pi_{0,t}(K_b^a(k+r))u_b(t)\bar{u}_a(t) + \pi_{\sigma(a,b),t}(Q_b^a(k+r))\tilde{u}_b(t)u_a(t)^*)$$

$$= 0, m = 0, 1, k \geq 1 - - - (d)$$

[c] and [d] constitute an infinite system of linear equations to solve for $F_{m,t}(X), G_{m,r,t}(X), m = 0, 1, r \geq 1$

5 Conclusions

We have determined the basic quantum filter equation for estimating on a real time basis the state of a quantum system coupled to a quantum noisy bath when the bath supplies both Bosonic and Fermionic noise. The non-demolition measurement process in full generality turns out to be a superposition of Bosonic creatiion, annihilation and conservation/counting processes and Fermionic counting processes passed through the HP system. It turns out that measurements on the Fermionic creation and annihilation processes do not yield non-demolition measurements owing to them having memory in the sense that their increments over non-overlapping time intervals do not mutually commute.

The crucial steps involves computation of the quantum stochastic differentials $dj_t(X(-1)^{\Lambda_t})$ apart from $dj_t(X)$ where X is a system operator and Λ_t is a quantum Poisson noise. Using these differentials, we calculate the coupled stochastic differential equation for the conditional expectations for $\pi_{0,t}(X) = \mathbb{E}[j_t(X)|\eta_0(t)]$ and $\pi_{1,t}(X) = \mathbb{E}[j_t(X(-1)^{\tilde{\Lambda}_t})|\eta_0(t)]$ where X is a system operator and $\eta_0(t)$ is the output measurement Abelian algebra upto time t. These differential equations are obtained by applying the orthogonality principle with the hypothesis that $d\pi_{k,t}(X), k = 0, 1$ can be expressed as polynomials in the output measurement differentials $dY_0(t)$. In order to do so, we have determined a scheme for obtaining, $(dY_0(t))^m, m = 0, 1, 2, 3....$ in terms of $dA, dA^\dagger, d\Lambda$ with coefficients of the form $j_t(K_j(m))$ where $K_j(m)$ are system operator algebra elements. The final result is an algorithm for determining the coefficients $F_{mt}(X), G_{m,k,t}(X)$ in the filter $d\pi_{m,t}(X) = F_{mt}(X)dt + \sum_{k\geq 1} G_{m,k,t}(X)(dY_0(t))^k, m = 0, 1$.

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