Research Article

Dirac Theory of Electron According to Spacetime Algebra

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A simple and straightforward formulation of the Dirac theory of electron exclusively according to Spacetime algebra (and calculus) is presented. The `Dirac algebra', is expressed in identical compact combination of the commutation and anticommutation relations both for Dirac matrices and the spacetime basis vectors. The formulation restates the theory with a set of 'local observables' and provides both comprehensive and coherent description revealing new insights. The noncollinearity of momentum and velocity, arising from a link between spin and momentum is also discussed. According to this reformulation, spin appears as a dynamical property of electron motion and plays fundamental role in exhibiting the quantum behavior.

1. Introduction

In search of a more general, relativistic quantum equation of motion, Dirac[1] was looking for a square root of the d'Alembert operator $\square^2$ and using four (4×4) 'γ' matrices finally obtained his celebrated linearized equation of motion for electron in terms of the 'Dirac operator' $\square = \gamma_\mu \partial_\mu$ $(\partial_\mu = \partial / \partial x_\mu)$; $\mu = 0, 1, 2, 3$ and the 'Dirac ket' $|\psi\rangle$ representing the spinor-valued statefunction. Introducing the 'minimal coupling' with the spacetime vector potential $\mathbf{a}$ $(= \{a_\mu\})$ of an external electromagnetic field, the equation reads:

$$(i\hbar \square - e\,\mathbf{a})|\psi\rangle = mc|\psi\rangle\,, \tag{1}$$

where m is the mass and $e = -|e|$ is the charge of the electron.

The algebra of Dirac matrices – the socalled 'Dirac algebra', can be expressed in a compact combination of the commutation and anticommutation relations using Einstein summation convention as:

$$\gamma_\mu \gamma_\nu = g_{\mu\nu}\, \gamma_4 + g_{\mu\mu}\, g_{\nu\nu}\, \epsilon_{\mu\nu\eta\lambda}\, \gamma_5\, \gamma_\eta \gamma_\lambda\,, \tag{2}$$

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where $g_{\mu\nu}$ $(= 0, \pm 1)$ are the elements of the 4-D pseudo-Euclidean Minkowski spacetime metric, γ_4 represents the (4×4) identity matrix, $\epsilon_{\mu\nu\lambda\eta}$ $(= 0, \pm 1)$ are the elements of Levi-Civita tensor and γ_5 $(= \gamma_0\gamma_1\gamma_2\gamma_3)$ is the product of the four gamma matrices.

Every particle described by the Dirac equation has to have a corresponding antiparticle, which differs only in the sign of its charge. The additional states of the electron predicted by the theory got experimental confirmation with the discovery of the positron in 1932 and this spectacular success was hailed by Heisenberg as 'perhaps the biggest change of all the big changes in physics of our century . . . because it changed our whole picture of matter'[2]. Interestingly, Dirac himself once jovially remarked that his equation is more intelligent than its author[3]! And indeed, carrying out an extensive reformulation using *spacetime algebra* (the geometric algebra and calculus of Minkowski spacetime) in a series of papers, Hestenes has delved deep to reveal further the 'mysteries and insights of Dirac theory'.

Dirac's matrix formulation uses complex numbers, whereas Hestenes[4][5] has clearly demonstrated that there is no justification in using complex numbers. Also, unnecessary complexity arises from the use of admixture of matrix and tensor algebras in the mathematical formalism. In the following, we note that the Dirac algebra is isomorphic to the spacetime algebra (STA) which offers a more compact, coherent and comprehensive reformulation – the 'Real Dirac theory'[5][6]. STA also provides the appropriate representation of spinors, the spin angular momentum states of spin-half elementary particles. Quaternions and their isomorphic cousins, spinors are different from both vectors and tensors and change sign under rotation of 2π and a rotation of 4π is equivalent to the absence of rotation.

These conceptual inputs facilitate a comprehensive introduction to the reformulated Dirac's theory, exclusively in the straightforward STA framework and to provide a broad based exposure to the advanced undergraduate students. After a breif introduction to the spacetime algebra and calculus, the appropriate reformulation of the Dirac theory is discussed.

1.1. Spacetime algebra and calculus as the appropriate Dirac algebra

Retaining the same symbols (with added hats) of the matrices for the four orthogonal spacetime basis vectors ($\hat{\gamma}_\mu$), the full STA spanned by 16 multivector bases are generated by multiplications of the four as:

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$$1,\ \{\hat{\gamma}_\mu\},\ \{\hat{\gamma}_\mu\,\hat{\gamma}_\nu\},\ \{\hat{\imath}\,\hat{\gamma}_\mu\}\text{ and }\hat{\imath},\quad \mu\neq\nu.$$

Geometrically, the unit pseudoscalar $\hat{\imath}\ (=\hat{\gamma}_0\hat{\gamma}_1\hat{\gamma}_2\hat{\gamma}_3)$ represents a unit oriented 4-volume for spacetime with the following basic algebraic properties: (i) it squares to -1, and (ii) anticommutes with vectors $\{\hat{\gamma}_\mu\}$ and trivectors $\{\hat{\imath}\,\hat{\gamma}_\mu\}$ and commutes with bivectors $\{\hat{\gamma}_\mu\,\hat{\gamma}_\nu\}$. The six unit bivectors together with the unit scalar and the unit pseudoscalar comprise the eight bases of the even subalgebra of STA.

The properties of spacetime metric can be similarly represented in a compact form in terms of Clifford's associative geometric product, combining both inner (dot) and exterior (wedge) products of Grassmann algebra, of any two of the four basis vectors as:

$$\hat{\gamma}_\mu\,\hat{\gamma}_\nu=\hat{\gamma}_\mu.\,\hat{\gamma}_\nu+\hat{\gamma}_\mu\wedge\hat{\gamma}_\nu=g_{\mu\nu}+g_{\mu\mu}\,g_{\nu\nu}\epsilon_{\mu\nu\eta\lambda}\,\hat{\imath}\,\hat{\gamma}_\eta\,\hat{\gamma}_\lambda, \tag{3}$$

here, the pseudo-Eucleadian Minkowski spacetime metric elements $g_{\mu\nu}$, represented by the scalar product of the basis vectors $(\hat{\gamma}_\mu.\,\hat{\gamma}_\nu$, produce the sequence $(+---)$ of algebraic signs on the main diagonal of this spacetime metric. The opposite signature of $(-+++)$ is also used and it is always possible to translate between the two sets of basis vectors. $\hat{\gamma}_\mu\wedge\hat{\gamma}_\nu$'s represent the unit bivectors and $\hat{\imath}\,=\hat{\gamma}_0\,\hat{\gamma}_1\,\hat{\gamma}_2\,\hat{\gamma}_3$ is the unit pseudoscalar of the spacetime algebra. Equation (3) exactly reproduces the algebra of Dirac matrices (eq.2), where the (4×4) unit matrix γ_4 and γ_5 – the product of the four Dirac matrices are replaced by the unit scalar 1 and the unit spacetime pseudoscalar $\hat{\imath}$ respectively.

Secondly the Dirac operator $(\Box=\hat{\gamma}_\mu\partial_\mu)$, with the replacement of the γ matrices by spacetime basis vectors, actually represents the spacetime gradient operator, since $\Box x_\mu=\hat{\gamma}_\mu$. The 'spontaneous emergence' of spin in Dirac's theory of electron is generally attributed to the derivation of its linearized relativistic wave equation and spin has been said to be 'a quantum phenomenon'. However, the redefined Dirac operator is equally applicable in the formulation of the field equations of both electromagnetism and fluid mechanics[7][8]. Hestenes has, therefore, rightly concluded that the Dirac algebra arises from spacetime geometry rather than anything special about quantum theory and the origin of spin must lie somewhere else. Moreover, it may also be pointed out that, with the introduction of the geometric product, the spacetime algebra removes much of the mathematical divide among classical, quantum, and relativistic physics.

Finally, the even subalgebra of STA expresses appropriately the theory of spinors. The spinor-valued statefunction – an even spacetime multivector $\Psi=a_0+a_j\,\hat{\gamma}_k\,\hat{\gamma}_l+b_j\,\hat{\gamma}_j\hat{\gamma}_0+b_0\,\hat{\imath}\,,$ is conveniently recomposed in the most succinct polar form:

$$\Psi = (\rho \exp(\hat{\imath}\beta))^{1/2}\mathcal{R}, \tag{4}$$

where $\rho\ (= \rho(x))$ and $\beta\ (= \beta(x))$ are scalar fields. The bivector part is represented by the rotor $\mathcal{R}$, the generator of pure rotation – a normalized spinor[9][10].

Using the generalisation of the concept of exponential function of multivectors introduced by Hestenes[9], the rotors can be represented by elliptic functions of the bivector field B as $\mathcal{R} = \exp(\mathbf{B})$ to generate rotation through a bilinear transformation. The 'canonical form' (eq.4), as introduced by Hestenes, is an invariant composition of the Dirac wave function into a 2-parameter statistical factor $(\rho \exp(\hat{\imath}\beta))^{1/2}$ and a 6-parameter kinematical factor – the rotor $\mathcal{R}$. The bilinear covariants for the 16 multivector bases of STA (constructed using the wave function Ψ having only 8 parameters) are not all mutually independent. It may be noted that, the invariant polar form of Ψ provides simple expressions for the interdependence of the bilinear covariants and dispense with the need for the 'Fierz identities' and complicated index manipulations of the conventional formulation[5]. For the even multivector Ψ, we have:

$$\Psi\tilde{\Psi} = \rho \exp(\hat{\imath}\beta),\ \tilde{\Psi}\,\{= \rho^{1/2}\exp(\hat{\imath}\beta)^{1/2}\tilde{\mathcal{R}}\} \text{ and } \tilde{\mathcal{R}} \text{ being the reverses of } \Psi \text{ and } \mathcal{R} \text{ respectively,}$$

with $\mathcal{R}\tilde{\mathcal{R}} = \tilde{\mathcal{R}}\mathcal{R} = 1 \Rightarrow \tilde{\mathcal{R}} \equiv \mathcal{R}^{-1}$. At each spacetime point, $\mathcal{R}$ determines a Lorentz rotation of a given fixed frame of vectors $\{\hat{\gamma}_\mu\}$ into new set of field vectors $\{\hat{\alpha}_\mu\}$ and bivectors $\{\hat{\alpha}_\mu\,\hat{\alpha}_\nu\}$ given by $\hat{\alpha}_\mu = \mathcal{R}\hat{\gamma}_\mu\tilde{\mathcal{R}}$ and $\hat{\alpha}_\mu\,\hat{\alpha}_\nu = \mathcal{R}\hat{\gamma}_\mu\,\hat{\gamma}_\nu\tilde{\mathcal{R}}$ etc. – the bilinear operation with the rotor being grade preserving. It turns out that the rotor field $\mathcal{R}$, which determines the comoving frame $\hat{\alpha}_\mu = \mathcal{R}\hat{\gamma}_\mu\tilde{\mathcal{R}}$ on the streamline (along $\hat{\alpha}_0$), is the descriptor of the kinematics of electron motion in this formulation. The physical interpretation given to $\{\hat{\alpha}_\mu\}$ and $\{\hat{\alpha}_\mu\,\hat{\alpha}_\nu\}$ in defining the 'local observables' is the key to the interpretation of the Real Dirac theory.

This introduction provides the required straightforward STA framework for the present study. A concise and coherent reformulation of the Dirac theory in the proper language of STA, dispensing with the matrix and tensor algebras, is discussed in the following sections.

2. Reformulation of Dirac theory according to Spacetime algebra

In STA both operators and states are real-space multivectors, and the distinction between states and operators is removed naturally, appearing as an important conceptual simplification. From the expression $\Psi\tilde{\Psi} = \rho \exp(\hat{\imath}\beta)$, ρ is identified as the position probability density (proper) that the electron is at the given spacetime point and β represents the phase factor (akin to the Yvon-

Takabayasi angle[11] defined to generate transformation of internal degrees of freedom). Also $\Psi\hat{\gamma}_\mu\tilde{\Psi} = \rho\hat{\alpha}_\mu$ and $\Psi\hat{\gamma}_\mu\hat{\gamma}_\nu\tilde{\Psi} = \rho\exp(\hat{\imath}\,\beta)\hat{\alpha}_\mu\hat{\alpha}_\nu$ representing a set of linearly independent vector and bivector field densities respectively. Since every timelike vector field, say $\hat{\alpha}_0$, written in the bilinear form with a spinor field obey the conservation law $\Box.\,\rho\,\hat{\alpha}_0 = 0$, the timelike vector $\hat{\alpha}_0$ defines the direction of velocity (proper) $\mathbf{v} = c\,\hat{\alpha}_0$ of the electron. The vector field density $\rho\hat{\alpha}_0$ is aptly interpreted as the Dirac probability current density in accordance with the standard Born interpretation. The angular momentum is actually a bivector quantity and according to STA, the spin angular momentum is given by the bivector:

$$\mathbf{S} = \frac{\hbar}{2}\mathcal{R}\hat{\gamma}_2\hat{\gamma}_1\tilde{\mathcal{R}} \equiv \frac{\hbar}{2}\hat{\alpha}_2\hat{\alpha}_1\,. \tag{5}$$

Justification for taking this bivector form of spin comes from the angular momentum conservation law. Equation (5) expresses that at each spacetime point the reference representation of the spin bivector $\hbar(\hat{\gamma}_2\hat{\gamma}_1)/2$ is rotated by the kinematical factor $\mathcal{R}$ into the local spin $\mathbf{S}$. The spin bivector is dual to the bivector $\hbar(\hat{\alpha}_3\,\hat{\alpha}_0)/2 = \mathbf{s}\,\hat{\alpha}_0$ i.e. $\mathbf{S} = \hat{\imath}\;\mathbf{s}\,\hat{\alpha}_0$, where the vector $\mathbf{s}\,(\equiv \hbar\hat{\alpha}_3/2)$ replaces the spin pseudovector of the standard formulation.

Of the six parameters implicit in the rotor $\mathcal{R}$, five are needed to determine the directions of velocity ($\hat{\alpha}_0$) and spin ($\hat{\alpha}_3$) vectors, since the two are constrained by by the three conditions that they are orthogonal and have constant magnitudes. By duality, these vectors also determine the ‘spin plane’ containing $\hat{\alpha}_1$ and $\hat{\alpha}_2$. The remaining parameter determines the directions of the two vectors in this plane, an angle of rotation and the spin bivector $\mathbf{S}$ is the generator of the rotation. Hestenes thus arrived at a geometrical interpretation of the phase of the wave function inherent in the Dirac theory, but concealed in the conventional matrix formulation.

From an analysis of the Schrödinger, Pauli and Dirac equations, Hestenes[5][12] has come to the conclusion that the imaginary ‘i’ in the usual quantum theories actually ‘smuggles spin into the expressions’. The justification for including ‘i’ in conventional theories is to make operators hermitian to get real observables, which actually hides the fact that the distinctive factor $i\hbar$ is exactly twice the spin of electron. The spacetime formulation of Dirac theory reveals this hidden significance and the unit imaginary of equation (1) is replaced with the unitary bivector $\hat{\gamma}_2\,\hat{\gamma}_1$, the ‘generator of spin’ (rotation) in a spacelike plane. The Dirac equation according to STA – also called the Dirac-Hestenes equation (DHE) is written in terms of the spacetime gradient operator and the real, spinor-valued wavefunction Ψ as:

$$\hbar \Box \Psi \hat{\gamma}_2 \hat{\gamma}_1 = m c \Psi \hat{\gamma}_0 + e \mathbf{a} \Psi . \tag{6}$$

The equivalence to the standard matrix form (eq.1) is obtained[12] by interpreting $\hat{\gamma}_\mu$ as matrices γ_μ and appropriately transforming the even multivector Ψ to the Dirac ket function $|\psi>$. The correspondence is ensured by the right multiplication of the generator bivector $\hat{\gamma}_2 \hat{\gamma}_1$ of spin to the STA spinor Ψ, which also plays the role of an operator generating observables in the theory.

In the standard formulation of quantum mechanics, observables and the observed values are described by hermitian operators and their eigenvalues respectively. However, Hestenes[5] have argued that certain features of the Dirac theory conflict with the view that it is a universal principle of quantum mechanics. It is contended that the success of this principle is derived from a set of operators only, namely, the kinetic energy-momentum operators and STA clarifies and leads to a new view of the significance of these operators in quantum mechanics.

Dispensing with the operator representation, in the STA approach observables of the Dirac theory are redefined with 'local observables' in terms of bilinear covariants of the multivector bases of STA and associated directly with the Dirac wave function. One interesting aspect of this approach is that the conservation laws can be established directly from the Dirac equation, without recourse to the standard lagrangian formalism. A complete set of observables is determined by the conservation laws providing all mathematical features of the wave function. The formulation greatly simplifies the derivation of conservation laws and may be regarded as a field theoretic description of Dirac's theory of electron.

The Dirac theory describes the electron as a point particle, however, the description is statistical and the position probability current is to be identified with the Dirac current $\rho \hat{\alpha}_0$. The conservation of the Dirac current can be established as follows. Multiplying DHE (eq.6) by $\hat{\gamma}_1 \hat{\gamma}_2 \hat{\gamma}_\mu \tilde{\Psi}$ $(\equiv \hat{\imath} \hat{\gamma}_0 \hat{\gamma}_3 \hat{\gamma}_\mu \tilde{\Psi})$ on the right and recalling that $\Psi \hat{\gamma}_\mu \tilde{\Psi} = \rho \hat{\alpha}_\mu$ etc., we get:

$$\begin{aligned} \hbar (\Box \Psi) \hat{\gamma}_2 \hat{\gamma}_1 \hat{\gamma}_1 \hat{\gamma}_2 \hat{\gamma}_\mu \tilde{\Psi} = & \quad m c \Psi \hat{\gamma}_0 \hat{\imath} \hat{\gamma}_0 \hat{\gamma}_3 \hat{\gamma}_\mu \tilde{\Psi} + e \mathbf{a} \Psi \hat{\gamma}_1 \hat{\gamma}_2 \hat{\gamma}_\mu \tilde{\Psi} \\ \Rightarrow \hbar (\Box \Psi) \hat{\gamma}_\mu \tilde{\Psi} = & \quad -m c \Psi \hat{\imath} \hat{\gamma}_3 \hat{\gamma}_\mu \tilde{\Psi} + e \mathbf{a} \Psi \hat{\gamma}_1 \hat{\gamma}_2 \hat{\gamma}_\mu \tilde{\Psi} \\ = & \quad -\hat{\imath} m c \rho \exp(\hat{\imath} \beta) \hat{\alpha}_3 \hat{\alpha}_\mu - e \rho \mathbf{a} \hat{\alpha}_2 \hat{\alpha}_1 \hat{\alpha}_\mu , \end{aligned} \tag{7}$$

Taking the scalar part of this expression:

$$\begin{aligned} \hbar < (\Box \Psi) \hat{\gamma}_\mu \tilde{\Psi} >_0 = & \quad -\rho < \hat{\imath} m c \exp(\hat{\imath} \beta) \hat{\alpha}_3 \hat{\alpha}_\mu + e \mathbf{a} \hat{\alpha}_2 \hat{\alpha}_1 \hat{\alpha}_\mu >_0 \\ \Rightarrow \Box . (\rho \hat{\alpha}_\mu) = & \quad \frac{2\rho}{\hbar} [m c \sin \beta \hat{\alpha}_3 . \hat{\alpha}_\mu - e (\mathbf{a} \hat{\alpha}_\mu) : (\hat{\alpha}_2 \hat{\alpha}_1)], \end{aligned}$$

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since $\hbar < (\Box\,\Psi)\hat{\gamma}_\mu\,\tilde{\Psi} >_0 = 2^{-1}\hbar\Box.\,(\rho\,\hat{\alpha}_\mu)$. The double dot ':' implies double contraction[10], giving the scalar product of the two bivectors. The divergences of various current densities are obtained by putting $\mu = 0,\,1,\,2$ and 3 respectively as:

$$\begin{aligned} \Box.\,(\rho\,\hat{\alpha}_0) = &\quad 0,\;\; \Box.\,(\rho\,\hat{\alpha}_1) = -\frac{2}{\hbar}\rho\,e\,\mathbf{a}.\,\hat{\alpha}_2,\;\; \Box.\,(\rho\,\hat{\alpha}_2) = \frac{2}{\hbar}\rho\,e\,\mathbf{a}.\,\hat{\alpha}_1 \\ \text{and } \Box.\,(\rho\,\hat{\alpha}_3) = &\quad -\frac{2}{\hbar}\rho\,m\,c\,\sin\beta \Rightarrow \Box.\,(\rho\,\mathbf{s}) = -\rho\,m\,c\,\sin\beta. \end{aligned} \tag{8}$$

Defining $\rho\,\hat{\alpha}_0$ as the local probability current (Dirac current), the vanishing divergence in accordance with the Dirac equation correctly reproduces the conservation of (local) probability. The conservation law implies the existence of a unique integral curve, passing through each spacetime point and tangent to $\mathbf{v}$, called electron streamline[1]. Along the tangents flows the probability current $\rho\,\hat{\alpha}_0$. In any spacetime region where $\rho \neq 0$, a solution of the Dirac equation determines a family of streamlines that fills the region with exactly one streamline through each point. Also, both the mass and the charge current densities ($m\,\rho\,\hat{\alpha}_0$, $e\,\rho\,\hat{\alpha}_0$ respectively) are proportional to the Dirac current and both the mass and charge conservations are immediate consequences of the probability conservation.

The future-pointing timelike unit vector $\hat{\gamma}_0$, tangent to the world line of an observer at rest, represents the observer in STA. The novel spacetime split method of Hestenes[5] amounts comparing the motion of a given system relative to the observer by compactifyng the spacetime to a relative 3-D space with respect to $\hat{\gamma}_0$. Splitting up the six spacetime bivectors into three relative vectors $\{\hat{\sigma}_j\} = \{\hat{\gamma}_0\hat{\gamma}_j\}$ and three bivectors $\{\hat{\sigma}_j\hat{\sigma}_k\} \equiv \{\hat{\gamma}_j\hat{\gamma}_k\}$; $j \neq k$ (relative to the $\hat{\gamma}_0$ observer), the eight multivector bases of the even subalgebra of STA, including the unit scalar and the unit pseudoscalar $\hat{\iota} = \hat{\gamma}_0\,\hat{\gamma}_1\,\hat{\gamma}_2\,\hat{\gamma}_3 \equiv \hat{\sigma}_1\,\hat{\sigma}_2\,\hat{\sigma}_3$, generates the eight multivector bases of the relative 3-D space. The geometric product of any two of the three basis vectors of this relative 3-D space exactly reproduces the algebra of Pauli matrices (or simply the Pauli algebra): $\hat{\sigma}_j\,\hat{\sigma}_k = \delta_{jk} + \hat{\iota}\,\varepsilon_{jkl}\,\hat{\sigma}_l$; $j, k, l = 1, 2, 3$ and the Pauli spinor ($\Psi' = \rho'^{1/2}\mathcal{R}'$) is appropriately represented by a four component even multivector.

The Pauli equation as the nonrelativistic approximation of the Dirac equation correctly brings out the nonrelativistic limit of the Dirac (charge) current, distinguishing it from the nonrelativistic limit of the convection current part that arises from the motion of the center of mass of the charged particle only, excluding the part that arises from gradients of the spin density (the socalled 'Gordon current'[13]). Failure to identify the 'Darwin term' as the s-state spin-orbit energy in conventional treatments of the hydrogen atom is thus traced to a failure to distinguish between charge and momentum flow in the theory. As a consistent approximation to the Dirac theory, the Schrödinger

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equation can also be derived successively from the Pauli equation with vanishingly small external magnetic field[12]. At variance with the standard view, the limiting 'Schrödinger equation' describes a particle in a spin eigenstate and not a spinless particle. This important difference is a consequence of deriving the Schrödinger equation from the Pauli equation which leads to an expression for the charge current containing nonvanishing term from the spin magnetization current. The usual Schrödinger current appears as the nonrelativistic limit of the 'Gordon' current rather than the Dirac current. However, the hydrogen s-state electron motion is more correctly described by the Dirac current, than the Gordon-Schrödinger current.

2.1. New insights from the reformulated Dirac-Hestenes equation

The importance of equation (6) lies in the fact that it provides a more comprehensive description of the theory and provides new insights to gain an edge over the conventional formulation. In this formulation, the momentum includes a contribution from the spin, rendering it noncollinear with the local velocity that corresponds to the 'Weyssenhoff behaviour'[14] of Dirac's electron (see eq.11). Beauregard and Imbert have initiated the theoretical and experimental study on noncollinearity of velocity and momentum in electron theory and optics[15].

In this formulation, the variables $\mathbf{v}$, $\mathbf{p}$, $\mathbf{S}$ and so on are field quantities and represent local observables velocity, momentum and spin respectively. These basic observables are completely characterized by the kinematical factor, the rotor field $\mathcal{R}$ in the wave function, whereas ρ and β provides the statistical interpretation in this formulation.

Though the physical interpretation of β is rather a debatable issue, it may be noted that from a superposition of solutions with $\beta = 0$, composite solution with $\beta \neq 0$ to the Dirac-Hestenes equation can be obtained. Therefore, β characterizes a more general class of statistical superpositions than particle-antiparticle mixtures. The many particle aspects of Dirac equation, as it admits negative energy solutions, is usually dealt with by enforcing a second quantization. However, proper recognition of the parameter β (reffered to as the 'β-problem' by Hestenes[5]) provides an alternative, new perspective rendering second quantization unnecessary. Moreover, as it will be seen presently, the magnetization $\mathbf{M}$ can be defined in this formulation in terms of the spin by a duality rotation represented by the factor $\exp(i\,\beta)$, also provides some justification for referring β as the duality parameter.

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A relation among the local observables $\mathbf{v}$, $\mathbf{p}$ and $\mathbf{S}$ can be obtained from equation (6), first by multiplying it on the right by $\tilde{\Psi}$ to get

$$\hbar(\Box\Psi)\hat{\gamma}_2\,\hat{\gamma}_1\tilde{\Psi} = \rho\, m\mathbf{v} + \rho\, e\,\mathbf{a}\,\exp(\iota\,\beta)\,. \tag{9}$$

Expressing Ψ, $\Box\Psi$ in polar form, the l.h.s. of the above equation may be written as:

$$\begin{aligned}
\hbar(\Box\Psi)\hat{\gamma}_2\,\hat{\gamma}_1\tilde{\Psi} &= 2^{-1}\hbar\{\Box\rho\,\exp(\iota\,\beta)\}(\mathcal{R}\hat{\gamma}_2\,\hat{\gamma}_1\tilde{\mathcal{R}}) + \hbar\rho\,\exp(-\iota\,\beta)(\Box\mathcal{R})\hat{\gamma}_2\,\hat{\gamma}_1\tilde{\mathcal{R}} \\
&= \Box\{\rho\,\exp(\iota\,\beta)\mathbf{S}\} + \rho\,\exp(-\iota\,\beta)\{(\Box\mathcal{R})\tilde{\mathcal{R}}\mathbf{S} - \mathbf{S}\,\mathcal{R}(\Box\tilde{\mathcal{R}})\},\ \mathbf{S} = 2^{-1}\hbar(\mathcal{R}\hat{\gamma}_2\,\hat{\gamma}_1\tilde{\mathcal{R}}) \\
\text{Since, } \Box\mathbf{S} &= \{(\Box\mathcal{R})\tilde{\mathcal{R}}\mathbf{S} + \mathbf{S}\mathcal{R}(\Box\tilde{\mathcal{R}})\} \\
\Rightarrow & \quad 2(\Box\mathcal{R})\tilde{\mathcal{R}}\,\mathbf{S} = \Box\mathbf{S} + \{(\Box\mathcal{R})\tilde{\mathcal{R}}\mathbf{S} - \mathbf{S}\,\mathcal{R}(\Box\tilde{\mathcal{R}})\}.
\end{aligned}$$

Finally, equation (9), becomes:

$$\rho\,\exp(-\iota\,\beta)\{(\Box\mathcal{R})\tilde{\mathcal{R}}\mathbf{S} - \mathbf{S}\,\mathcal{R}(\Box\tilde{\mathcal{R}}) - e\,\mathbf{a}\} = \rho\, m\,\mathbf{v} - \Box\{\rho\,\exp(\iota\,\beta)\mathbf{S}\}. \tag{10}$$

Both the entire l.h.s. and the factor $(\Box\mathcal{R})\tilde{\mathcal{R}}\mathbf{S} - \mathbf{S}\,\mathcal{R}(\Box\tilde{\mathcal{R}})$ in equation (10) are composed of vectors and trivectors only. The rotor field $\mathcal{R}$ is the descriptor of the kinematics of electron motion in this formulation and writing: $(\Box\mathcal{R})\tilde{\mathcal{R}}\mathbf{S} - \mathbf{S}\,\mathcal{R}(\Box\tilde{\mathcal{R}}) = \mathbf{p} + \iota\,\mathbf{q}$, where $\mathbf{p}$ and $\mathbf{q}$ are two vector fields, Hestenes[6] has finally identified $\mathbf{p}$ as the total local momentum field $\mathbf{p}_e + e\,\mathbf{a}$. Multiplying both sides of equation (10) with $\rho^{-1}\,\exp(\iota\,\beta)$ and equating only the vector parts, the local momentum of electron is finally obtained as a function velocity and spin as:

$$\mathbf{p}_e = m\mathbf{v}\cos\beta - \rho^{-1}\Box.\,(\rho\mathbf{S}) + (\hat{\iota}\;\mathbf{S}).\,\Box\beta, \tag{11}$$

in the simplest way, which expresses the general noncollinearity of electron momentum and velocity in Dirac theory. The noncollinearity, introduced by the electron spin, means that *charge and energy flows are not concurrent* – the Weyssenhoff behaviour or motion. While the charge current, like the probability current follows the timelike trajectories along $\mathbf{v}$, the energy-momentum density given by $\rho\mathbf{p}_e$ flows along the electron streamlines. The factor $\cos\beta$ in equation (11), reduces the contribution of the 'mass density' to the energy-momentum density. The last term:

$$(\hat{\iota}\;\mathbf{S}).\,\Box\beta = (\hat{\alpha}_0\wedge\mathbf{s}).\,\Box\beta$$

shows a dependence of momentum on the rate of change of β in the $\hat{\alpha}_0\wedge\mathbf{s}$ plane. Also a number of important algebraic relations with momentum, velocity and spin follows from equation (11). Here we consider first:

$$\mathbf{p}_e.\,\mathbf{v} = \quad m\mathbf{v}^2\cos\beta - \rho^{-1}(\mathbf{v}\wedge\Box) : (\rho\mathbf{S}) - (\hat{\iota}\;\mathbf{S}) : (\mathbf{v}\wedge\Box\beta),$$

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expressing the flow of local energy along a streamline. The first term contains reduced rest mass $m\cos\beta$ and the remaining terms involve the ‘normal gradient’ $\mathbf{v}\wedge\Box$, which shows that their contribution to the local energy is determined by the flow of $\mathbf{S}$ and β onto the streamline.

The measure of the noncollinearity of momentum and velocity in (11) can be expressed as:

$$\mathbf{p}_e\wedge\mathbf{v}=\rho^{-1}\mathbf{v}\wedge\Box.(\rho\mathbf{S})+\hat{\iota}\,\mathbf{S}\dot{\beta}=\dot{\mathbf{S}}-\rho^{-1}\Box.(\rho\,\mathbf{v}\,\mathbf{S})-(\mathbf{S}.\Box)\wedge\mathbf{v}+\hat{\iota}\,\mathbf{S}\,\dot{\beta},$$

which may be regarded as defining the relative momentum in the electron rest frame. This equation may also be compared with the equivalent of Weyssenhoff’s classical equation for angular momentum conservation $\dot{\mathbf{S}}=\mathbf{p}_e\wedge\mathbf{v}$[6][14].

Equating the trivector parts similarly, we get from equation (10) the expression for the dual vector field $\mathbf{q}$ as:

$$\mathbf{q}=m\mathbf{v}\sin\beta+\rho^{-1}\Box.(\rho\,\hat{\iota}\,\mathbf{S})-\mathbf{S}.\Box\beta. \tag{12}$$

Equation (12) for $\mathbf{q}$ together with the last equation of equation (8), gives the rate of change of spin along a velocity streamline. A number of other important auxiliary formulas are easily derived from equations (11) and (12) by utilizing algebraic properties of the velocity and spin[6].

Alternatively, the local field velocity is composed of two distinct parts: (i) the center of mass (CM) part $\mathbf{v}_{cm}$ and (ii) $\mathbf{v}_{zbw}$, the spin dependent (or the internal zitterbewegung motion resulting in the electron spin) part in the CM frame:

$$\mathbf{v}=\mathbf{v}_{cm}+\mathbf{v}_{zbw}=\mathbf{p}_e/(m\cos\beta)+(\rho^{-1}\Box.(\rho\mathbf{S})+(\hat{\iota}\,\mathbf{S}).\Box\beta)/(m\,\cos\beta).$$

Thus the Weyssenhoff behaviour of a spinning particle, corresponds to Zitterbewegung (zbw) of Dirac’s electron. Analysing the Dirac equation, the rapid oscillatory motion was first predicted by Breit[16] and subsequently by Schrödinger[17], who coined the term Zitterbewegung from the German word for ‘trembling motion’ (where the fermion executes a zig-zag, back-and-forth motion at the speed of light). Although zbw of a free relativistic particle is not observable directly, it has been simulated in model systems[18]. Zitterbewegung is also invoked to explain the Darwin term, a small correction of the energy level, a spin-orbit energy for the s-orbitals of the hydrogen atom[19].

On the basis of an old calculation by Balinfante, Ohanian[20] has argued that the (electron) spin may be regarded as an angular momentum generated due to a circulating flow of energy, or momentum density, in the electron wave field. Subsequently Esposito[21] has generalized the result for arbitrary spin system and reiterated the claim that the 'quantum behavior of microsystems is a direct consequence of the fundamental existence of spin'. The new zbw interpretation, as advanced by Hestenes[22] from DHE (eq.6), claims spin as determining the ground-state or the zero-point (kinetic) energy. Spin – the zero-point angular momentum is associated with the zero-point energy of the electron which determines the dispersion in electron momentum. The position-momentum Heisenberg uncertainty relations for an electron can, therefore, be interpreted as a property of the electron spin motion[23]. Also, the noncollinearity of local momentum and velocity urges that the position-momentum uncertainty relation is not equivalent, contrary to the usual supposition, to the uncertainty relation for position and velocity. Indeed, the inequivalence of velocity and momentum persists when the nonrelativistic limit is carried out correctly, even in the Schrödinger *regime*. This important fact has gone unnoticed and the necessary relation of spin with the Schrödinger equation as well as to the uncertainty principle has been overlooked.

Putting $\beta = 0$ for Pauli-Schrödinger case, Recami and Salesi[24] have defined the Bohm quantum potential[25] as the kinetic energy of the zbw (internal) motion:

$$Q = \frac{1}{2} m \mathbf{v}_{zbw}^2 = \frac{\hbar^2}{8m} (\frac{\Box \rho}{\rho})^2 \text{ with } |\mathbf{S}| = \hbar/2$$

– in a very similar form, but as a mere consequence of the kinetic interpretation of spin. However, all the time the authors of Ref. 23 have erroneously expressed the zbw part of the velocity vector as a bivector, in terms of wedge product between two vectors. The origin of the quantum potential is also hidden in the Pauli-Schrödinger theory. This STA formulation suggests the dependence of the quantum potential on $\hbar$ as coming entirely from the spin motion. The complete general expresion of Q [5][26][27] for the Dirac-Hestenes equation differs considerably from the usual Bohm quantum potential, where the mass parameter $m\cos(\beta)$ involves the angle β.

Moreover, we note that the vector part of equation (10), on multiplication with e/m, gives the Gordon current (density):

$$\frac{e\,\rho}{m}(\mathbf{p}_e \cos\beta + \mathbf{q} \sin\beta) = \mathbf{j}_G = e\,\rho\,\mathbf{v} - \frac{e}{m}\Box.\{\rho\,\exp(\iota\,\beta)\mathbf{S}\}, \tag{13}$$

and the celebrated Gordon decomposition of Dirac current (density) $e\,\rho\,\mathbf{v}$ in the language of STA:

$$\mathbf{j}_D = \mathbf{j}_G + \frac{e}{m}\Box.\{\rho\,\exp(\iota\,\beta)\mathbf{S}\} = \mathbf{j}_G + \Box.\mathbf{M}, \tag{14}$$

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which played an important role in the interpretation of Dirac equation. The Gordon current differs from the charge current by the magnetization current associated with the spin density[4]. The last term of equation (14) is consistently identified as the divergence of spin magnetic moment density i.e. the magnetization field $\mathbf{M} = \frac{e}{m}\rho\,\exp(\iota\,\beta)\mathbf{S}$. This relation between magnetization and spin is much simpler than any other to be found in standard formulations and also is another indication that equation (5) provides an appropriate representation for electron spin.

Equation (13) shows that in the nonrelativistic limit ($\beta = 0$) the Gordon current, representing the Schrödinger current in the conventional formulation, is given by in terms of the local momentum density only as $e\,\rho\,(\mathbf{p}_e/m)$ – and is purely a convection current. Also, in this limit: $\mathbf{p}_e = m\,\mathbf{v} - \rho^{-1}\Box.\,\{\rho\,\mathbf{S}\}$. From the mathematical identity $\Box.\,(\Box.\,\mathbf{M}) = (\Box\wedge\Box) : \mathbf{M} = 0$, it follows that the conservation law $\Box.\,\mathbf{j}_D = 0$ (eq.8) also implies that $\Box.\,\mathbf{j}_G = 0$, i.e. both Dirac and Gordon currents are conserved quantities. However, as mentioned earlier, the interpretation of the Dirac current as proportional to the charge current is consistent with the experiment.

2.2. *The force equation and the energy-momentum conservation law*

Taking the gradient of the Dirac-Hestenes equation and multiplying on the right by $c\,\hat{\gamma}_0\,\tilde{\Psi}$, one gets after some manipulation[12]:

$$\begin{aligned}\hbar\,(\Box^2\,\Psi)\hat{\gamma}_2\,\hat{\gamma}_1\,c\,\hat{\gamma}_0\,\tilde{\Psi} &= m\,c(\Box\,\Psi)c\,\tilde{\Psi} + e(\Box\,\mathbf{a}\Psi)c\,\hat{\gamma}_0\,\tilde{\Psi}\\ &= m\,c(\Box\,\Psi)c\,\tilde{\Psi} + e(\Box\,\mathbf{a})\Psi\,c\,\hat{\gamma}_0\,\tilde{\Psi} + e\,\mathbf{a}(\Box\,\Psi)c\,\hat{\gamma}_0\,\tilde{\Psi}\\ &= [\{m\,c(\Box\,\Psi) - e\,\mathbf{a}(\Box\,\Psi)\hat{\gamma}_0\} + 2\,e\,\mathbf{a}(\Box\,\Psi)\hat{\gamma}_0 + e(\Box\,\mathbf{a})\Psi\hat{\gamma}_0]c\,\tilde{\Psi}\end{aligned}$$

Substituting for $\Box\,\Psi$ using equation (6) in the first two terms of r.h.s., one gets:

$$\hbar\,(\Box^2\,\Psi)\hat{\gamma}_2\,\hat{\gamma}_1\,c\,\hat{\gamma}_0\,\tilde{\Psi} = (e^2\mathbf{a}^2 - m^2c^2)\,\hat{\iota}\,\rho\,\hat{\alpha}_3\,c + e\,\mathbf{a}\,\Box(\rho\,\mathbf{v}) + e(\Box\,\mathbf{a})\rho\,\mathbf{v},$$

and the vector part of which gives:

$$\begin{aligned}<\hbar\,(\Box^2\,\Psi)\hat{\gamma}_2\,\hat{\gamma}_1\,c\,\hat{\gamma}_0\,\tilde{\Psi}>_1 &= e\,\mathbf{a}\Box.\,(\rho\mathbf{v}) + e(\Box.\,\mathbf{a})\rho\,\mathbf{v} + e(\Box\wedge\mathbf{a}).\,\rho\,\mathbf{v}\\ \Rightarrow<\hbar\,(\Box^2\,\Psi)\hat{\gamma}_2\,\hat{\gamma}_1\,c\,\hat{\gamma}_0\,\tilde{\Psi}>_1 - & e\,\Box.\,(\rho\,\mathbf{a}\mathbf{v}) = \rho\,e\,\mathbf{F}.\,\mathbf{v},\end{aligned} \tag{15}$$

where, $\mathbf{F} = \Box\wedge\mathbf{a}$ is the electromagnetic bivector field[8]. The r.h.s term $\rho\,e\,\mathbf{F}.\,\mathbf{v}$ of the above equation representing a local force density ρf on a charge/velocity streamline, is just the familiar classical Lorentz force. However, the force equation must include a Stern-Gerlach force term to establish the energy-momentum conservation law. The apparent absence of a Stern-Gerlach force in the presence of an intrinsic magnetic moment also indicates that the electron magnetic moment arises from a

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circulation of electron charge[22] and equation (15) does not reveal the full effect of the electromagnetic field **F** on the momentum flow. On the other hand, equation (13) indicates that the force on a momentum streamline (along Gordon current) consists of a Lorentz force supplemented by a Stern-Gerlach type force, arising from the circulation of charge relative to the momentum streamlines. Same general conclusion may be reached in other ways too and this problem requires further analysis and we intend to address it in a subsequent work. However, it is important to note that the electron spin (density) $\rho\mathbf{S}$ determines this circulation locally, so we can regard it as describing a property of electron motion rather than electron structure.

3. Concluding remarks

In two earlier papers[8] we have noted the profound advatages of using STA in describing Electromagnetic field equations and Fluid mechanics. In this paper similarly new insights are found from Dirac equation with a spacetime geometric reformulation. The Dirac theory of the electron contains important geometric and physical information, which is obscured in standard matrix-based approaches. The spacetime metric in terms of geometric product of any pair of the four basis vectors exactly reproduces the Dirac algebra (algebra of Dirac matrices). Also, both the Dirac operator □ and the spinors are convenient representations in STA. Finally, identifying the unit imaginary of equation (1) with the unitary bivector, the generator of spin, the Dirac equation is reformulated exclusively according to STA – the Dirac-Hestenes equation. The flexible representation offers a far reaching powerful theory for a more detailed investigation of various aspects of the physics of electron than what is given by the conventional expositions of quantum mechanics. For example it elicits the noncollinearity of velocity and momentum – the Weyssenhoff motion of Dirac's electron providing the Gordon decomposition of Dirac current in a simple way. The expression for the spin magnetization current term in equation (14) endorses the bivector representation of the electron spin (eq.8).

The spacetime algebraic reformulation facilitates direct calculation of various conservation laws for probability current or charge current, energy-momentum and angular momentum densities from the Dirac equation. Also, in conjunction with the nouvel spacetime split concept, the transition from Dirac to Pauli and then to Schrödinger equation becomes straightforward and explains the elusive Darwin term in exact accordance with the original argument of Thomas. The reformulated theory also offers a new interpretation of zitterbewegung and Bohm's quantum potential.

The geometric algebra of spacetime is the best available mathematical tool for theoretical physics, classical, relativistic or quantum. The redefined Dirac operator, i.e. the spacetime gradient operator is equally applicable in the formulation of field theoretic studies of both electromagnetism and fluid mechanics, as discussed extensively in two recent papers[8]. According to Baylis 'Had the electrodynamics and relativity of Maxwell and Einstein been originally formulated in algebra of physical space (APS – geometric algebra of 3-D), the transition to quantum theory would have been less of a quantum leap'[28]. Unfortunately though, mainstream physics is yet to appreciate and embrace this development fully. Apart from the historical reasons, it appears that the introduction needs a more familiar and nonaxiomatic direct approach for the research and advanced graduate students.

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