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A Novel Computational Approach for Solving Fully Implicit Singular Systems of Ordinary Differential Equations

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October 4, 2023

Abstract. This paper presents a novel computational approach to solve fully implicit singular nonlinear systems of ordinary differential equations. These systems have a two fold difficulty: being fully implicit and singular at the same time. Such systems cannot be solved in general by software packages such as Maple due to their fully implicit structure. Furthermore, numerical methods like Runge-Kutta cannot be applied. The proposed method here is based on the idea of applying the differential transform method (DTM) directly to these systems while exploiting an important property of Adomian polynomials. This new idea has led to a general and efficient algorithm that can be easily implemented using Maple, Mathematica or Matlab. We stress here that our technique does not require transforming the implicit system in hands to an explicit differential system. Also our technique equips the DTM with a powerful tool to solve other fully implicit differential systems. To illustrate the capability and efficiency of the proposed method, four numerical examples that are not solvable by software packages like Maple are given. Numerical results show that our method has successfully solved these examples by providing the exact solutions in a convergent power series form.

Keywords: Fully implicit systems of ordinary differential equations; differential transform method; singular ordinary differential equations; Adomian polynomials

1 Introduction

Systems of differential equations are used to model many important problems in different fields of science and engineering. One can find applications in unmanned aerial vehicles (UAVs) [1, 2], robotics [3], aircraft landing gears [4] and vehicle industry [5, 6]. One can find also applications in other fields such as chemistry and biology. Explicit systems of ordinary differential equations can be easily solved by several methods such as Runge-Kutta method [7, 8], the residual power series method [9], Adomian decomposition method [10, 11, 12, 13], the homotopy perturbation method [14], the differential transform method (DTM) [15, 16]. A comprehensive review of the DTM is given in [17]. All these approximation methods were limited to solving explicit ordinary differential equations and their systems. Furthermore, there are many software packages such as Maple, Mathematica or Matlab which can easily solve explicit equations or systems. By contrast, fully implicit systems of ordinary differential equations that cannot be written in an explicit form, have received less focus [18]. To our knowledge, implicit ordinary differential equations were discussed only in [18]. These systems are difficult to solve both analytically and numerically due to their implicit structure. Many integration methods, like Runge-Kutta method, do not apply to these systems as these methods require the

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https://doi.org/10.32388/X4S2ZL

system in hands to be in an explicit form. This forms a strong restriction that limits the use of these methods to fully implicit systems of ordinary differential equations. Furthermore, some software packages like Maple cannot deal, in general, with implicit systems. Implicit ordinary differential equations and their systems can arise from the semi-analytical transformations of partial differential equations to ordinary differential equations. They can also arise from the application of the method of lines to implicit partial differential equations. There are many works on existence and uniqueness for fractional implicit differential equations [19, 20, 21, 22]. However, a literature review revealed that no previous attempts were made to solve fully implicit ordinary differential systems using of semi-analytical methods in general and the DTM in particular. The DTM was recently used to solve implicit differential algebraic systems [28]. The DTM has found a large application in solving explicit differential equations [15, 16, 17]. The method yields analytical solutions in form of power series in a straightforward manner by constructing a recursion for the series expansion coefficients. The method was also successfully used to solve explicit nonlinear singular ordinary differential equations and their systems [23, 24, 25, 26, 27].

The goal of this paper is to present a novel technique using the DTM and Adomian polynomials [29, 30, 31, 32, 33, 34] to solve fully implicit nonlinear singular systems of ordinary differential equations. These systems have a two fold difficulty: being fully implicit and singular at the same time. To illustrate the proposed method, two nonlinear classes of implicit first and second order singular systems of ordinary differential equations are considered. Two simple and efficient algorithms that can be easily implemented using software packages such Maple, Mathematica or Matlab are given.

The first class consists of the following fully implicit singular nonlinear first order systems of ordinary differential equations

$$F\left(\mathbf{x}', \mathbf{x}/t, \mathbf{x}, t\right) = 0, \tag{1}$$

where $t > 0$. Here $\mathbf{x} := \mathbf{x}(t) \in \mathbb{R}^n$ is the sought solution, the dash denotes the derivative of $\mathbf{x}$ with respect to t and the function F is such that $F : (\mathbb{R}^n)^3 \times (0, +\infty) \longrightarrow \mathbb{R}^n$. This system is subject to the initial condition

$$\mathbf{x}(0) = 0. \tag{2}$$

For system (1), we assume that the matrix $M_k := kF_{\mathbf{x}'} + F_{\mathbf{x}/t} \in \mathbb{R}^{n\times n}$ is nonsingular for all $k \geq 1$, where $F_{\mathbf{x}'}$ and $F_{\mathbf{x}/t}$ are the Jacobians of F with respect to $\mathbf{x}'$ and $\mathbf{x}/t$ respectively.

The second class consists of the following fully implicit singular nonlinear second order systems of ordinary differential equations

$$F\left(\mathbf{x}'', \mathbf{x}'/t, \mathbf{x}', \mathbf{x}, t\right) = 0, \tag{3}$$

where $t > 0$. Here the function F is such that $F : (\mathbb{R}^n)^4 \times (0, +\infty) \longrightarrow \mathbb{R}^n$. This system is subject to the initial conditions

$$\mathbf{x}(0) = \mathbf{x}_0,\ \mathbf{x}'(0) = 0, \tag{4}$$

where $\mathbf{x}_0 \in \mathbb{R}^n$ is a given vector.

For this second system, we assume that the matrix $N_k := kF_{\mathbf{x}''} + F_{\mathbf{x}'/t} \in \mathbb{R}^{n\times n}$ is nonsingular for all $k \geq 1$, where $F_{\mathbf{x}''}$ and $F_{\mathbf{x}'/t}$ are the Jacobians of F with respect to $\mathbf{x}''$ and $\mathbf{x}'/t$ respectively. Throughout this paper, we assume the function F to be analytical with respect to its variables and that the initial values problems (1)-(2) and (3)-(4) have unique solutions. The method we present in this paper can be easily extended to fully implicit nonlinear systems of ordinary differential equations with orders higher than two.

Two algorithms to solve the implicit singular systems (1) and (3) are given. These algorithms are based on an effective combination of the DTM and the Adomian polynomials. The main idea behind our technique is to apply the DTM directly to systems (1) and (3) without transforming them to a semi-explicit form. Then use an important property of the Adomian polynomials to derive two simple and efficient algorithms for the DTM. Furthermore, it is worth pointing out that our technique can provide the exact solution in a convergent power series if all computations were performed exactly. The algorithms of our technique can be easily implemented using Maple, Mathematica or Matlab. To demonstrate the effectiveness and accuracy of the DTM algorithms, four numerical examples of nonlinear implicit singular systems are solved by the new technique. Here we should emphasise that all the numerical examples solved in this paper are not solvable by software packages like Maple. Also the Runge-Kutta method cannot be applied to these examples. The numerical results show that our technique is successful in solving all these examples by providing the solutions in convergent power series form.

This manuscript is organised as follows: in Section 2, we review the Adomian polynomials and the DTM to solve explicit ordinary differential equations. Next, in Section 3, we give two theorems that provide the new algorithms to solve the fully implicit singular first and second order initial-value problems (1)-(2) and (3)-(4). Then, in Section 4, four fully implicit singular nonlinear systems of ordinary differential equations are solved to illustrate the efficiency and accuracy of this new method. Finally, we give a conclusion in the Section 5.

2 Adomian polynomials and the differential transform method

In this section we give a brief review to the Adomian polynomials [29, 30, 31, 32, 33, 34] which are useful for the expansion of a nonlinear function F, then we review the differential transform method. Usually, a nonlinear function $F(\mathbf{x}, \mathbf{y})$ is decomposed as

$$F(\mathbf{x}, \mathbf{y}) = \sum_{k=0}^{\infty} F_k, \tag{5}$$

where the Adomian polynomials F_k are computed, for all nonlinearities, from

$$F_k := F_k\left(x_0, x_1, \ldots, x_k; y_0, y_1, \ldots, y_k\right) = \frac{1}{k!}\frac{d^k}{d\xi^k}\left(F\left(\sum_{i=0}^{\infty}\xi^i x_i, \sum_{i=0}^{\infty}\xi^i y_i\right)\right)_{\xi=0}, \; k \geq 0. \tag{6}$$

Here $x_k,\ y_k,\ k \geq 0$ are the terms used in the expansions

$$\mathbf{x}(t) = \sum_{k=0}^{\infty} x_k,\ \mathbf{y}(t) = \sum_{k=0}^{\infty} y_k. \tag{7}$$

Making use of (6), for a function $F(\mathbf{x},\mathbf{y})$ with two variables, we can calculate the following first few Adomian polynomials:

$$
\begin{aligned}
F_0 &= F(x_0,y_0),\\
F_1 &= x_1F^{(1,0)} + y_1F^{(0,1)},\\
F_2 &= x_2F^{(1,0)} + y_2F^{(0,1)} + \frac{x_1^2}{2}F^{(2,0)} + \frac{y_1^2}{2}F^{(0,2)} + x_1y_1F^{(1,1)},\\
F_3 &= x_3F^{(1,0)} + y_3F^{(0,1)} + x_1x_2F^{(2,0)} + y_1y_2F^{(0,2)} + (x_1y_2+x_2y_1)F^{(1,1)}\\
&\quad + \frac{x_1^3}{6}F^{(3,0)} + \frac{y_1^3}{6}F^{(0,3)} + \frac{x_1^2y_1}{2}F^{(2,1)} + \frac{x_1y_1^2}{2}F^{(1,2)},\\
F_4 &= x_4F^{(1,0)} + y_4F^{(0,1)} + \left(x_1x_3+\frac{x_2^2}{2}\right)F^{(2,0)} + \left(y_1y_3+\frac{y_2^2}{2}\right)F^{(0,2)} + (x_1y_3+x_2y_2+x_3y_1)F^{(1,1)}\\
&\quad + \frac{x_1^2x_2}{2}F^{(3,0)} + \frac{y_1^2y_2}{2}F^{(0,3)} + \left(x_1x_2y_1+\frac{x_1^2y_2}{2}\right)F^{(2,1)} + \left(x_1y_1y_2+\frac{x_2y_1^2}{2}\right)F^{(1,2)} + \frac{x_1^4}{4!}F^{(4,0)}\\
&\quad + \frac{x_1^3y_1}{6}F^{(3,1)} + \frac{x_1^2y_1^2}{4}F^{(2,2)} + \frac{x_1y_1^3}{4}F^{(1,3)} + \frac{y_1^4}{4!}F^{(0,4)},
\end{aligned}
$$

where $F^{(k,l)} := \dfrac{\partial^{k+l}F\,(x_0,y_0)}{\partial x_0^k \partial y_0^l}$, $k \geq 0$, $l \geq 0$.
The differential transform of a function $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are defined by

$$
x_k = \frac{1}{k!}\left[\frac{d^k\mathbf{x}(t)}{dt^k}\right]_{t=0},\ y_k = \frac{1}{k!}\left[\frac{d^k\mathbf{y}(t)}{dt^k}\right]_{t=0},\ k \geq 0, \tag{8}
$$

and the inverse differential transforms of $x_k,\ y_k,\ k \geq 1$ are given by

$$
\mathbf{x}(t) = \sum_{k=0}^{\infty} x_k t^k,\ \mathbf{y}(t) = \sum_{k=0}^{\infty} y_k t^k. \tag{9}
$$

From (8)-(9), we have

$$
\mathbf{x}(t) = \sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{d^k\mathbf{x}(t)}{dt^k}\right]_{t=0} t^k,\ \mathbf{y}(t) = \sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{d^k\mathbf{y}(t)}{dt^k}\right]_{t=0} t^k. \tag{10}
$$

An approximate solution is given by

$$
\mathbf{x}(t) = \sum_{k=0}^{K} x_k t^k,\ \mathbf{y}(t) = \sum_{k=0}^{K} y_k t^k, \tag{11}
$$

where K is the number of terms in the approximation. If $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are expanded as in (9), then the nonlinear function $F(\mathbf{x},\mathbf{y})$ can be expanded using the Adomian polynomials as

$$
F\Big(\sum_{k=0}^{\infty} x_k t^k, \sum_{k=0}^{\infty} y_k t^k\Big) = \sum_{k=0}^{\infty} F_k\,(x_0,x_1,\ldots,x_k;y_0,y_1,\ldots,y_k)\,t^k, \tag{12}
$$

where now x_k and y_k, $k \geq 1$ are the coefficients of expansion (11). One important property of the Adomian polynomials which we will exploit to develop our algorithms is the fact that the Adomian polynomials $F_k,\ k \geq 1$ are affine with respect to x_k and y_k.

3 Proposed approach for fully implicit singular ode systems

The proposed algorithms are based on an effective combination of the differential transform method (DTM) with the Adomian polynomials [29, 30, 31, 32, 33, 34]. The main idea of our technique is to apply the DTM directly to the classes (1)-(2) and (3)-(4), where the nonlinear vector functions are expanded in power series using the Adomian polynomials [35]. Then, by using the fact that the Adomian polynomial F_k is affine with respect to x_k and y_k, an algebraic recursion system for the differential transforms of the solution is obtained. The main advantage of our technique is that it does not require transforming system (1) or (3) into an explicit first order system before applying the DTM. This has considerably helped in simplifying the algorithms. The following new theorems are important for derivation of our technique. These two theorems apply for the general fully implicit singular systems (1) and (3).

Theorem 1.
Consider the fully implicit singular nonlinear system of ordinary differential equations $F(\mathbf{x}', \mathbf{x}/t, \mathbf{x}, t) = 0$, where $F : (\mathbb{R}^n)^3 \times (0, +\infty) \longrightarrow \mathbb{R}^n$, with $\mathbf{x}(0) = 0 \in \mathbb{R}^n$. Assume that the function F is analytical and let $\mathbf{x}(t) = \sum_{k=0}^{\infty} x_k t^k$, where x_k is the differential transform of the solution $\mathbf{x}(t)$.
Let $F_{k-1} := F_{k-1}(x_1, \ldots, (k-1)x_{k-1}, kx_k; x_1, \ldots, x_{k-1}, x_k; x_1, \ldots, x_{k-1})$, $k \geq 2$ be the vector of $(k-1)$-th Adomian polynomials of the components of the vector F. Furthermore, assume that the matrix $M_k := kF_{\mathbf{x}'} + F_{\mathbf{x}/t} \in \mathbb{R}^{n\times n}$, $k \geq 1$ is nonsingular at $(x_1, x_1, x_0, 0)$. Then the differential transform x_k of the solution $\mathbf{x}(t)$ of the above system of ordinary differential equations is given by the recursion $M_k x_k = -r_{k-1}$ where $r_{k-1} := F_{k-1}(x_1, \ldots, (k-1)x_{k-1}, 0; x_1, \ldots, x_{k-1}, 0; x_1, \ldots, x_{k-1})$, $k \geq 2$ and where the first recursion term x_1 is uniquely determined from the algebraic system $F(x_1, x_1, x_0, 0) = 0$.

Proof:
Assume that the solution $\mathbf{x}(t)$ can be expanded as

$$\mathbf{x}(t) = \sum_{k=0}^{\infty} x_k t^k, \tag{13}$$

where x_k is the differential transform of the solution $\mathbf{x}(t)$. Let $\mathbf{y} = \mathbf{x}/t$, then $\mathbf{x} = t\mathbf{y}$ and taking the differential transform of both sides of the latter equation, we get

$$x_k = \sum_{l=0}^{k} \delta(l-1) y_{k-l} = y_{k-1},\ k \geq 1, \tag{14}$$

where $\delta_k = \begin{cases} 1, & \text{if } k = 0, \\ 0, & \text{if } k \neq 0. \end{cases}$
From this, we obtain the differential transform of $\mathbf{y}$ in terms of that of $\mathbf{x}$ as

$$y_k = x_{k+1},\ k \geq 0. \tag{15}$$

Then, we expand the left side of the equation $F(\mathbf{x}', \mathbf{y}, \mathbf{x}, t) = 0$ in terms of the Adomian polynomials to obtain

$$F\left(\sum_{k=0}^{\infty}(k+1)x_{k+1}t^k, \sum_{k=0}^{\infty} x_{k+1}t^k, \sum_{k=0}^{\infty} x_k t^k, t\right) = \sum_{k=0}^{\infty} F_k t^k = 0. \tag{16}$$

This gives
for $k = 0$:

$$F(x_1, x_1, x_0, 0) = 0, \tag{17}$$

and for $k \geq 1$:

$$F_k\left(x_1, \ldots, kx_k, (k+1)x_{k+1}; x_1, \ldots, x_k, x_{k+1}; x_1, \ldots, x_k\right) = 0, \tag{18}$$

or

$$M_{k+1}x_{k+1} + F_k(x_1, \ldots, kx_k, 0; x_1, \ldots, x_k, 0; x_1, \ldots, x_k) = 0, \tag{19}$$

where the matrix $M_{k+1} = (k+1)F_{\mathbf{x}'} + F_{\mathbf{x}/t}$ is nonsingular. This leads to the following recursion for the differential transform x_k

$$M_k x_k = -F_{k-1}(x_1, \ldots, (k-1)x_{k-1}, 0; x_1, \ldots, x_{k-1}, 0; x_1, \ldots, x_{k-1}),\ k \geq 2. \tag{20}$$

The above system determines x_k in terms of $x_0, x_1, \ldots, x_{k-2}, x_{k-1}$ for $k \geq 2$. The first recursion term x_1 is uniquely determined from the algebraic system (17). This completes the proof of theorem 1.

Algorithm 1: DTM solution of $F(\mathbf{x}', \mathbf{x}/t, \mathbf{x}, t) = 0,\ \mathbf{x}(0) = 0$:
input: $K,\ n,\ F \in \mathbb{R}^n$
output: approximate solution $\mathbf{x}(t) = \sum_{k=0}^{K} x_k t^k$
for $2 \leq k \leq K$ do
compute $r_{k-1} := F_{k-1}(x_1, \ldots, (k-1)x_{k-1}, 0; x_1, \ldots, x_{k-1}, 0; x_1, \ldots, x_{k-1})$
end do
initialization: $x_0 := 0$
solve for x_1 the algebraic system: $F(x_1, x_1, x_0, 0) = 0$
for $2 \leq k \leq K$ do
compute M_k
solve for x_k the linear algebraic system: $M_k x_k = -r_{k-1}$
end do

Theorem 2.
Consider the fully implicit singular nonlinear system of ordinary differential equations $F(\mathbf{x}'', \mathbf{x}'/t, \mathbf{x}', \mathbf{x}, t) = 0$, where $F : (\mathbb{R}^n)^4 \times (0, +\infty) \longrightarrow \mathbb{R}^n$, with $\mathbf{x}(0) = \mathbf{x}_0 \in \mathbb{R}^n$, $\mathbf{x}'(0) = 0 \in \mathbb{R}^n$. Assume that the function F is analytical and let $\mathbf{x}(t) = \sum_{k=0}^{\infty} x_k t^k$, where x_k is the differential transform of the solution $\mathbf{x}(t)$.
Let $F_{k-2} := F_{k-2}\left(2x_2, \ldots, (k-1)kx_k; 2x_2, \ldots, kx_k; x_1, \ldots, (k-1)x_{k-1}; x_1, \ldots, x_{k-2}\right)$, $k \geq 3$ be the vector of $(k-2)$-th Adomian polynomials of the components of the vector function F. Assume that the matrix $N_k := kF_{\mathbf{x}''} + F_{\mathbf{x}'/t} \in \mathbb{R}^{n \times n}$, $k \geq 1$ is nonsingular at $(2x_2, 2x_2, x_1, x_0, 0)$. Then the differential transform x_k of the solution $\mathbf{x}(t)$ of the above system of ordinary differential equations is given by the recursion $kN_{k-1}x_k = -r_{k-2}$, where $r_{k-2} := F_{k-2}\left(2x_2, \ldots, (k-2)(k-1)x_{k-1}, 0; 2x_2, \ldots, (k-1)x_{k-1}, 0; x_1, \ldots, (k-1)x_{k-1}; x_1, \ldots, x_{k-2}\right)$, $k \geq 3$, where the second recursion term x_2 is computed from the algebraic system $F(2x_2, 2x_2, x_1, x_0, 0) = 0$.

Proof:
Assume that the solution $\mathbf{x}(t)$ can be expanded as in (13). Let $\mathbf{y} = \mathbf{x}'/t$, then $\mathbf{x}' = t\mathbf{y}$ and taking the differential transform of both sides of the latter equation, we get

$$(k+1)x_{k+1} = \sum_{l=0}^{k} \delta(l-1)y_{k-l} = y_{k-1}, \; k \geq 1. \tag{21}$$

From this, we obtain the differential transform of $\mathbf{y}$ in terms of that of $\mathbf{x}$ as

$$y_k = (k+2)x_{k+2}, \; k \geq 0. \tag{22}$$

Then we expand the left side of the equation $F(\mathbf{x}'', \mathbf{y}, \mathbf{x}', \mathbf{x}, t) = 0$ in terms of the Adomian polynomials to obtain

$$F\left(\sum_{k=0}^{\infty}(k+1)(k+2)x_{k+2}t^k, \sum_{k=0}^{\infty}(k+2)x_{k+2}t^k, \sum_{k=0}^{\infty}(k+1)x_{k+1}t^k, \sum_{k=0}^{\infty}x_k t^k, t\right) = \sum_{k=0}^{\infty} F_k t^k = 0. \tag{23}$$

This gives
for $k = 0$:

$$F(2x_2, 2x_2, x_1, x_0, 0) = 0, \tag{24}$$

and for $k \geq 1$:

$$F_k\left(2x_2, \ldots, (k+1)(k+2)x_{k+2}; 2x_2, \ldots, (k+2)x_{k+2}, x_2, \ldots, (k+1)x_{k+1}; x_1, \ldots, x_k\right) = 0, \; k \geq 1, \tag{25}$$

or

$$(k+2)N_{k+1}x_{k+2} + F_k(6x_3, \ldots, k(k+1)x_{k+1}, 0; 3x_3, \ldots, (k+1)x_{k+1}, 0; x_2, \ldots, (k+1)x_{k+1}; x_1, \ldots, x_k), \; k \geq 1, \tag{26}$$

where the matrix $N_{k+1} = (k+1)F_{\mathbf{x}''} + F_{\mathbf{x}'/t}$ is nonsingular. This leads to the following recursion for the differential transform x_k

$$kN_{k-1}x_k = -F_{k-2}\left(2x_2, \ldots, (k-2)(k-1)x_{k-1}, 0; 2x_2, \ldots, (k-1)x_{k-1}, 0; x_2, \ldots, (k-1)x_{k-1}; x_2, \ldots, x_{k-1}\right), \; k \geq 3. \tag{27}$$

The above system determines x_k in terms of $x_0, x_1, \ldots, x_{k-2}, x_{k-1}$ for $k \geq 3$. The second recursion term x_2 is uniquely determined from the algebraic system (24). This completes the proof of theorem 2.

Algorithm 2: DTM solution of $F(\mathbf{x}'', \mathbf{x}'/t, \mathbf{x}', \mathbf{x}, t) = 0,\ \mathbf{x}(0) = \mathbf{x}_0,\ \mathbf{x}'(0) = 0$:
input: $K,\ n,\ \mathbf{x}_0 \in \mathbb{R}^n,\ F \in \mathbb{R}^n$
output: approximate solution $\mathbf{x}(t) = \sum_{k=0}^{K} x_k t^k$
for $3 \leq k \leq K$ do
compute $r_{k-2} := F_{k-2}(2x_2, \ldots, (k-2)(k-1)x_{k-1}, 0; 2x_2, \ldots, (k-1)x_{k-1}, 0; x_2, \ldots, (k-1)x_{k-1}; x_2, \ldots, x_{k-1})$
end do
initialization: $x_0 := \mathbf{x}_0,\ x_1 := 0$
solve for x_2 the algebraic system: $F(2x_2, 2x_2, x_1, x_0, 0) = 0$
for $3 \leq k \leq K$ do
compute N_{k-1}
solve for x_k the linear algebraic system: $kN_{k-1}x_k = -r_{k-2}$
end do

4 Numerical results and discussion

In this section, four numerical examples of fully implicit singular nonlinear systems of ordinary differential equations are solved by the proposed technique to demonstrate its effectiveness and accuracy. All these examples cannot be solved by software packages like Maple. Furthermore, methods like Runge-Kutta cannot be applied to these types of systems as these systems cannot be written in an explicit form. Thanks to theorems 1 and 2, our technique has successfully solved all these examples in power series form. We emphasise here that our technique gives exactly the power series expansion of the exact solution if all computations are performed exactly. The implementation of our algorithms was performed in Maple 15.

Example 1
In this example, we consider the following fully implicit nonlinear singular system of first order ordinary differential equations

$$\ln\left(\mathbf{x}_1' + te^{-t}\right) + \mathbf{x}_1' + \frac{2\mathbf{x}_2}{t} + t - (1-t)e^{-t} - e^{-2t} = 0, \tag{28}$$

$$\ln\left(2\mathbf{x}_2' + 2te^{-2t}\right) + 2\mathbf{x}_2' + \frac{\mathbf{x}_1}{t} + 2t - (1-2t)e^{-2t} - e^{-t} = 0, \tag{29}$$

where $t > 0$. The sought solution is $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)^{\mathsf{T}}$. The differential system (28)-(29) has the form (1), with $n = 2$. This system is subject to the initial condition

$$\mathbf{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{30}$$

We should emphasise here that Maple software cannot not solve this system. However, our technique has successfully obtained the exact solution of this implicit singular system in power series. By applying the DTM for solving equation (28)-(29) with initial condition (30), we use theorem 1. Thanks to this theorem, we can recursively determine the DTM expansion coefficients x_k for $k \geq 2$. According to the initial conditions (30), we have

$$x_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{31}$$

Then, we determine the differential transform $x_1 = (x_{11}, x_{12})^{\mathsf{T}}$ from the algebraic system

$$F(x_1, x_1, x_0, 0) = 0. \tag{32}$$

The above system has a unique solution and simplifies to

$$\ln(x_{11}) + x_{11} + 2x_{12} - 2 = 0, \tag{33}$$
$$\ln(2x_{12}) + 2x_{12} + x_{11} - 2 = 0. \tag{34}$$

Solving this algebraic system, we obtain

$$x_1 = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix}. \tag{35}$$

Then, we compute the Jacobians $A := F_{\mathbf{x}'}$ and $B := F_{\mathbf{x}/t}$ of the function F with respect to $\mathbf{x}'$ and $\mathbf{x}/t$ at the point $(x_1, x_1, x_0, 0)$. We get

$$A = \begin{pmatrix} \frac{1}{x_{11}} + 1 & 0 \\ 0 & \frac{1}{x_{12}} + 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}. \tag{36}$$

Using theorem 1, we have the following recursion for the differential transform x_k

$$x_k = -\left(kA + B\right)^{-1} r_{k-1}, \ k \geq 2, \tag{37}$$

where

$$r_{k-1} := F_{k-1}(x_1, \ldots, kx_k, 0; x_1, \ldots, kx_k, 0; x_1, \ldots, \ldots, x_{k-1}; 0). \tag{38}$$

Using our algorithm, we compute the inverse of the matrix and the right hand sides r_{k-1} for $k \geq 2$ and deduce:
For $k = 2$:

$$x_2 = -\left(2A + B\right)^{-1} r_1 = -\begin{pmatrix} \frac{4}{15} & -\frac{1}{15} \\ -\frac{1}{30} & \frac{2}{15} \end{pmatrix} \begin{pmatrix} \frac{1}{x_{11}} + 5 \\ \frac{1}{x_{12}} + 7 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}. \tag{39}$$

For $k = 3$:

$$x_3 = -\left(3A + B\right)^{-1} r_2 = -\begin{pmatrix} \frac{6}{35} & -\frac{1}{35} \\ -\frac{1}{70} & \frac{3}{35} \end{pmatrix} \begin{pmatrix} \frac{1}{x_{11}} + \frac{(2x_{21}+1)^2}{2x_{11}^2} + \frac{7}{2} \\ \frac{2}{x_{12}} + \frac{(4x_{22}+2)^2}{8x_{12}^2} + \frac{13}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}. \tag{40}$$

For $k = 4$:

$$x_4 = -\left(4A + B\right)^{-1} r_3 = -\begin{pmatrix} \frac{8}{63} & -\frac{1}{63} \\ -\frac{1}{126} & \frac{4}{63} \end{pmatrix} \begin{pmatrix} \frac{1}{2x_{11}} - \frac{(6x_{31}-2)(2x_{21}+1)}{2x_{11}^2} + \frac{(2x_{21}+1)^3}{3x_{11}^3} + 2 \\ \frac{2}{x_{12}} - \frac{(12x_{32}-8)(4x_{22}+2)}{8x_{12}^2} + \frac{(4x_{22}+2)^3}{24x_{12}^3} + \frac{11}{2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{6} \\ -\frac{2}{3} \end{pmatrix}. \tag{41}$$

For sake of presentation, we give now only the numerical values of r_{k-1} rather than its algebraic expressions.
For $k = 5$:

$$x_5 = -\left(5A + B\right)^{-1} r_4 = -\begin{pmatrix} \frac{10}{99} & -\frac{1}{99} \\ -\frac{1}{198} & \frac{5}{99} \end{pmatrix} \begin{pmatrix} \frac{-13}{12} \\ \frac{-161}{24} \end{pmatrix} = \begin{pmatrix} \frac{1}{24} \\ \frac{1}{3} \end{pmatrix}. \tag{42}$$

For $k = 6$:

$$x_6 = -\left(6A + B\right)^{-1} r_5 = -\begin{pmatrix} \frac{12}{143} & -\frac{1}{143} \\ -\frac{1}{286} & \frac{6}{143} \end{pmatrix} \begin{pmatrix} \frac{11}{30} \\ \frac{77}{24} \end{pmatrix} = \begin{pmatrix} -\frac{1}{120} \\ \frac{2}{15} \end{pmatrix}. \tag{43}$$

Finally, we construct the following approximate solution based on six terms

$$\mathbf{x}(t) = t \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} + t^2 \begin{pmatrix} -1 \\ -1 \end{pmatrix} + t^3 \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} + t^4 \begin{pmatrix} -\frac{1}{6} \\ \frac{-2}{3} \end{pmatrix} + t^5 \begin{pmatrix} \frac{1}{24} \\ \frac{1}{3} \end{pmatrix} + t^6 \begin{pmatrix} \frac{1}{120} \\ \frac{-2}{15} \end{pmatrix}. \tag{44}$$

One can easily check that the above approximation forms the first terms of the Maclauran power series of the exact solution. It is worth noticing here that we obtain the exact values of the successive DTM expansion coefficients.

Example 2

In this example, we consider the following fully implicit nonlinear singular system of first order ordinary differential equations

$$3\mathbf{x}_1' + \sin\left(\mathbf{x}_1' - t^4\right) + \frac{\mathbf{x}_2}{t} - 7t - \frac{14}{5}t^4 - \sin(2t) = 0, \tag{45}$$

$$3\mathbf{x}_2' - \cos\left(\mathbf{x}_2' + t^4\right) + \frac{\mathbf{x}_1}{t} - 7t + \frac{14}{5}t^4 + \cos(2t) = 0, \tag{46}$$

where $t > 0$. The sought solution is $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)^{\mathsf{T}}$. The differential system (45)-(46) has the form (1), with $n = 2$. This system is subject to the initial condition

$$\mathbf{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{47}$$

We emphasise here that Maple software cannot not solve this system numerically or in power series in its present form. However, our technique has successfully obtained the exact solution of this implicit singular system in power series. By applying the DTM for solving equation (45)-(46) with initial condition (47), we use theorem 1. Thanks to this theorem, we can recursively determine the DTM expansion coefficients x_k for $k \geq 2$. According to the initial condition (47), we have

$$x_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{48}$$

Then, we determine the DTM expansion coefficient $x_1 = (x_{11}, x_{12})^{\mathsf{T}}$ from the algebraic system

$$F(x_1, x_1, x_0, 0) = 0. \tag{49}$$

The above system has a unique solution and simplifies to

$$3x_{11} + \sin(x_{11}) + x_{12} = 0, \tag{50}$$

$$3x_{12} - \cos(x_{12}) + x_{11} + 1 = 0. \tag{51}$$

Solving this algebraic system, we obtain

$$x_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{52}$$

Then, we compute the Jacobians $A := F_{\mathbf{x}'}$ and $B := F_{\mathbf{x}/t}$ of the function F with respect to $\mathbf{x}'$ and $\mathbf{x}/t$ at the point $(x_1, x_1, x_0, 0)$. We get

$$A = \begin{pmatrix} 3 + \cos(x_{11}) & 0 \\ 0 & 3 + \sin(x_{12}) \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{53}$$

Using theorem 1, we have the following recursion for the differential transform x_k

$$x_k = -\left(kA + B\right)^{-1} r_{k-1}, \ k \geq 2, \tag{54}$$

where

$$r_{k-1} := F_{k-1}(x_1, \ldots, kx_k, 0; x_1, \ldots, kx_k, 0; x_1, \ldots, \ldots, x_{k-1}; 0). \tag{55}$$

Using our algorithm, we compute the inverse of the matrix and the right hand sides r_{k-1} for $k \geq 2$ and deduce:
For $k = 2$:

$$x_2 = -\left(2A + B\right)^{-1} r_1 = -\begin{pmatrix} \frac{6}{47} & -\frac{1}{47} \\ -\frac{1}{47} & \frac{8}{47} \end{pmatrix} \begin{pmatrix} -9 \\ -7 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{56}$$

For $k = 3$:

$$x_3 = -\left(3A + B\right)^{-1} r_2 = -\begin{pmatrix} \frac{9}{107} & -\frac{1}{107} \\ -\frac{1}{107} & \frac{12}{107} \end{pmatrix} \begin{pmatrix} -2x_{21}^2 \sin(x_{11}) \\ -2 + 2x_{22}^2 \cos(x_{12}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{57}$$

For $k = 4$:

$$x_4 = -\left(4A + B\right)^{-1} r_3 = -\begin{pmatrix} \frac{12}{191} & -\frac{1}{191} \\ -\frac{1}{191} & \frac{16}{191} \end{pmatrix} \begin{pmatrix} \frac{4}{3} - 6x_{31}x_{21}\sin(x_{11}) - \frac{4}{3}x_{21}^3\cos(x_{11}) \\ 6x_{32}x_{22}\cos(x_{12}) - \frac{4}{3}x_{22}^3\sin(x_{12}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{58}$$

For sake of presentation, we give now only the numerical values of r_{k-1} rather than its algebraic expressions.
For $k = 5$:

$$x_5 = -\left(5A + B\right)^{-1} r_4 = -\begin{pmatrix} \frac{15}{299} & -\frac{1}{299} \\ -\frac{1}{299} & \frac{20}{299} \end{pmatrix} \begin{pmatrix} \frac{-19}{5} \\ \frac{14}{5} \end{pmatrix} = \begin{pmatrix} \frac{1}{5} \\ -\frac{1}{5} \end{pmatrix}. \tag{59}$$

For $k \geq 6$, we can show that $x_k = 0$. Finally, we construct the following approximate solution based on six terms

$$\mathbf{x}(t) = t^2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + t^5 \begin{pmatrix} \frac{1}{5} \\ -\frac{1}{5} \end{pmatrix}. \tag{60}$$

One can easily check that the above approximation is the exact solution. It is worth noting here that we obtain the exact values of the successive DTM expansion coefficients.

Example 3
In this example, we consider the following fully implicit nonlinear singular system of second order ordinary differential equations

$$\mathbf{x}''^3_1 + \mathbf{x}''_1 + \left(\frac{\mathbf{x}'_1}{t}\right)^3 + \mathbf{x}_2^2 - t^6 - 239t^3 - 6t - 4 = 0, \tag{61}$$

$$\mathbf{x}''^3_2 + \mathbf{x}''_2 + \left(\frac{\mathbf{x}'_2}{t}\right)^3 + \mathbf{x}_1^2 - t^6 - 247t^3 - 6t - 4 = 0, \tag{62}$$

where $t > 0$. The sought solution is $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)^\mathsf{T}$. The differential system (61)-(62) has the form (3), with $n = 2$. This system is subject to the initial conditions

$$\mathbf{x}(0) = \begin{pmatrix} 2 \\ -2 \end{pmatrix}, \ \mathbf{x}'(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{63}$$

This implicit singular system cannot be solved by Maple software numerically or in power series in its present form. However, our technique has successfully obtained the exact solution of this system in power series. By applying the DTM for solving system (61)-(62) with initial condition (63), we use theorem 2 and recursively determine the DTM expansion coefficients x_k for $k \geq 3$. According to the initial conditions (63), we have

$$x_0 = \begin{pmatrix} 2 \\ -2 \end{pmatrix}, \ x_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{64}$$

Then, we determine the DTM expansion coefficient $x_2 = (x_{21}, x_{22})^\mathsf{T}$ from the algebraic system

$$F(2x_2, 2x_2, x_1, x_0, 0) = 0. \tag{65}$$

This system has a unique solution $x_2 = (x_{21}, x_{22})^\mathsf{T}$ and simplifies to

$$16x_{21}^3 + 2x_{21} + x_{02}^2 - 4 = 0, \tag{66}$$
$$16x_{22}^3 + 2x_{22} + x_{01}^2 - 4 = 0, \tag{67}$$

which gives

$$8x_{21}^3 + x_{21} = 0, \tag{68}$$
$$8x_{22}^3 + x_{22} = 0. \tag{69}$$

Solving this algebraic system, we obtain

$$x_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{70}$$

Then, we compute the Jacobians $A := F_{\mathbf{x}''}$ and $B := F_{\mathbf{x}'/t}$ of the function F with respect to $\mathbf{x}''$ and $\mathbf{x}'/t$ at the point $(2x_2, 2x_2, x_1, x_0, 0)$. We get

$$A = \begin{pmatrix} 12x_{21}^2 + 1 & 0 \\ 0 & 12x_{22}^2 + 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; B = \begin{pmatrix} 12x_{21}^2 & 0 \\ 0 & 12x_{22}^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{71}$$

Using theorem 1, we have the following recursion for the differential transform x_k

$$x_k = -\left(\frac{1}{k}\right)\left((k-1)A + B\right)^{-1} r_{k-2}, \; k \geq 3, \tag{72}$$

where

$$r_{k-2} := F_{k-2}(2x_2, \ldots, (k-2)(k-1)x_{k-1}, 0; 2x_2, \ldots, (k-1)x_{k-1}, 0; x_1, \ldots, (k-1)x_{k-1}; x_1, \ldots, \ldots, x_{k-2}; 0). \tag{73}$$

This leads to

$$x_k = -\frac{1}{k(k-1)} r_{k-2}, \; k \geq 3. \tag{74}$$

Using our algorithm, we compute the right hand sides r_{k-2} for $k \geq 3$ and deduce:
For $k = 3$:

$$x_3 = -\frac{1}{6} r_1 = -\frac{1}{6}\begin{pmatrix} 2x_{02}x_{12} - 6 \\ 2x_{01}x_{11} - 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{75}$$

For $k = 4$:

$$x_4 = -\frac{1}{12} r_2 = -\frac{1}{12}\begin{pmatrix} 270x_{21}x_{31}^2 + 2x_{02}x_{22} + x_{12}^2 \\ 270x_{22}x_{32}^2 + 2x_{01}x_{21} + x_{11}^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{76}$$

For $k = 5$:

$$x_5 = -\frac{1}{20} r_3 = -\frac{1}{20}\begin{pmatrix} 1008x_{21}x_{31}x_{41} + 243x_{31}^3 + 2x_{02}x_{32} + 2x_{12}x_{22} - 239 \\ 1008x_{22}x_{32}x_{42} + 243x_{32}^3 + 2x_{01}x_{31} + 2x_{11}x_{21} - 247 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{77}$$

In a similar way, we can show that $x_k = 0$ for $k \geq 7$. Finally, we construct the following approximate solution based on seven terms

$$\mathbf{x}(t) = \begin{pmatrix} 2 \\ -2 \end{pmatrix} + t^3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 + t^3 \\ -2 + t^3 \end{pmatrix}. \tag{78}$$

One can easily check that the above approximation is the exact solution. It is worth noting here that we obtain the exact values of the successive DTM expansion coefficients.

Example 4
In this example, we consider the following fully implicit nonlinear second order system of singular ordinary differential equations

$$\mathbf{x}''_1 - \mathbf{x}''^3_2 + \frac{\mathbf{x}'_1}{t} + \mathbf{x}_2 - \left(4 - 5t + t^2 + t^3\right) e^{-t} + \left(6t - 6t^2 + t^3\right)^3 e^{-3t} = 0, \tag{79}$$

$$\mathbf{x}''^3_1 + \mathbf{x}''_2 + \frac{\mathbf{x}'_2}{t} + \mathbf{x}_1 - \left(9t - 6t^2 + t^3\right) e^{-t} - \left(2 - 4t + t^2\right)^3 e^{-3t} = 0, \tag{80}$$

where $t > 0$. The sought solution is $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)^\mathsf{T}$. The differential system (79)-(80) has the form (3), with $n = 2$. This system is subject to the initial conditions

$$\mathbf{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \; \mathbf{x}'(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{81}$$

This implicit singular system cannot be solved by Maple software numerically or in power series in its present form. However, our technique has successfully obtained the exact solution of this system in power series. By applying the DTM for solving the system (79)-(80) with initial condition (81), we use theorem 2. Thanks to this theorem, we can recursively determine the DTM expansion coefficients x_k for $k \geq 3$. According to the initial conditions (81), we have

$$x_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \; x_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{82}$$

Then, we determine the differential transform x_2 from the algebraic system

$$F(2x_2, 2x_2, x_1, x_0, 0) = 0. \tag{83}$$

The above system has a unique solution $x_2 = (x_{21}, x_{22})^\mathsf{T}$ and simplifies to

$$4x_{21} - 8x_{22}^3 + x_{02} - 4 = 0, \tag{84}$$

$$4x_{22} + 8x_{21} + x_{01} - 8 = 0, \tag{85}$$

which gives

$$4x_{21} - 8x_{22}^3 - 4 = 0, \tag{86}$$

$$4x_{22} + 8x_{21} - 8 = 0. \tag{87}$$

Solving this algebraic system, we obtain

$$x_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{88}$$

Then, we compute the Jacobians $A := F_{\mathbf{x}''}$ and $B := F_{\mathbf{x}'/t}$ of the function F with respect to $\mathbf{x}''$ and $\mathbf{x}'/t$ at the point $(2x_2, 2x_2, x_1, x_0, 0)$. We get

$$A = \begin{pmatrix} 1 & -12x_{22} \\ 12x_{21} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 12 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{89}$$

Using theorem 1, we have the following recursion for differential transform x_k

$$x_k = -\left(\frac{1}{k}\right)\left((k-1)A + B\right)^{-1} r_{k-2},\ k \geq 3, \tag{90}$$

where

$$r_{k-2} := F_{k-2}(2x_2, \ldots, (k-2)(k-1)x_{k-1}, 0; 2x_2, \ldots, (k-1)x_{k-1}, 0; x_1, \ldots, (k-1)x_{k-1}; x_1, \ldots, \ldots, x_{k-2}; 0). \tag{91}$$

This leads to

$$x_k = -\frac{1}{k^3}\begin{pmatrix} k & 0 \\ -12(k-1) & k \end{pmatrix} r_{k-2},\ k \geq 3. \tag{92}$$

Using our algorithm, we compute the right hand sides r_{k-2} for $k \geq 3$ and deduce:
For $k = 3$:

$$x_3 = -\frac{1}{9}\begin{pmatrix} 1 & 0 \\ -8 & 1 \end{pmatrix} r_1 = -\frac{1}{9}\begin{pmatrix} 1 & 0 \\ -8 & 1 \end{pmatrix}\begin{pmatrix} x_{12}+9 \\ x_{11}+63 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \tag{93}$$

For $k = 4$:

$$x_4 = -\frac{1}{16}\begin{pmatrix} 1 & 0 \\ -9 & 1 \end{pmatrix} r_2 = -\frac{1}{16}\begin{pmatrix} 1 & 0 \\ -9 & 1 \end{pmatrix}\begin{pmatrix} -216x_{22}x_{32}^2 + x_{22} - 8 \\ 216x_{21}x_{32}^2 + x_{21} - 273 \end{pmatrix} = -\frac{1}{16}\begin{pmatrix} 1 & 0 \\ -9 & 1 \end{pmatrix}\begin{pmatrix} -8 \\ -56 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -1 \end{pmatrix}. \tag{94}$$

In a similar way, we can compute

$$r_3 = \begin{pmatrix} \frac{25}{6} \\ \frac{55}{2} \end{pmatrix},\ r_4 = \begin{pmatrix} -\frac{3}{2} \\ -9 \end{pmatrix},\ r_5 = \begin{pmatrix} \frac{49}{20} \\ \frac{259}{120} \end{pmatrix}. \tag{95}$$

Using the recursion (90), we find

$$x_5 = \begin{pmatrix} -\frac{1}{6} \\ \frac{1}{2} \end{pmatrix},\ x_6 = \begin{pmatrix} \frac{1}{24} \\ -\frac{1}{6} \end{pmatrix},\ x_7 = \begin{pmatrix} -\frac{1}{120} \\ \frac{1}{24} \end{pmatrix}. \tag{96}$$

Finally, we construct the following approximate solution based on seven terms

$$\mathbf{x}(t) = t^2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t^3\begin{pmatrix} -1 \\ 1 \end{pmatrix} + t^4\begin{pmatrix} \frac{1}{2} \\ -1 \end{pmatrix} + t^5\begin{pmatrix} -\frac{1}{6} \\ \frac{1}{2} \end{pmatrix} + t^6\begin{pmatrix} \frac{1}{24} \\ -\frac{1}{6} \end{pmatrix} + t^7\begin{pmatrix} -\frac{1}{120} \\ \frac{1}{24} \end{pmatrix}. \tag{97}$$

One can easily check that the above approximation forms the first terms of the Maclauran power series of the exact solution $\mathbf{x}(t) = (t^2e^{-t}, t^3e^{-t})^\mathsf{T}$. It is worth noting here that we obtain the exact values of the successive DTM coefficients.

5 Conclusion

This manuscript presents a new method for solving two classes of fully implicit systems of singular nonlinear ordinary differential equations. These types of systems are difficult to solve, and existing software packages like Maple cannot solve them due to their implicit structure. The proposed method combines the differential transform method (DTM) with an important property of the Adomian polynomials to develop two simple and efficient algorithms that can be easily implemented using software packages like Maple, Mathematica, or Matlab. The new approach provides the DTM with a powerful technique to solve fully implicit differential systems, giving it an advantage over other methods like Runge-Kutta, Adomian decomposition method, and other semi-analytical methods. Unlike other techniques, this new method does not require transforming the system into a first-order explicit system, making it a more efficient and accurate solution technique. The paper includes numerical examples of four fully implicit nonlinear singular systems of ordinary differential equations that are not solvable at least by Maple. The numerical results show that the proposed method has successfully solved these examples by providing the exact solutions in a convergent power series form. To solve these implicit classes of differential equations over large intervals, a multi-stage algorithm can be used [28]. We emphasize that this new technique can be also applied to solve other fully implicit systems of ordinary differential equations, delay differential equations, differential-algebraic equations, fractional differential equations, and time-dependent algebraic equations. Future work will be on developing algorithms to solve such implicit classes. Finally, we believe that the findings of this paper will broaden the application and popularity of the DTM.

Data Availability

This manuscript includes all the data used to support its findings.

Funding

The author received no financial support for the research and the publication of this article.

Conflicts of Interest

The author declares that no conflicts of interest exist regarding the publication of this manuscript.

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