Essential Calculus, a Revolutionary Approach to Teaching Calculus

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Classical physics is based on the mathematical analysis of differential equation models of physical systems. Yet, because differential equations are an advanced topic, they are typically not taught until the second or third year at the university. This paper describes a new approach to teaching calculus that covers differential equations in the tenth week.

1. Introduction

Calculus and modern physics were developed simultaneously by Newton with his study of gravity and planetary motion.

Since Newton, the paradigm for the analysis of physical systems has been:

- State the laws of physics governing the system. Laws of physics governing how things change are written as differential equations.
- Derive a differential equation model of the system from the laws of physics.
- Solve the differential equation model, with the goal of predicting the performance of the system.

The model of a physical system consists of a set of variables, called state variables, that determine the state of the system, and a differential, i.e., rate, equation for each state variable.

The laws governing planetary motion are taught in high school, they are Newton's law of gravity $F= - G \cdot m \cdot m_E/r^2$, and the second law of motion $F = m \cdot A$. Starting with the second law of motion, and substituting the force of gravity for F, yields $G \cdot m \cdot m_E/r^2 = m \cdot A$, where A is the acceleration of the falling object, G is the gravitational constant, m is the mass of the object, m_E is the mass of the earth, and r is the distance from the center of the earth to the object. Dividing by m yields:

$A = - G \cdot m_E/r^2$.

The 1-d model for a falling object, e.g., Newton's apple, consists of state variables r for position and v for velocity, and the following rate equations:
$r'(t) = v(t)$
$v'(t) = G \cdot m_E/r(t)^2$
where r' is the variable representing the rate of change of r, and v' is the variable representing the rate of change of v.

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The solution is very difficult and is an infinite series, its derivation is beyond the scope of even university physics. A 1-D analytic solution is (https://en.wikipedia.org/wiki/Free_fall):

$$r(t) = \sum_{n=1}^{\infty} \left[\lim_{q \to 0} \left(\frac{x^n}{n!} \frac{d^{n-1}}{dq^{n-1}} \left[r^n \left(\frac{7}{2} \left(\arcsin(\sqrt{q} - \sqrt{q - q^2}) \right)^{-\frac{2}{3}} \right] \right) \right]$$

Evaluating this expression yields

$$r(t) = y_0 \left(x - \frac{1}{5} x^2 - \frac{3}{175} x^3 - \frac{23}{7875} x^4 - \frac{1894}{3931875} x^5 - \frac{3293}{21896875} x^6 - \frac{2418092}{62077640625} x^7 - \ldots \right)$$

$$with\ x = \left[\frac{3}{2} \left(\frac{\pi}{2} - t \sqrt{\frac{2\mu}{r_0^3}} \right) \right]^{\frac{2}{3}}$$

The orbit problem is a two-body problem. The three-body problem, e.g., calculating the trajectory of a rocket from the Earth to the moon, is analytically unsolvable. This is the problem with analytic calculus, the models of almost all physical systems are unsolvable (https://pubs.aip.org/aapt/ajp/article/91/4/256/2878657/A-revolution-in-physics-education-was-forecast-in).

2. Essential calculus versus traditional calculus

The problems with the way calculus is currently taught are #1 – it is unmotivated because interesting problems are unsolvable, #2 – it is abstract and too rigorous, and #3 – it takes two years. From 1985 – 2000 there was a major NSF-sponsored effort to reform calculus education, that came to naught. The most popular calculus reform textbook is ‘Calculus’ by Hughes-Hallett, et. al. (https://archive.org/details/HuguesHallettCalculusSingleMultivariable6thEdText_201805)

Quick overviews of the calculus reform effort are given here (https://www.math.arizona.edu/~dhh/NOVA/calculus-conceptual-understanding.pdf) and here (https://peer.asee.org/calculus-reform-differential-equations-and-engineering.pdf)

The primary purpose of calculus is to solve differential equations, but calculus is taught at the university for two or three semesters before differential equations are introduced. The reason is that differential equations are typically very difficult to solve, and what’s worse, most are unsolvable.

This paper demonstrates a new approach to calculus education by taking a direct route to solving differential equations and eliminating unnecessary material. The following table lists the topics covered along with page numbers in this paper and the Hughes-Hallett book.

Topic	starting page # this paper	starting page # in Hughes/Hallett
Differentiation	2	65
Product rule, etc.	3	121
Integration	5	239
FTOC	5	256

Differential equations	6	521
Taylor's theorem	7	486
Trig functions	10	133
Exponential functions	13	540
Euler's formula	13	591
Complex numbers	14	990
Linear 2nd order systems	15	580

The following is *everything* that the student needs to learn to be able to reach the goal of being able to solve linear second-order differential equations. There are only two theorems, the Fundamental Theorem of Calculus and Taylor's theorem, and the proofs are easy and intuitively clear. While the many equations in the paper are intimidating at first glance, there is nothing complex or 'advanced' about the derivations, they represent simple algebraic reasoning.

3. Differentiating polynomials

3.1 Differentiation

Given a function *p*(t), *p*'(t) is a function for the rate of change of *p* or the velocity of *p*. In analytic calculus *p*'(t) is called the derivative of *p*(t), and p(t) is an anti-derivative of *p*'(t). The formal definition for the derivative of *p*(t) is:

$$p'(t) = \lim_{\Delta t \to 0} \frac{p(t+\Delta t) - p(t)}{\Delta t}$$

Given *p*(t) = t,

$$p'(t) = \lim_{\Delta t \to 0} \frac{(t+\Delta t) - t}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta t}{\Delta t} = \lim_{\Delta t \to 0} 1 = 1$$

Given *p*(t) = t^2,

$$p'(t) = \lim_{\Delta t \to 0} \frac{(t+\Delta t)^2 - t^2}{\Delta t} = \lim_{\Delta t \to 0} \frac{t^2 + 2*t*\Delta t + \Delta t^2 - t^2}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{t^2 + 2*t*\Delta t + \Delta t^2 - t^2}{\Delta t} = \lim_{\Delta t \to 0} (2*t + \Delta t) = 2*t$$

Given *p*(t) = t^3,

$$p'(t) = \lim_{\Delta t \to 0} \frac{(t+\Delta t)^3 - t^3}{\Delta t} = \lim_{\Delta t \to 0} \frac{t^3 + 3*t^2*\Delta t + 3*t*\Delta t^2 + \Delta t^3 - t^3}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{3*t^2*\Delta t + 3*t*\Delta t^2 + \Delta t^3}{\Delta t} = \lim_{\Delta t \to 0} (3*t^2 + 3*t*\Delta t + \Delta t^2) = 3*t^2$$

And so on, so that for *p*(t) = t^n

$$p'(t) = n*t^{n-1}$$

The general case follows by using the binomial theorem to expand $(p(t) + \Delta t)^n$ and evaluating the resulting expression by setting Δt to 0.

____________optional technical details ____

$$\textit{Binomial}\ \text{theorem}: \frac{(t+\Delta t)^n - t^n}{\Delta t} = \frac{\sum_{k=0}^{n} \left(\binom{n}{k} \cdot t^{n-k} \cdot \Delta t^k \right) - t^n}{\Delta t} \quad \text{where } \binom{n}{k} = \frac{n!}{(n-k)! \cdot k!}$$

$$e.g. \frac{(t+\Delta t)^4 - t^4}{\Delta t} = \frac{\frac{4!}{4! \cdot 0!} \cdot t^4 \cdot \Delta t^0 + \frac{4!}{3! \cdot 1!} \cdot t^3 \cdot \Delta t^1 + \frac{4!}{2! \cdot 2!} \cdot t^2 \cdot \Delta t^2 + \frac{4!}{1! \cdot 3!} \cdot t^1 \cdot \Delta t^3 + \frac{4!}{0! \cdot 4!} \cdot x^0 \cdot \Delta t^4 - t^4}{\Delta t}$$

$$= \frac{4!}{3! \cdot 1!} \cdot t^3 + \frac{4!}{2! \cdot 2!} \cdot t^2 \cdot \Delta t^1 + \frac{4!}{1! \cdot 3!} \cdot t^1 \cdot \Delta t^2 + \frac{4!}{0! \cdot 4!} \cdot x^0 \cdot \Delta t^3$$

$$= 4 \cdot t^3 \quad \text{when } \Delta t = 0$$

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3.2 Product and chain rules

The product rule for the derivative of $p(t)=q(t)\cdot r(t)$:

$$p'(t) = \lim_{\Delta t \to 0} \frac{p(t+\Delta t) - p(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{q(t+\Delta t) \cdot r(t+\Delta t) - q(t) \cdot r(t)}{\Delta t}$$

$$\text{add 0 to the numerator,} \quad -q(t+\Delta t) \cdot r(t) + q(t+\Delta t) \cdot r(t)$$

$$= \lim_{\Delta t \to 0} \frac{q(t+\Delta t) \cdot r(t+\Delta t) - q(t+\Delta t) \cdot r(t) + q(t+\Delta t) \cdot r(t) - q(t) \cdot r(t)}{\Delta t}$$

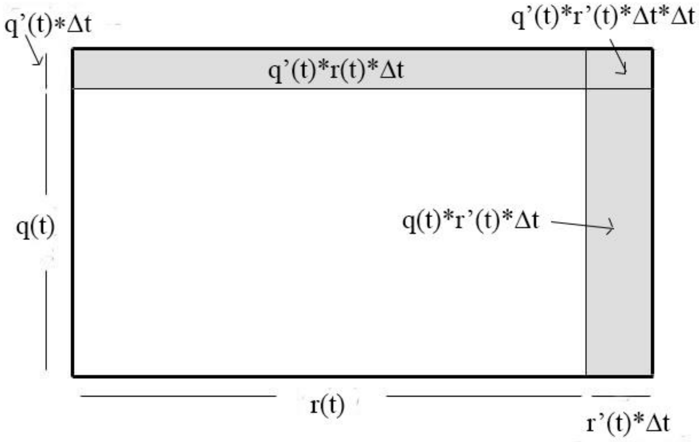
$$= \lim_{\Delta t \to 0} \frac{q(t+\Delta t) \cdot r(t+\Delta t) - q(t+\Delta t) \cdot r(t)}{\Delta t} + \lim_{\Delta t \to 0} \frac{q(t+\Delta t) \cdot r(t) - q(t) \cdot r(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{q(t+\Delta t) \cdot (r(t+\Delta t) - r(t))}{\Delta t} + \lim_{\Delta t \to 0} \frac{(q(t+\Delta t) - q(t)) \cdot r(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} q(t+\Delta t) \frac{(r(t+\Delta t) - r(t))}{\Delta t} + \lim_{\Delta t \to 0} \frac{(q(t+\Delta t) - q(t))}{\Delta t} \cdot r(t)$$

$$= q(t) \cdot r'(t) + q'(t) \cdot r(t)$$

Graphically



$$\frac{q(t+\Delta t)\cdot r(t+\Delta t)-q(t)\cdot r(t)}{\Delta t}$$
$$=\frac{r(t)\cdot q'(t)\cdot\Delta t+q(t)\cdot r'(t)\cdot\Delta t+r'(t)\cdot q'(t)\cdot\Delta t^2}{\Delta t}$$
$$=r(t)\cdot q'(t)+q(t)\cdot r'(t)+r'(t)\cdot q'(t)\cdot\Delta t$$

and

$$\lim_{\Delta t\to 0}(r(t)\cdot q'(t)+q(t)\cdot r'(t)+r'(t)\cdot q'(t)\cdot\Delta t)$$
$$= q(t)\cdot r'(t)+q'(t)\cdot r(t)$$

The chain rule for the derivative of p(t)=q(r(t)):

$$p'(t)=\lim_{\Delta t\to 0}\frac{p(t+\Delta t)-p(t)}{\Delta t}=\lim_{\Delta t\to 0}\frac{q(r(t+\Delta t))-q(r(t))}{\Delta t}$$
$$=\lim_{\Delta t\to 0}\frac{q(r(t+\Delta t))-q(r(t))}{\Delta t}\cdot\frac{r(t+\Delta t)-r(t)}{r(t+\Delta t)-r(t)}$$
$$=\lim_{\Delta t\to 0}\frac{q(r(t+\Delta t))-q(r(t))}{r(t+\Delta t)-r(t)}\cdot\frac{r(t+\Delta t)-r(t))}{\Delta t}$$
$$=q'(r(t))\cdot r'(t)$$

Intuitively $q(r(t+\Delta t)) = q(r(t) + r'(t)\cdot\Delta t) = q(r(t)) + q'(r(t)) \cdot r'(t)\cdot\Delta t$

So $(q(r(t+\Delta t)) - q(r(t)) / \Delta t = q'(r(t)) \cdot r'(t)$

If C is a constant, and $p(t) = C*q(t)$, then $p'(t) = C*q'(t)$.

If $p(t) = q(t) + r(t)$, then $p'(t) = q'(t) + r'(t)$.

Now, we can differentiate polynomials.

Example: $p(t) = 10\cdot t^4 + 5\cdot t^2 + 1$

The first order derivative of p is $p'(t) = 4\cdot 10\cdot t^3 + 2\cdot 5\cdot t + 0 = 40\cdot t^3 + 10\cdot t$

The second order derivative of p is $p''(t) = 3\cdot 40\cdot t^2 + 10 = 120\cdot t^2 + 10$

The third order derivative of p is $p'''(t) = 2\cdot 120\cdot t = 240\cdot t$

2.3 Quotient rule and rational functions, i.e ratios of polynomials

The quotient rule for the derivative of p(t)=q(t)/r(t)?

$$p'(t) = \lim_{\Delta t \to 0} \frac{p(t+\Delta t) - p(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{q(t+\Delta t)/r(t+\Delta t) - q(t)/r(t)}{\Delta t}$$
$$= \lim_{\Delta t \to 0} \frac{q(t+\Delta t)}{r(t+\Delta t)\cdot \Delta t} - \frac{q(t)}{r(t)\cdot \Delta t} = \lim_{\Delta t \to 0} \frac{q(t+\Delta t)\cdot r(t) - q(t)\cdot r(t+\Delta t)}{r(t+\Delta t)\cdot r(t)\cdot \Delta t}$$
$$= \lim_{\Delta t \to 0} \frac{1}{r(t+\Delta t)\cdot r(t)} \cdot \frac{q(t+\Delta t)\cdot r(t) - q(t)\cdot r(t+\Delta t)}{\Delta t}$$
$$= \frac{1}{r(t)\cdot r(t)} \cdot (q'(t)\cdot r(t) - q(t)\cdot r'(t)) = \frac{r(t)\cdot q'(t) - q(t)\cdot r'(t)}{r(t)\cdot r(t)}$$

The mnemonic device for the derivative of a quotient is: down-d-up-minus-up-d-down- over-down-down.

Given $p(t) = t^n$

$$p'(t) = n * t^{n-1}$$

For $q(t) = t^{-n}$

$p(t) \cdot q(t) = 1$,

differentiating both sides,

$p'(t) \cdot q(t) + p(t) \cdot q'(t) = 0$

hence,

$$q'(t) = \frac{-p'(t)\cdot q(t)}{p(t)} = -\frac{n * t^{n-1} \cdot t^{-n}}{t^n} = -\frac{n * t^{-1}}{t^n} = -n * t^{-n-1}$$

Thus $(t^n)' = n \cdot t^{n-1}$ for positive and negative values of n.

Examples:

$p(t) = 10 \cdot t^4$

$p'(t) = 4 \cdot 10 \cdot t^3 = 40 \cdot t^3$

$p(t) = 10 \cdot t^{-3}$

$p'(t) = -3 \cdot 10 \cdot t^{-4} = 30 \cdot t^{-4}$

4. Integration and The Fundamental Theorem of Calculus

The integral of a function $f(t)$ over the interval t_1 to t_2 is denoted by

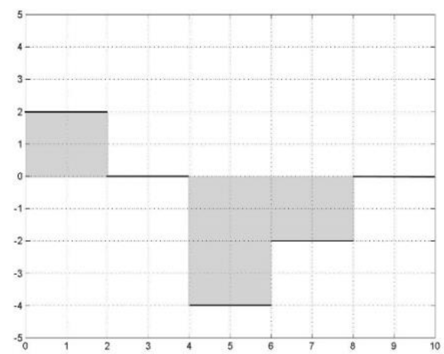
$$\int_{t_1}^{t_2} f(t)$$

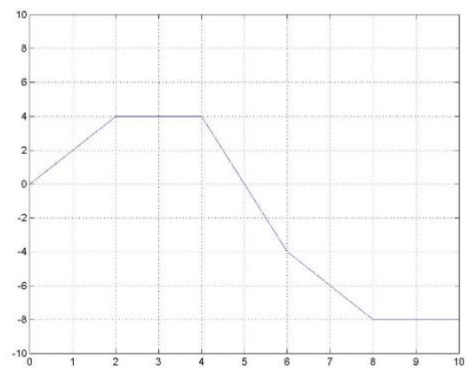
The value of the integral is the signed area under *f*(t) over the interval t_1 to t_2.

The Fundamental Theorem of Calculus states that if g is an anti-derivative of *f*, i.e., *g'*(t) = *f*(t), then

$$\int_{t1}^{t2} f(t) = g(t_2) - g(t_1)$$

Given a step function *f*(t) we can construct a corresponding piece-wise linear anti-derivative *g*(t). Note that the slope of *g*(t) equals *f*(t).





The signed area for each subinterval of the graph equals the value of *f* in the subinterval times the length of the subinterval, and this equals the change in *g* over the subinterval, from the formula for constant velocity motion, i.e., distance equals velocity times time.

t_i - t_{i+1}	signed area	$g(t_i)$	$g(t_i)$	$g(t_{i+1}) - g(t_i)$
0 - 2	4	0	4	4
2-4	0	4	4	0
4-6	-8	4	-4	-8
6-8	-4	-4	-8	-4
8-10	0	-8	-8	0

Thus the signed area under the graph above is given by the sum of the signed areas for each subinterval, which is given by

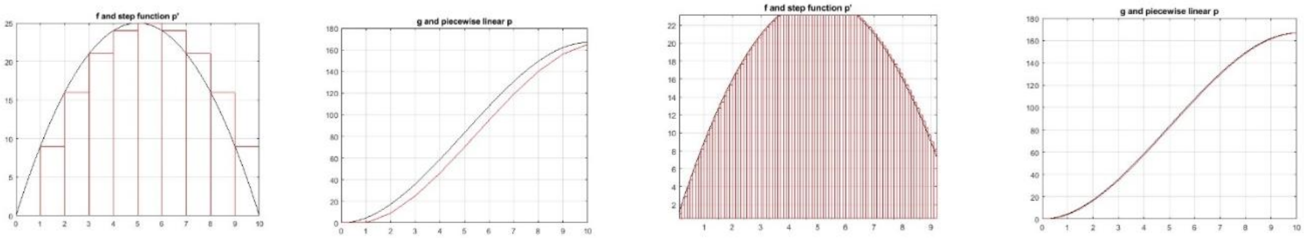
$g(2)$-$g(0)$ + $g(4)$- $g(2)$ + $g(6)$- $g(4)$ + $g(8)$- $g(6)$ + g(10)- $g(8)$

= $g(10) – g(0)$.

Hence the FTOC, $\int_{t1}^{t2} f(t) = g(t_2) - g(t_1)$, is true when *f*(t) is a step function.

Let f(t) be any function, e.g., $f(t) = -t^2 + 10 \cdot t$. Then we can approximate *f*(t) with a step function *p'*(t) by dividing the time interval 0 - 10 into n subintervals of length $\Delta t = 10/n$, with $0 = t_1$, t_2 =

Δt, $t_3 = 2\cdot\Delta t$ …. $t_n\cdot\Delta t = 1$ and construct its piecewise linear anti-derivative *p*(t), with $p(t_i) = f(t_i)$ for i = 1 to n. The graphs show f and p', and g and p, for n = 10, and n = 100.



By the FTOC the signed are under p'(t) = p(10) – p(0).

As n increases the limit of the signed area under *p*' equals the signed area under *f*.
As n increases the velocity of p (i.e., *p*') approaches the velocity of *g* (i.e., *f*), therefore the 'distance traveled' by *p* equals the distance traveled by *g*, and *p*(10) – *p*(0) = *g*(10) – *g*(0).

So area under *f*(t) ~ area under *p*'(t) = *p*(10) – *p*(0) ~ *g*(10) – *g*(0), and so

$$\int_{t_1}^{t_2} f(t) = g(t_2) - g(t_1)$$

and the FTOC is true for all functions.

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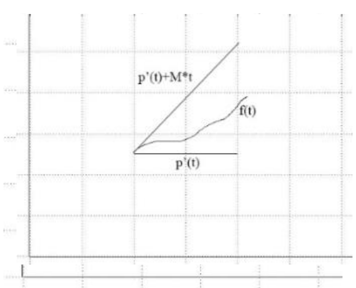
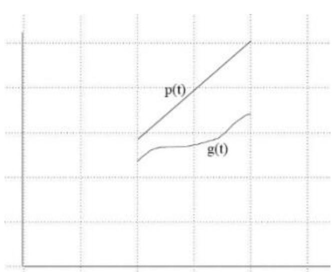


Given that f '(t) < M, then over 1 subinterval |f(t) – p'(t)| < M·Δt, and hence the difference in the areas under f(t) and p'(t) for that subinterval is < M·Δt·Δt

With n subintervals Δt = 1/n, so the sum of the difference in the area under f(t) and p'(t) < n· M·Δt·Δt = M·Δt which goes to 0 as n increases.



At the start of a subinterval, p(t) doesn't necessarily equal g(t), but p'(t) = g'(t). Given that g''(t) < M, then over 1 subinterval |g'(t) – p'(t)| < M·Δt and so the difference between g(tk+1) – g(tk) and p(tk+1) – p(tk) is < M·Δt·Δt, so the sum of the difference in distance traveled by g(t) and p(t) < n· M·Δt·Δt = M·Δt which goes to 0 as n increases.

Note that a function *p*(t) has only one derivative, *p*'(t), but *p*'(t) has many antiderivatives, since *g*(t) = *p*(t) + C, where C is a constant, is also an antiderivative of *p*'(t).

Example:

$$\int_0^{10}(5\cdot t^2-8)=?$$

An antiderivative of $5\cdot t^2 - 8$ has the form g(t) = (5/3) · t^3 - 8·t + C, so:

$$\int_0^{10}(5\cdot t^2-8)=g(10)-g(0)=((5/3)\cdot 10^3-8\cdot 10+C)-((5/3)\cdot 0^3-8\cdot 0+C)=(5/3)\cdot 10^3-80$$

$$\int_1^{10}2\cdot t^{-4}=?$$

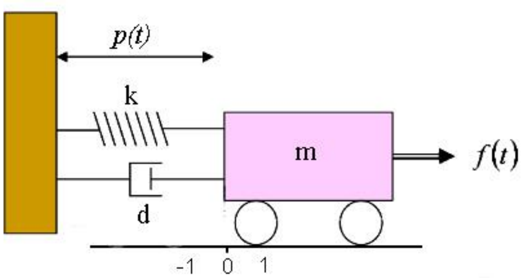
An antiderivative of $2\cdot t^{-4}$ has the form g(t) = 2/-3 · t^{-3} + C, so:

$$\int_1^{10}2\cdot t^{-4}=g(10)-g(1)=-2/3\cdot 10^{-3}+C-(-2/3\cdot 1^{-3}+C)=-2/3000+2/3\cdot$$

5 Differential equations.

The primary purpose of calculus is to solve differential equations. A differential equation is an equation for a derivative function, in physics the differential equations are first order and second order.

For example, a spring-block-damper system is shown in the figure: the block is at position p(0) = 0 and the spring is relaxed. We will ignore gravity and assume that the rolling friction is 0.



There are only two forces acting on the block, the spring force and the damper force.

The spring force is a function of the position of the block, and is given by

$f_s(t) = -k \cdot p(t)$

where k is the spring constant.

The damper force is proportional to the velocity of the block and is given by

$f_d(t) = -d \cdot p'(t)$ where d is a coefficient of damping.

From Newton's Second Law of Motion, the acceleration of the block at time t is given by

$p''(t) = (f_s(t) + f_d(t)) / m = (-k \cdot p(t) - d \cdot p'(t)) / m$

So, with m = 1 for simplicity,

$p''(t) = - k \cdot p(t) - d \cdot p'(t))$

We will solve this differential equation later in this paper. Note that it is a functional equation, and the function p(t) and its derivatives are unknown. A differential equation is solved by finding a definition of p(t) that satisfied the differential equation, that is, makes it true.

As you might expect, the differential equation above does not have a polynomial function solution. Note that the unknown function appears on both sides of the equation, this suggests that solution p(t) and its derivatives have a similar shape.

Looking forward a bit, we'll see that the functions that resemble their derivatives are the trigonometric sine and cosine functions, and the exponential function. We need the theorem in the following section to use these functions.

6. Polynomial approximation and Taylor's theorem

Going beyond polynomials, the next step is to differentiate and integrate trigonometric and exponential functions. In order to even calculate, like your calculator does, values for these functions, we need Taylor's theorem, because there are no closed-form expressions that calculate these functions exactly, they are calculated using Taylor series polynomial approximations.

We start with an arbitrary function g(t) that has 1^{st}, 2^{nd}, … $(n+1)^{st}$ order derivatives. The idea is to approximate g in a neighborhood of 0 (for convenience, it could be any point) with an n^{th}-order polynomial whose value at 0 matches g and whose derivatives at 0 up to order n match those of g. We will define the approximating polynomial, p, such that $p(0) = g(0)$, and derivatives of p up to order n match those of g at 0.

With p(t) defined as $p(t) = P_0 + P_1 \cdot t + P_2 \cdot t^2 + P_3 \cdot t^3 + \ldots + P_n \cdot t^n$

$p(0) = P_0$,

$p'(0) = P_1$

$p''(0) = P_2 \cdot 2$

$p'''(0) = P_3 \cdot 3 \cdot 2$

$p''''(0) = P_3 \cdot 4 \cdot 3 \cdot 2$

…

$p^{(n)}(0) = P_k \cdot n!$

note: $p^{(n)}(0)$ is the notation for the n^{th} derivative of p.

So, if we define $P_k = g^{(k)}(0)/k!$ for k = 0,1,2,3, …. n, then

$p^{(k)}(0) = g^{(k)}(0)$ for k = 0,1,2,3, …. n and we have our polynomial approximating g.

The error function for our approximation is $err(t) = g(t) - p(t)$. Note that:

$err^{(k)}(0) = g^{(k)}(0) - p^{(k)}(0) = 0$

for k = 0,1,2,3, …. n, and

$err^{(n+1)}(0) = g^{(n+1)}(0) - p^{(n+1)}(0) = g^{(n+1)}(0) – 0 = g^{(n+1)}(0)$

Now we will determine a limit for the error of the approximation. We need a limit for the n+1st derivative of *g*, call it M. We'll start with a first-order approximation *p*(t), a linear approximation.

$$For\ n = 1,\ \mathrm{p(t) = g(0) + g'(0) \cdot t}$$

$$\mathrm{err''(t) < M\ so\ err'(t)} = \int_0^t \mathrm{err''(x)} < \int_0^t \mathrm{M} \ = M \cdot t$$

$$\text{and } \mathrm{err(t)} = \int_0^t \mathrm{err'(x)} < \int_0^t \mathrm{M} \cdot x \ = M \cdot t^2 / 2$$

$$For\ n = 2,\ \mathrm{p(t) = g(0) + g'(0) \cdot t + g''(0) \cdot t/2}$$

$$\mathrm{err'''(t) < M\ so\ err''(t)} = \int_0^t \mathrm{err'''(x)} < \int_0^t \mathrm{M} \ = M \cdot t$$

$$\mathrm{err'(t)} = \int_0^t \mathrm{err''(x)} < \int_0^t \mathrm{M} \cdot x \ = M \cdot t^2 / 2$$

$$\text{and } \mathrm{err(t)} = \int_0^t \mathrm{err'(x)} < \int_0^t \mathrm{M} \cdot x^2/2 \ = M \cdot t^3 / 3 \cdot 2 = M \cdot t^3 / 3!$$

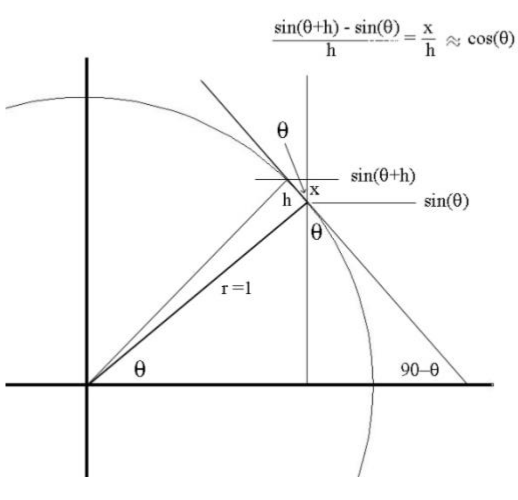
And so on, thus given that $g^{(n+1)}(t) < M$ on the interval of interest the error of the n^{th} order Taylor Taylor approximation for *g*, i.e.:

$p(t) = g(0) + g'(0) \cdot t^2/2 + g''(0) \cdot t^2/2 + g'''(0) \cdot t^3/(3 \cdot 2) + …. + g^{(n)}(0) \cdot t^n/n!$,

is bounded by $err(t) < M \cdot t^{n+1}/(n+1)!$

7. Trig functions

Consider the sine function, in the figure below as the radius of length 1 sweeps out h radians, the arc length of the corresponding section of the circle is h, and is approximately a straight line, and as h gets smaller the line gets straighter.



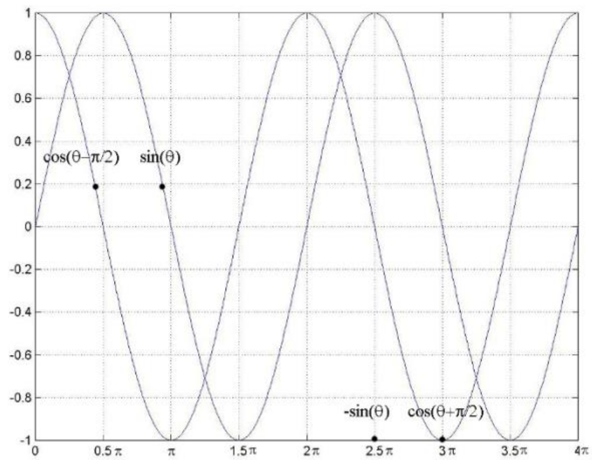
From the diagram, when h is small

sin(θ + h) – sin(θ) = x as shown in the diagram, and x/h ~ *cos*(θ), thus

$$\sin'(\theta) = \lim_{h \to 0} \frac{\sin(\theta + h) - \sin(h)}{h}$$

$$= \lim_{h \to 0} \frac{x}{h} = \cos(\theta)$$

The graph below shows plots of $\sin(\theta)$ and $\cos(\theta)$, note that $\sin(0) = 0$ and $\cos(0) = 1$.



Also note that the cos graph lags the sin graph by $\pi/2$ radians. That is $\cos(\theta) = \sin(\theta + \pi/2)$, so

$\cos'(\theta) = \sin'(\theta + \pi/2) = \cos(\theta + \pi/2)$

Also, the inverted sin graph lags the cos graph by p/2 radians, that is

$-\sin(\theta) = \cos(\theta + \pi/2)$.

Thus

$\cos'(\theta) = -\sin(\theta)$.

$\sin(0) = 0$, $\sin'(0) = \cos(0) = 1$, and $\cos(0) = 1$, $\cos'(0) = -\sin(0) = 0$.

Since we know the k^{th} derivative of *sin* at 0, we can write its Taylor series approximation at 0.

$\sin(t) = \sin(0) + \sin'(0)\cdot t + \sin''(0)\cdot t^2/2 + \sin'''(0)\cdot t^3/3! + \sin''''(0)\cdot t^4/4! + \sin'''''(0)\cdot t^5/5! + \ldots$

$= \sin(0) + \cos(0)\cdot t - \sin(0)\cdot t^2/2 - \cos(0)\cdot t^3/3! + \sin(0)\cdot t^4/4! + \cos(0)\cdot t^5/5! + \ldots$

$= 0 + 1\cdot t - 0\cdot t^2/2 - 1\cdot t^3/3! + 0\cdot t^4/4! + 1\cdot t^5/5! + \ldots$

$= t - t^3/3! + t^5/5! - t^7/7! + t^9/9! -$

Similarly, the Taylor series expansion of $\cos(t)$ is

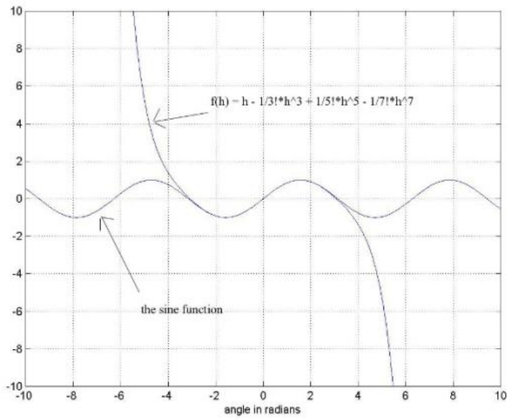
$\cos(t) = \cos(0) + \cos'(0)\cdot t + \cos''(0)\cdot t^2/2 + \cos'''(0)\cdot t^3/3!\ \cos''''(0)\cdot t^4/4! + \cos'''''(0)\cdot t^5/5! + \ldots.$

$= 1 - \sin(0)\cdot t - \cos(0)\cdot t^2/2 + \sin(0)\cdot t^3/3! + \cos(0)\cdot t^4/4! - \sin(0)\cdot t^5/5! +$

$= 1 - 0\cdot t + 1\cdot t^2/2 - 0\cdot t^3/3! + 1\cdot t^4/4! + 0\cdot t^5/5! +$

$= 1 - t^2/2! + t^4/4! - t^6/6! + t^8/8! - \ldots\ldots\ldots$

7^{th}-order Taylor series approximation to the sine function



With trig functions, we can solve differential equations of the form

$p''(t) = -C \cdot p(t)$

Let $\omega = \sqrt{C}$ and $p(t) = \sin(\omega \cdot t)$, then

$p''(t) = \sin''(\omega \cdot t) = [\omega \cdot \cos(\omega \cdot t)]' = -\omega^2 \cdot \sin(\omega \cdot t) = -C \cdot p(t)$

$p(t) = \cos(\omega \cdot t)$ also solves the differential equation

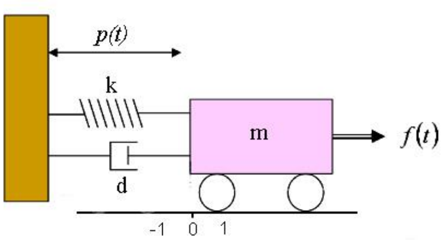
$p''(t) = \cos''(\omega \cdot t) = [-\omega \cdot \sin(\omega \cdot t)]' = -\omega^2 \cdot \cos(\omega \cdot t) = -C \cdot p(t)$

The general solution is

$A_1 \cdot \sin(\omega \cdot t) + A_2 \cdot \cos(\omega \cdot t)$

Example:

A spring-block-damper system is shown in the figure: the block is at position p(0) = 0 and the spring is relaxed. We will ignore gravity and assume that the rolling friction is 0.



As shown earlier, with m = 1 for simplicity, the differential for the system is:

$p''(t) = - k \cdot p(t) - d \cdot p'(t))$

With d = 0, this becomes:

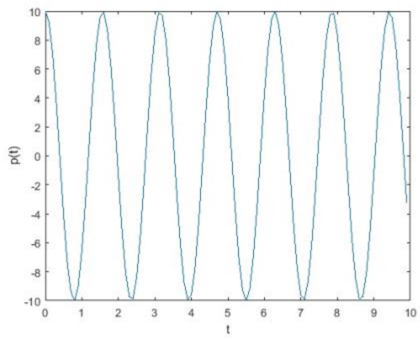
$p''(t) = - k \cdot p(t)$

The general solution to this differential equation is $A_1 \cdot \sin(\omega \cdot t) + A_2 \cdot \cos(\omega \cdot t)$.

If k = 9, p(0) = 10, and p'(0) = 0, we guess $\omega = \sqrt{k} = 3$ and solve for A_1 and A_2:
$p(0) = 10 = A_1 \cdot \sin(\omega \cdot t) + A_2 \cdot \cos(\omega \cdot t) = A_1 \cdot \sin(0) + A_2 \cdot \cos(0) = A_2$

$p'(0) = 0 = A_1 \cdot \omega \cdot \cos(0) - A_2 \cdot \omega \cdot \sin(0) = A_1 \cdot 3$ so $A_1 = 0$

Plotting $p(t) = 10 \cdot \cos(\omega \cdot t)$



8. Exponentials

A function $f(t) \equiv b^t$, where b is any positive number is called an exponential function with base b. The derivative of f at t, that is $f'(t)$, is

$$f'(t) = \lim_{\Delta t \to 0} \frac{b^{t+\Delta t} - b^t}{\Delta t} = \lim_{\Delta t \to 0} \frac{b^t (b^{\Delta t} - b^0)}{\Delta t} = b^t \cdot \lim_{\Delta t \to 0} \frac{(b^{\Delta t} - 1)}{\Delta t} = b^t \cdot f'(0)$$

We'd like for $f'(t) = f(t)$ which will be the case if $f'(0) = 1$, that is:

$$\lim_{\Delta t \to 0} \frac{(b^{\Delta t} - 1)}{\Delta t} = 1$$

$$\lim_{\Delta t \to 0} (b^{\Delta t} - 1 - \Delta t) = 0$$

$$\lim_{\Delta t \to 0} (b^{\Delta t} = 1 + \Delta t)$$

$$\lim_{\Delta t \to 0} (b = (1 + \Delta t)^{1/\Delta t})$$

which is equivalent to

$$b = \lim_{n \to \infty} (1 + 1/n)^n = 2.718...$$

which can be calculated to the desired degree of accuracy
e = 2.718… is known as Euler's number, and the exponential function *exp* is defined as

$exp(t) = e^t$,

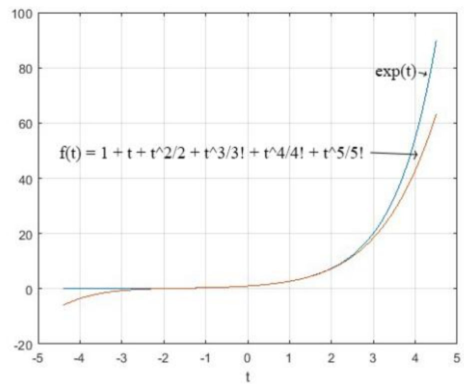
and $exp'(t) = exp(t)$

Using Taylor's theorem, we can approximate *exp*(t) by

$exp(t) = exp(0) + exp'(0) \cdot t + exp''(0) \cdot t^2/2! + exp'''(0) \cdot t^3/3! + \ldots + exp^{(n)}(0) \cdot t^n/n! \ldots$

$= 1 + t + t^2/2! + t^3/3! + \ldots + t^n/n! \ldots$

5th-order Taylor series approximation to the exp function



The exponential function can be used to solve a differential equation of the form

$p'(\mathrm{t}) = \mathrm{k}\cdot p(\mathrm{t})$

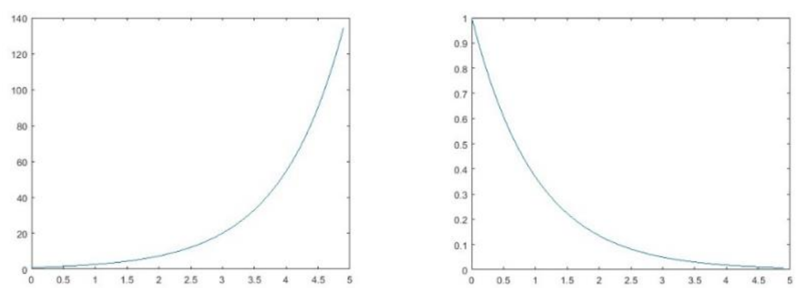
The solution is given by

$p(\mathrm{t}) = \mathrm{P}_0\cdot exp(\mathrm{k}\cdot\mathrm{t})$,

since

$p'(\mathrm{t}) = \mathrm{P}_0\cdot exp'(\mathrm{k}\cdot\mathrm{t}) = \mathrm{P}_0\cdot\mathrm{k}\cdot exp(\mathrm{k}\cdot\mathrm{t}) = \mathrm{k}\cdot p(\mathrm{t})$

The differential equation $p'(\mathrm{t}) = \mathrm{k}\cdot p(\mathrm{t})$ models exponential growth for $\mathrm{k} > 1$ and exponential decay if $\mathrm{k} < 1$.



Solutions to $p'(\mathrm{t}) = p(\mathrm{t})$ and $p'(\mathrm{t}) = - p(\mathrm{t})$

9. Logs

The natural log function ln(t) is the inverse of the exponential function, that is, ln(t) = x such that *exp*(x) = t.

Thus $ln(exp(\mathrm{t})) = \mathrm{t}$

Taking the derivative on both sides of ln(exp(t)) = t , and using the chain rule:

$[ln(exp(\mathrm{t}))]' = \mathrm{t}'$

$ln'(exp(\mathrm{t}))\cdot exp'(\mathrm{t}) = 1$

$ln'(exp(\mathrm{t}))\cdot exp(\mathrm{t}) = 1$

$ln'(exp(t)) = 1/exp(t)$

So, $ln'(y) = 1/y$

9. Roots and Radicals

An expression of the form x^y is evaluated using the *exp* and *ln* functions,

$x^y = exp(ln(x) \cdot y)$,

e.g, $8^{4.5} = exp(ln(8) \cdot 4.5)$

$8^{-4.5} = exp(ln(8) \cdot - 4.5)$

10. Complex numbers

A complex number has the form $a + b \cdot i$, where a and b are real numbers, and i is an imaginary number having the property that $i^2 = -1$, that is, i is the square root of -1.

A positive number times a positive number is a positive number, and a negative number times a negative number is a positive number, so it is impossible to make sense of an 'imaginary number' that is the square root of -1, so, we have an alternate definition - a complex number is an ordered pair of numbers, a and b, or equivalently (a, b) in a 2D plane. A complex number (a, b) can also be written in polar form, $(r, \theta)_p$ where r = the square root of $a^2 + b^2$, and θ = arctan(b/a).

Addition of complex numbers

$a_1 + b_1 \cdot i + a_2 + b_2 \cdot i = (a_1 + a_2) + (b_1 + b_2) \cdot i$

$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$

Multiplication of complex numbers

$(a_1 + b_1 \cdot i) + (a_2 + b_2 \cdot i) = a_1 \cdot a_2 - b_1 \cdot b_2 + (a_1 \cdot b_2 + b_1 \cdot a_2) \cdot i$

$(r_1, \theta_1)_p \cdot (r_2, \theta_2)_p = (r_1 \cdot r_2, \theta_1 + \theta_2)_p$

Euler's formula

For an imaginary number $z = b \cdot i = (0,b)$ define

$exp(z) = 1 + z + z^2/2 + z^3/3! + z^4/4! + z^5/5! \ldots$

$= 1 + b \cdot i + (b \cdot i)^2/2 + (b \cdot i)^3/3! + (b \cdot i)^4/4! - (b \cdot i)^5/5! \ldots.$

$= 1 + i \cdot b - b^2/2 - i \cdot b^3/3! + b^4/4! + i \cdot b^5/5! - b^6/6! - i \cdot b^7/7! + b^8/8! + i \cdot b^9/9! \ldots$

$= (1 - b^2/2 + b^4/4! - b^6/6! + b^8/8! - \ldots..) + i \cdot (b - b^3/3! + b^5/5! - b^7/7! + b^9/9! ...)$

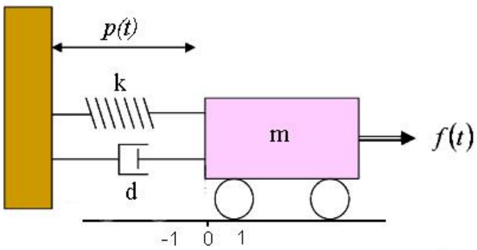
$= \cos(b) + i \cdot \sin(b)$

Thus $e^{a+b \cdot i} = e^{a} \cdot e^{b \cdot i} = exp(a) \cdot exp(b \cdot i) = exp(a) \cdot (cos(b) + i \cdot sin(b))$

This is known as Euler's formula.

11. Linear second-order systems

A spring-block-damper system is shown in the figure: the block is at position p(0) = 0 and the spring is relaxed. We will ignore gravity and assume that the rolling friction is 0.



As we've seen, the differential equations for this system, with m = 1 for simplicity, is

$p''(t) + d \cdot p'(t) + k \cdot p(t) = 0$

We expect the motion of the block to oscillate because of the spring, and we expect the oscillations to die out because of the damper, so will guess that the solution has the form

$p(t) = A \cdot e^{rt}$

where r is a complex number. Substituting this solution into the differential equation gives

$A \cdot (r^2 + d \cdot r + k) \cdot e^{rt} = 0,$

which is true iff $r^2 + b \cdot r + c = 0$. This is the characteristic equation for the system, and the solutions are determined using the quadratic formula

$$r = \frac{-d \pm \sqrt{d^2 - 4 \cdot 1 \cdot k}}{2}$$

If $d^2 - 4 \cdot k > 0$ then there are 2 distinct real roots and both are solutions to the differential equation. If $d^2 - 4 \cdot k = 0$ then there is 1 real root, $r = -d/2$, to the equation, and e^{rt} is the corresponding solution, $t \cdot e^{rt}$ is also a solution as we now check.

With $p = t \cdot e^{rt}$, $p'' + d \cdot p' + k \cdot p$ becomes:

$(r \cdot e^{rt} + r \cdot e^{rt} + r^2 \cdot t \cdot e^{rt}) + b \cdot (e^{rt} + r \cdot t \cdot e^{rt}) + c \cdot t \cdot e^{rt}$

$= e^{rt} \cdot (r + r + r^2 \cdot t + d + d \cdot r \cdot t + k \cdot t)$

$= e^{rt} \cdot (2r + d + t \cdot (r^2 + d \cdot r + k))$

$= 0$ since $2r + d = 0$ and $r^2 + d \cdot r + k = 0$

If $d^2 - 4 \cdot k <=$ there are two complex solutions.

Ex. d = 2 and k = 25, $r^2 + 2 \cdot r + 26 = 0$ and $r_1 = -1 + 5 \cdot i$ and $r_2 = -1 - 5 \cdot i$,

and the general solution is:

$p(\mathrm{t}) = \mathrm{A}_1 \cdot e^{(-1+5 \cdot i) \cdot t} + \mathrm{A}_2 \cdot e^{(-1-5 \cdot i) \cdot t}$

Given $p(0) = 10$ and $p'(0) = 0$

$p(0) = 10 = \mathrm{A}_1 \cdot e^{(-1+5 \cdot i) \cdot 0} + \mathrm{A}_2 \cdot e^{(-1-5 \cdot i) \cdot 0} = \mathrm{A}_1 + \mathrm{A}_2$

When we specify real initial values, the solution is real.

Given an initial position

$p(0) = 10 = \mathrm{A}_1 \cdot e^{(-1+5 \cdot i) \cdot 0} + \mathrm{A}_2 \cdot e^{(-1-5 \cdot i) \cdot 0} = \mathrm{A}_1 + \mathrm{A}_2$

and initial velocity

$p'(0) = 0 = \mathrm{A}_1 \cdot (-1+5 \cdot i) \cdot e^{(-1+\cdot 5 \cdot i) \cdot 0} + \mathrm{A}_2 \cdot (-1-5 \cdot i) \cdot e^{(-1-5 \cdot i) \cdot 0} = \mathrm{A}_1 \cdot (-1+5 \cdot i) \cdot + \mathrm{A}_2 \cdot (-1-5 \cdot i)$

Solving for A_1 and A_2 gives $\mathrm{A}_1 = 5 - i$ and $\mathrm{A}_2 = 5 + i$, so the solution is

$p(\mathrm{t}) = (5 - i) \cdot e^{(-1+5 \cdot i) \cdot t} + (5 + i) \cdot e^{(-1-5 \cdot i) \cdot t}$, which equals, using Euler's formula

$= (5 - i) \cdot e^{-t} \cdot (\cos(5 \cdot t) + i \cdot \sin(5 \cdot t)) + (5 + i) \cdot e^{-t} \cdot (\cos(-5 \cdot t) + i \cdot \sin(-5 \cdot t))$

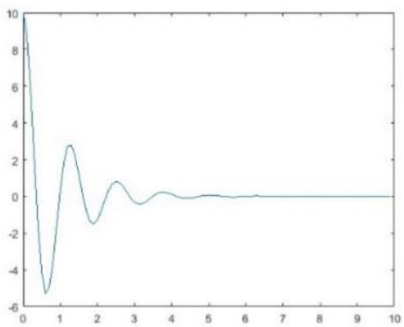
$= (5 - i) \cdot e^{-t} \cdot (\cos(5 \cdot t) + i \cdot \sin(5 \cdot t)) + (5 + i) \cdot e^{-t} \cdot (\cos(5 \cdot t) - i \cdot \sin(5 \cdot t))$

$= 5 \cdot e^{-t} \cdot \cos(5 \cdot t) + 5 \cdot e^{-t} \cdot i \cdot \sin(5 \cdot t) - i \cdot e^{-t} \cdot \cos(5 \cdot t) - i \cdot e^{-t} \cdot i \cdot \sin(5 \cdot t)$

$+ 5 \cdot e^{-t} \cdot \cos(5 \cdot t) - 5 \cdot e^{-t} \cdot i \cdot \sin(5 \cdot t) + i \cdot e^{-t} \cdot \cos(5 \cdot t) - i \cdot e^{-t} \cdot i \cdot \sin(5 \cdot t)$

$= 10 \cdot e^{-t} \cdot \cos(5 \cdot t) - 2 \cdot i \cdot e^{-t} \cdot i \cdot \sin(5 \cdot t)$

$= 10 \cdot e^{-t} \cdot \cos(5 \cdot t) + 2 \cdot e^{-t} \cdot \sin(5 \cdot t)$



A 10-week course in analytic calculus

1 - Differentiating polynomials

2 - Differential rules

3 - Rational functions

4 - Integration and the Fundamental Theorem of Calculus

5 - Differential Equations

6 - Taylor's Theorem

7 - Trig Functions

8 - Exponentials, Logs, Roots, and Radicals

9 - Complex Numbers, Euler's formula

10 - Second Order Systems