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https://doi.org/10.32388/X89HNM

Spanning Graph approach for AHP, BWM, DEMATEL and SWARA: Normality of Judgments

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Abstract. An approach for estimating the weight and estimating the consistency of the pairwise comparison matrix using a set of spanning graphs is proposed. The structure of the set of spanning graphs of the decision matrix for AHP, BWM, DEMATEL and SWARA processes is shown. Consistency of judgments is defined through the distribution of weight coefficients calculated on the set of spanning graphs, as the value of the standard deviation.

Keywords: Pairwise Comparison Matrix, Spanning Graph, AHP, BWM, DEMATEL, SWARA, Consistency index

Introduction

Pairwise Comparisons (PCs) of alternatives (objects, criteria) are among the most common tools for making decisions on multiple criteria. PCs are an integral part of the analytical hierarchy process (AHP) [1], methods for assessing the significance of criteria, such as the Best Worst Method (BWM) [2], methods for identifying the components of the causal chain (DEcision MAking Trial and Evaluation Laboratory, DEMATEL) [3], and Step-wise Weight Assessment Ratio Analysis (SWARA) method [4].

In practice, comparisons are never or almost never made directly, but through comparisons with a standard scale. Therefore, the validity of this decision tool depends on the choice of scale for pairwise comparisons. In AHP, the decision maker (DM) first gives linguistic pairwise comparisons, then receives numerical pairwise comparisons by choosing a specific numerical scale to quantify them, and finally derives a priority vector from the numerical pairwise comparisons. At all three stages, there are various choices and various methods with varying degrees of validity, which determines the design of the AHP model and is the subject of scientific discussions [5–10].

For the stage of choosing a linguistic scale, there is no consensus on the number of gradations or shades for pairwise comparison of alternatives, and this choice is justified in psychological studies, for example, [11]. The generally accepted approach to choosing the number of gradations is to determine the minimum perceived difference in the intensity of two exciters (stimuli), and the expert is actually able to distinguish only a certain limited number of its gradations.

At the stage of choosing a numerical scale for obtaining a numerical matrix of pairwise comparisons, there are also a large number of different options, such as the Saaty scale [1], the Ma-Zheng scale [12], the Geometrical scale [13–15], Salo-Hämäläinen scale [16], etc.

The process of obtaining the priority vector of objects from the numerical matrix of pairwise comparisons is also multivariate. There are a large number of methods for processing pairwise comparisons for prioritization [17, 18], among which the Eigenvector Method (EVM) [1] and the Logarithmic Least Squares Method (LLSM) [19] are the most commonly used.

Consistency of the judgments reflected in the matrix of paired comparisons is hardly the only sign of the adequacy of the weighing procedure. For the correct formation of the matrix of pairwise comparisons, criterial measures of the degree of consistency are used in the form of consistency indices. Several variants of such indices have been introduced into consideration:

– the Consistency index (CI) and the Consistency ratio (CR) by Saaty [1];
– Index KI by Koczkodaj [20];
– Index AI by Salo and Hämäläinen [16, 21];
– Index CI_H by Wu and Xu [22];
– Cosine consistency index (CCI) by Kou and Lin [23] et al. [24].

Multivariance is due to the fact that one number must reflect the degree of consistency of a large number of judgments reflected in the matrix of paired comparisons. For each index, an area of criterion values is determined, indicating the degree of consistency of the matrix of paired comparisons. If the consistency of the pairwise comparison matrix is low, the judgment needs to be reconsidered.

In this study, the author develops the notion of a spanning graph of a pairwise comparison matrix introduced in the fundamental work by T. Saaty [1]. The structure of the set of spanning graphs of the decision matrix for AHP, BWM, DEMATEL and SWARA processes is shown. Consistency of judgments is defined through the distribution of weight coefficients calculated on the set of spanning graphs, as the value of the standard deviation. Now the degree of inconsistency of judgments has a natural interpretation in the form of the amount of dispersion in the distribution of weight coefficients.

To be able to reproduce the results and facilitate subsequent research, in the appendix the author presented the main algorithms of the approach described, implemented in MatLab. The author will be very grateful if one of the researchers writes a recursive function for generating a set of spanning graphs for an arbitrary value of n based on the algorithm given by the author.

1. Methodology

It is assumed that the reader is familiar with the technique and problems of determining the weight of criteria based on the pairwise comparison matrix within the framework of the analytical hierarchy process (AHP) [1], the best-worst method (BWM) [2], the DEMATEL approach [3], and the method of Step-wise Weight Assessment Ratio Analysis (SWARA) [4].

1.1. General scheme of pairwise comparisons

The decision maker first gives linguistic pairwise comparisons in the selected scale of gradations, then receives numerical pairwise comparisons, choosing a certain numerical scale for their quantitative assessment, and finally derives a priority vector from numerical pairwise comparisons. The general scheme of the procedure for pairwise comparisons of alternatives in AHP (BWM, DEMATEL) is shown in Figure 1.

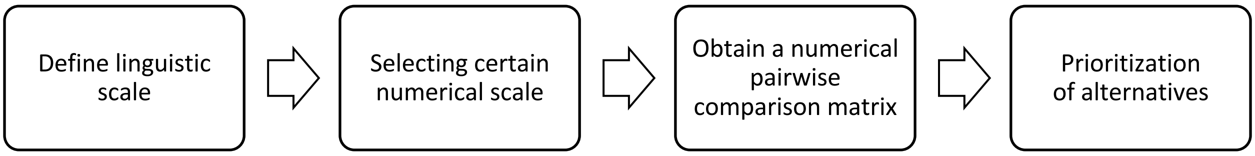

Fig. 1. General scheme of the procedure of pairwise comparisons

1.2 Paired comparisons and prioritization in AHP

The Pairwise Comparison matrix in AHP determines the relative importance of criteria w_i (i=1,…, n):

$$A = \begin{pmatrix} \frac{w_1}{w_1} & \frac{w_1}{w_2} & \cdots & \frac{w_1}{w_n} \\ \frac{w_2}{w_1} & \frac{w_2}{w_2} & \cdots & \frac{w_2}{w_n} \\ \cdots & & & \\ \frac{w_n}{w_1} & \frac{w_n}{w_2} & \cdots & \frac{w_n}{w_n} \end{pmatrix} \tag{1}$$

The ratio $w_i/w_j=a_{ij}$ is not a quantitative degree of superiority, but a conditional one.

A well-known example [25] of examination using judgements is presented in Table 1.

Table 1. Which drink is consumed more in the USA?

	Coffee	Wine	Tea	Beer	Sodas	Milk	Water
Coffee	1	9	5	2	1	1	1/2
Wine	1/9	1	1/3	1/9	1/9	1/9	1/9
Tea	1/5	3	1	1/3	1/4	1/3	1/9
Beer	1/2	9	3	1	1/2	1	1/3
Sodas	1	9	4	2	1	2	1/2
Milk	1	9	3	1	1/2	1	1/3
Water	2	9	9	3	2	3	1

When different scales are used for PCs, the same value from a given scale has a different meaning, and is known as the “pairwise comparison rating scale paradox” (Koczkodaj, [26–28]).

The prioritization method refers to the process of deriving a priority vector from a numeric pairwise comparison matrix. The two most common prioritization methods (EVM and LLSM) [1, 18, 19] are presented below.

1) Eigenvector Method is to take as weights the components of the eigenvector of the matrix A corresponding to the largest eigenvalue λ_{max}:

$$A \cdot \overline{w} = \lambda_{\max} \cdot \overline{w}, \tag{2}$$

This vector is normalized.

Consistency ratio (CR) by Saaty:

$$CI = \frac{\lambda_{\max} - n}{n-1},\ CR = \frac{CI}{RI} \tag{3}$$

Where *RI* is the average value of *CI* for random matrices using the Saaty scale. We accept a matrix as a consistent one iff $CR < 0.1$.

2) The Logarithmic Least Squares Method (LLSM) uses the L_2 metric to determine the objective function of the following optimization problem::

$$\min \sum_{i=1}^{n} \sum_{j>i} (\ln a_{ij} - \ln w_i - \ln w_j)^2, \tag{4}$$

$$s.t.\ \ w_i \geq 0, \quad \sum_{i=1}^{n} w_i = 1. \tag{5}$$

This solution can be found as the geometric mean of rows [19] and is equivalent to the method Geometric Mean Method (GMM):

$$w_i = \frac{(\prod_{j=1}^{n} a_{ij})^{1/n}}{\sum_{i=1}^{n} (\prod_{j=1}^{n} a_{ij})^{1/n}}. \tag{6}$$

1.3 Pairwise comparisons and prioritization in BWM

The Best-Worst Method performs paired comparisons of the best and worst criteria compared to other criteria. As a result, we have two preference vectors:

$A_B=(a_{B1}, a_{B2},\ldots, a_{Bn})$, $(a_{BB}=1)$ — Best-to-Other, (7)

$A_W=(a_{1W}, a_{2W},\ldots, a_{nW})$, $(a_{WW}=1)$ — Other-to-Worst. (8)

Numerical pairwise comparisons in the basic version of BWM are implemented by the author of the method [2] in the Saaty numerical scale with estimates of quantitative importance from 1 to 9.

To determine the priority vector, the following optimization problem is solved:

$$\min \xi^L, \tag{9}$$

$$s.t.\ \left| w_B - a_{Bj} w_j \right| \leq \xi^L,\ \forall j, \tag{10}$$

$$\left| w_j - a_{jW} w_W \right| \leq \xi^L, \forall j, \tag{11}$$

$$w_j \geq 0, \quad \sum_{j=1}^{n} w_j = 1. \tag{12}$$

1.4 Pairwise comparisons and prioritization in DEMATEL

The DEMATEL method uses a direct influence graph that expresses the mutual influence of the analyzed objects through cause-and-effect relationships. As in the AHP method, structural relationships arise between the analyzed elements, which served as a prerequisite for using DEMATEL when weighing criteria.

One of the famous AHP transformations in DEMATEL [3] uses the following technique:

– direct relation matrix $B=(b_{ij})_{n\times n}$ is a square matrix, the size of which is equal to the number of objects. Its rows correspond to the objects appearing first in the comparison.

– elements of the main diagonal are equal to zero, and non-zero elements b_{ij} ($i \neq j$), reflect the impact of the i-th object on the j-th object. When choosing an m-point scale, the direct influence matrix and the pairwise comparison matrix in AHP are related by:

$$b_{ij} = (a_{ij} - 1),\ \forall\ i, j\!: a_{ij} \geq 1, \tag{13}$$

$$b_{ij} = 0,\ \forall\ i, j\!: a_{ij} < 1. \tag{14}$$

Weight coefficients in DEMATEL are defined as follows [3]:

$$\widehat{B} = B / \max_i \left(\sum_{j=1}^{n} b_{ij}\right), \tag{15}$$

$$T = \widehat{B} \cdot (I - \widehat{B})^{-1}, \tag{16}$$

$$t_i^+ = \sum_{j=1}^{n} t_{ij} + \sum_{j=1}^{n} t_{ji},\quad t_i^- = \sum_{j=1}^{n} t_{ij} - \sum_{j=1}^{n} t_{ji},\quad \hat{t}_i = \frac{1}{2}(t_i^+ + t_i^-) = \sum_{j=1}^{n} t_{ij}\ , \tag{17}$$

as one of the possibilities, it is proposed [3] to determine the weights proportional to the average value of the corresponding pair of indicators t^+ и t^-:

$$w_i = \hat{t}_i / \sum_{i=1}^{n} \hat{t}_i\ . \tag{18}$$

where I is the identity matrix of dimension $n\times n$.

1.5 Pairwise comparisons and prioritization in SWARA

Stepwise Weight Assessment Ratio Analysis is a "direct" method of assigning weights based on quantitative priority in an ordered sequence [4]. (n–1) independent relations $w_i / w_j = q_k$ closes the normalization condition. A feature of the SWARA method in a procedure convenient for the decision maker:

- the criteria are ordered in descending order of their expected significance;
- the significance of the first criterion in the list is equal to 1;
- starting from the second criterion, the respondent expresses the relative importance of criterion j in relation to the previous (j–1) criterion ($w_1/w_2 = q_1$, $w_2/w_3 = q_2$, …, $w_{n-1}/w_n = q_{n-1}$).

If we take w_1=1, then $w_k = w_{k-1} / q_{k-1}$, $k = 2, \ldots, n$. (19)

Superiority values are well associated with percentages, which is very convenient, for example, w_1/w_2 =1,15 means that the quantitative significance of w_1 exceeds w_2 by 15%. Ordering is also important, because when comparing, there is no need to remember the background and track the transitivity of judgments. Normalization of weights closes the system of linear equations:

$$\overline{w}_j = \frac{w_j}{\sum_{j=1}^{n} w_j}, \ \forall j = 1, \dots, n. \tag{20}$$

The method is deterministic. Setting a quantitative priority uniquely determines the weights, and vice versa.

Obviously, any system of (n–1) independent relations w_i/w_j and the normalization condition are equivalent to SWARA.

2. Evaluation of the consistency of the matrix of pairwise comparisons

2.1 Spanning Graph approach

In the general case, if the condition of "cardinal" consistency of PCs of the matrix A ($a_{ij} \cdot a_{jk} \neq a_{ik}$, $\forall i, j, k$) violated for any row of A, we can determine the priority vector. This also holds for any set of entries whose graph is a spanning cycle of the graph of the matrix.

The spanning graph of the matrix A: any row of the matrix A or any set of elements taken one by one from each column of the matrix A without taking into account the repetition of reciprocal elements (a_{ij}=1/a_{ji}).

The scheme for selecting a spanning graph for AHP is shown in Figure 2 for the case n=5.

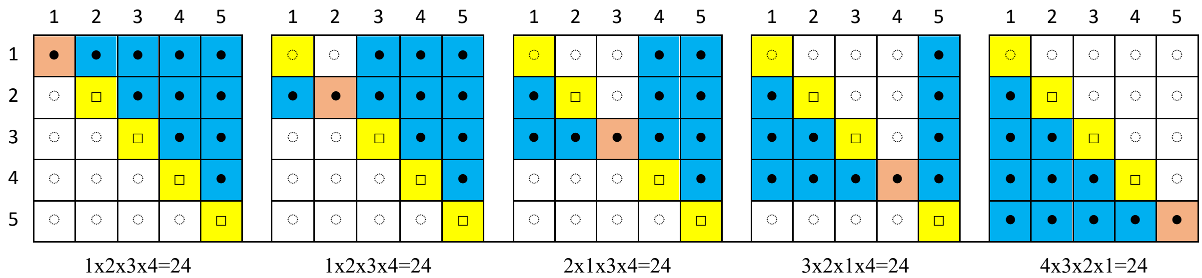

Fig. 2. Spanning graph selection scheme for AHP

Each diagonal element a_{ii} (brown) is combined with one of the elements of each column a_{ij} (blue) in a graph of n elements. Thus, n! spanning graphs. Each spanning graph is an n-dimensional vector and is a collection of w_i/w_j ratios. The normalization condition closes this system of relations, which makes it possible to determine the weights. The solutions differ by a multiplicative constant. However, normalization results in a unique solution no matter which graph is used.

Example 2:

	Spanning Graph (iXY)	Calculation
P=[1 1/4 1/6 1/4 1/8 4 1 1/3 3 1/7 6 3 1 4 1/2 4 1/3 1/4 1 1/7 8 7 2 7 1];	iXY=(21 22 13 24 15), $q=(p_{21}\ p_{22}\ p_{13}\ p_{24}\ p_{15})=$ (4 1 1/6 3 1/8) $w_i/w_j=p_{ij}$	$w_2=1$, $w_2/w_1=4 \Rightarrow w_1=1/4$, $w_1/w_3=1/6 \Rightarrow w_3=3/2$, $w_2/w_4=3 \Rightarrow w_4=1/3$, $w_1/w_5=1/8 \Rightarrow w_5=2$, After Normalization: w=(0,049 0,197 0,295 0,066 0,393)

The general algorithm for determining the set of spanning graphs based on the pairwise comparison matrix is given in Appendix A.

Each spanning graph reflects one of the possible interactions of a set of n objects.

Thus, $N=n!$ weight coefficients $\boldsymbol{w}^{(k)}=(w_1^{(k)}, w_2^{(k)}, \ldots, w_n^{(k)})$.

Each set of weight coefficients reflects a certain inconsistency in the decision maker's preferences.

As a result, we have a spectrum of N weight values for each object.

For example 2, there are 120 spanning graphs, for each of which the corresponding set of weight coefficients is defined:

```
 # Pij elements    Weight
-- ------------ -----------------------
1: 11, 12, 13, 14, 15,    0.043, 0.174, 0.261, 0.174, 0.348,
 2: 11, 12, 13, 14, 25,   0.023, 0.093, 0.140, 0.093, 0.651,
 3: 11, 12, 13, 14, 35,   0.037, 0.148, 0.222, 0.148, 0.444,
…
 23: 11, 12, 23, 34, 35,   0.023, 0.091, 0.273, 0.068, 0.545,
 24: 11, 12, 23, 34, 45,   0.024, 0.098, 0.293, 0.073, 0.512,
 25: 21, 22, 13, 14, 15,   0.043, 0.174, 0.261, 0.174, 0.348,
…
 27: 21, 22, 13, 14, 35,   0.037, 0.148, 0.222, 0.148, 0.444,
…
 71: 31, 32, 33, 34, 35,   0.044, 0.089, 0.267, 0.067, 0.533,
 72: 31, 32, 33, 34, 45,   0.048, 0.095, 0.286, 0.071, 0.500,
 73: 21, 32, 43, 44, 15,   0.036, 0.143, 0.429, 0.107, 0.286
…
118: 51, 42, 53, 54, 55,   0.057, 0.195, 0.228, 0.065, 0.455,
119: 51, 52, 43, 54, 55,   0.063, 0.072, 0.288, 0.072, 0.505,
120: 51, 52, 53, 54, 55,   0.065, 0.075, 0.262, 0.075, 0.523,
Mean:                      0.034  0.123  0.276   0.074  0.493
-----------------------------------------------------------------------
```

The distributions of the values of the weight coefficients of various objects (Drinks) for the decision matrix of the example in Table 1 are shown in the Figure 3.

To isolate the "true" solution in such situations, averaging of the results is used. As you know, the mean value is the optimal solution that minimizes the square of the deviation:

$$\sum_{k=1}^{N}\left\|\boldsymbol{w}^{(k)}-\boldsymbol{w}\right\|=\sum_{k=1}^{N}\sum_{i=1}^{n}(w_i^{(k)}-w_i)^2 \rightarrow \min\,. \qquad (21)$$

After finding the average values for each component, re-normalization is required.

Let's denote the proposed approach as MSG (Mean of Spanning Graphs).

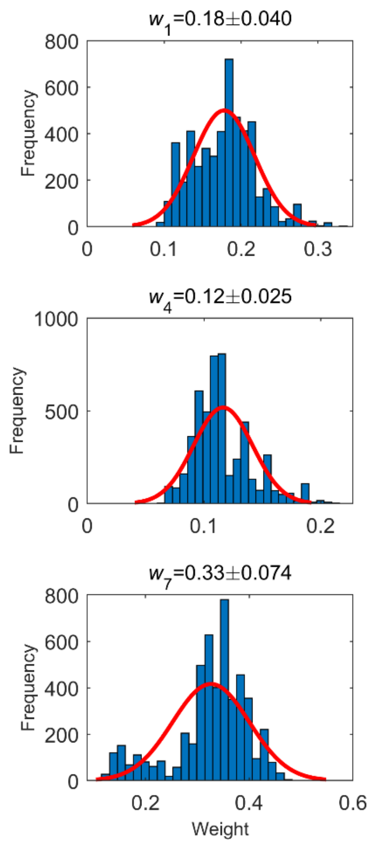

Fig. 3. The distributions of the values of the weight coefficients of various objects for the decision matrix of the example according to the Table 1.

The main advantage of MSG is the natural interpretation of the consistency of judgments of the decision maker, which is expressed by the distribution of weight coefficients. Now, as a measure of consistency, you can use the amount of dispersion of weight values from the average - the standard deviation. The smaller the standard deviation, the more consistent the pairwise comparison matrix.

The results of the MSG weight calculation in comparison with the weighting methods EVM (Eigenvector Method), LSM (Least Squares Method) and GMM (Geometric Mean Method) are presented in Table 2.

Table 2. Weights of objects of example 1 obtained in various prioritization methods

	w_1	w_2	w_3	w_4	w_5	w_6	w_7
AHP EVM (*CR*=0,022)	0,177	0,019	0,042	0,116	0,190	0,129	0,327
AHP LSM	0,178	0,021	0,039	0,111	0,190	0,121	0,340
AHP GMM (LLSM)	0,179	0,018	0,042	0,116	0,191	0,129	0,324
AHP MSG	0,178	0,017	0,043	0,116	0,194	0,126	0,326
standart deviation, *s*	0,040	0,005	0,009	0,025	0,039	0,031	0,074

The basis of the MSG method is the lack of "cardinal" consistency of the PCs matrix due to the following objective and subjective factors:

1) the numerical scale of paired comparisons is discrete and limited. Indeed, if the quantitative importance is $a_{12} = 3$ and $a_{23} = 5$, then in the case of a consistent judgment, $a_{13} = 15$. However, for example, for a 9 point scale, the closest value to 15 is 9. We will have to accept $a_{13} = 9$,

2) transitivity of judgments is not always observed. If, for example, the relative importance of criterion C_1 is greater than that of C_2, and the relative importance of C_2 is greater than C_3, then the relative importance of C_1 need not be greater than the importance of C_3,

3) in reality, people's feelings and preferences remain inconsistent, since people's feelings do not correspond to an exact formula.

Hypothesis: The weight distribution of each criterion is Normal

Significant deviations from normality (multimodality, skewness and kurtosis in distribution, fat tails) should indicate a conflict in comparisons of different pairs or groups of objects.

2.2 Consistency of CR score and standard deviation in MSG

Let's perform a numerical experiment to determine the consistency of the CR indicator in AHP and the standard deviation in MSG. To do this, we scale the numerical scale of pairwise comparisons used at stage 2 (see the diagram in Fig. 1). It is a well-known fact (see, for example, [29]) that shrinking the scale $[1, m]$ leads to:

$\lambda_{max} \rightarrow n$ and $CR \rightarrow 0$.

The task is to determine the consistency of the CR and standard deviation in MSG when scaling. To do this, we use the Pearson correlation between CR and std(w_i).

The linear scaling of the F scale must satisfy the conditions:

1) F: $[1, m] \rightarrow [1, t]$, $m \in N$, $t \in R$,
2) F is strictly increasing,
3) $F(1) = 1$, $F(m) = t$.

Simple expansion-contraction using a fixed point allows you to do this conversion:

$$F(x) = (x-1)\cdot(t-1)/(m-1) + 1 \tag{22}$$

Note: the linear transformation does not preserve the transitivity of judgments: the condition $a_{ij} \cdot a_{jk} = a_{ik}$ does not imply that $F(a_{ij}) \cdot F(a_{jk}) = F(a_{ik})$, $\forall i, j, k$.

However, strict transitivity can be neglected if the consistency of judgments in the matrix of pairwise comparisons is maintained.

The elements of the matrix of paired comparisons are transformed for $a_{ij}>1$ according to the formula (22) for $x=a_{ij}$

$$a^*_{ij} = F(a_{ij}) = (a_{ij}-1)\cdot(t-1)/(m-1)+1 \tag{23}$$

For other values, we use the principle of reciprocity: $a^*_{ji} = 1/a^*_{ij}$.

The main problem of choosing a numerical scale is to determine the quantitative superiority of one object of comparison over another. If in the Saaty scale (m=9) the linguistic term "extremely more important" corresponds to the numerical value 9, then in the new scale at t=2 the same term will correspond to the value 2. As the author of AHP pointed out, it should not be taken literally that the quantitative superiority of one criterion over another will be 9 or 2 times. This is just a conditional indicator for the subsequent assessment of the weight of the criteria, and the choice of the scale is determined by the correspondence to empirical data. A formal description of the quantitative importance of criteria, based on a strict definition of the concept "One criterion is more important than another in so many times" was given by V. Podinovsky [30, 31].

It is unequivocally true that contraction of the scale [1, m] leads to CR→0.

Figure 4 shows the dynamics of the degree of consistency of the PCs matrix and the standard deviation when the numerical scale Saaty (m=9) is compressed from 1 (t=9) to 7.5 (t=1,2) times for the problem of example 1. The scale compression ratio is equal to m/t.

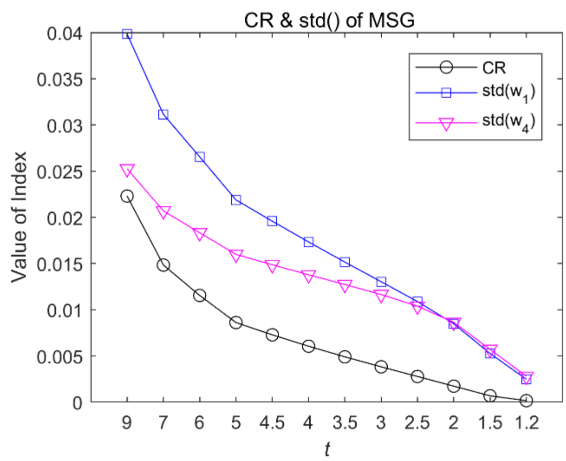

Fig. 4. Dynamics of the degree of consistency ratio (CR) of PCs matrix and the standard deviation when compressing the numerical scale from 1 (t=9) to 8 (t=1,2) times for the task of example 1.

The Pearson correlation values between CR and std(w_i) are respectively:
0.985, 0.115, 0.711, 0.958, 0.994, 0.942, 0.980.

Correlations close to 1 indicate consistency between the CR score and standard deviation in MSG. The low values of correlations corr(CR,w_2) are due to the small value of std(w_2) and the low variability of the standard deviation when the scale is compressed (Fig. 3).

Thus, the numerical experiment indicates the correctness of using the standard deviation defined in MSG as a measure of the consistency of the PCs matrix.

2.3 Sorting of objects of analysis and matrix of paired comparisons

The ordering of the criteria in the SWARA schema is a useful prioritization tool for the expert, because when comparing, there is no need to remember the background and it is easier to track the transitivity of judgments. It is obvious that when the sequence of the list of criteria is changed, the result of the evaluation of the weight coefficients will not change.

Each of n! the system of n independent ratios w_i/w_j and the normalization condition are equivalent to SWARA. Therefore, we re-sort both the objects and the corresponding system of spanning graphs.

If the researcher has at his disposal a matrix of paired comparisons on an unsorted list (as in example 2), then when changing the sequence of criteria, it is necessary to re-sort the matrix of pairwise comparisons according to a simple algorithm:

```
%-- input P0 -Pairwise Comparison Matrix
P0=[1 1/4 1/6 1/4 1/8
    4 1   1/3 3   1/7
    6 3   1   4   1/2
    4 1/3 1/4 1   1/7
    8 7   2   7   1];
n=size(P0,2);
for i=1:n  %-- iXY0-index Matrix for P0
     for j=1:n
          iXY0(i,j)=10*i+j;
     end
end
%-- EigenVector w0  for max EigenValue:
w0=eigs(P,1);
%-- new selection of criteria order ps
[w1 ps]=sort(w0,'descend');
[P iXY]=Fun_Seq(P0,p);
```

```
function [P iXY]=Fun_Seq(P0,p)

n=size(P,2);
for i=1:n
     for j=1:n
          iXY(i,j)=10*p(i)+p(j);
          P(i,j)=P0(p(i),p(j));
     end
end
%  iXY-index of Matrix for new PCs matrix
```

```
>>
 P=[ 1    2    7    7    8
    1/2  1    3    4    6
    1/7  1/3  1    3    4
    1/7  1/4  1/3  1    4
    1/8  1/6  1/4  1/4  1]
```

```
iXY=[55, 53, 52, 54, 51,
     35, 33, 32, 34, 31,
     25, 23, 22, 24, 21,
     45, 43, 42, 44, 41,
     15, 13, 12, 14, 11,]
```

To sort the list in the above algorithm, the MatLab functions are used: eigs() and sort().

After sorting, the pairwise comparison matrix in AHP is such that all elements above the main diagonal are >1 and all elements below the main diagonal are <1. Additionally, the elements in each row form a non-decreasing list, and the elements in each column a non-increasing list.

2.4 Spanning Graph for BWM, DEMATEL and SWARA

The set of spanning graphs in the BWM method for a list of criteria sorted in descending order of weight is formed according to the following scheme (Fig. 5) and represents $2{\cdot}(n–1)!$ sets of n elements, taken one from each column.

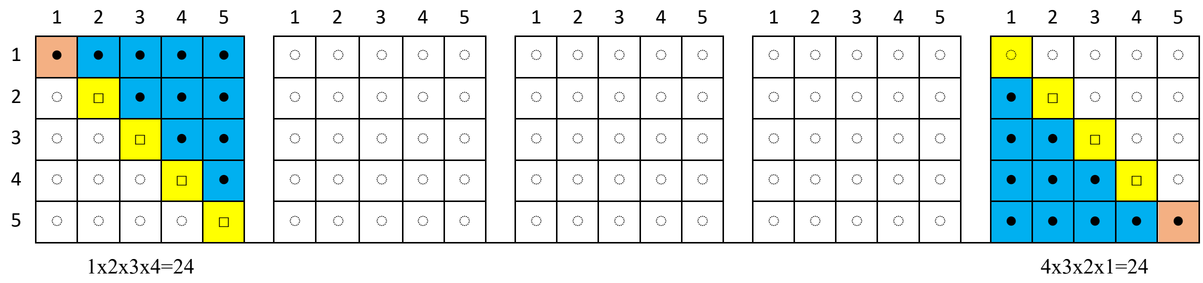

Fig. 5. Scheme for selecting spanning graphs in the BWM method

In the general algorithm for determining the set of spanning graphs based on the pairwise comparison matrix given in Appendix A, the set of spanning graphs of the BWM method is the first and last $(n–1)!$ sets of the general list of spanning graphs:

```
iXY(i,1:n)),  i=1,…,K & i=(n-1)·K+1,…, N, N=n!, K=(n-1)!
```

The set of spanning graphs in the DEMATEL method for a list of criteria sorted in descending order of weight is formed according to the following scheme (Fig. 6) and represents $(n\text{-}1)!$ sets of n elements, taken one from each column.

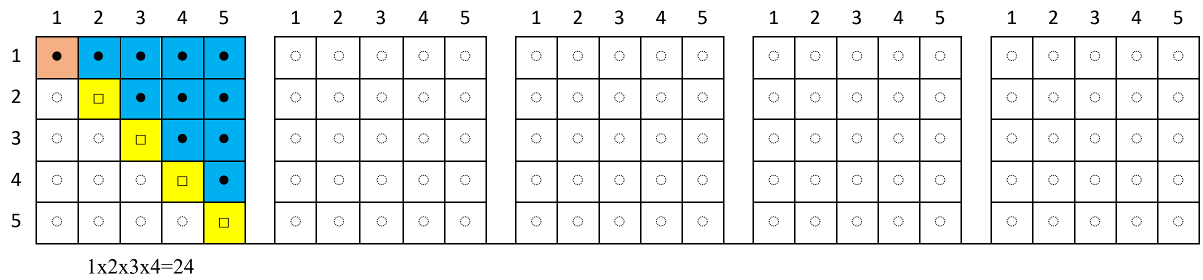

Fig. 6. Scheme for selecting spanning graphs in the DEMATEL method

In the general algorithm for determining the set of spanning graphs based on the pairwise comparison matrix given in Appendix A, the set of spanning graphs of the DEMATEL method is the first $(n–1)!$ sets of the general list of spanning graphs:

```
iXY(i,1:n)),  i=1,…,K, K=(n-1)!
```

It is easy to see that the SWARA scheme uses only one of the possible spanning graphs, namely:

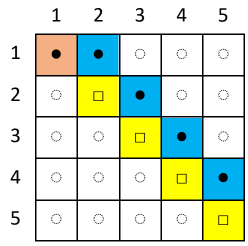

Fig. 7. Scheme for selecting spanning graphs in the SWARA method

Thus, the authors of [4] of the SWARA approach excluded preference errors from consideration and determined the procedure for directly assigning weights to decision makers.

3 Numerical example of weight estimation using MSG method within AHP, BWM, DEMATEL and SWARA

Below (Table 3 and Table 4) are sequential results of weight estimation using the MSG approach for the example 1 task in the AHP, BWM, DEMATEL and SWARA methods.

The results of all methods (except SWARA) are in good agreement.

Table 3. Results of successive steps of the MSD method

PCs	iXY, *ps*=(7, 5, 1, 6, 4, 3, 2)	P1
1 9 5 2 1 1 1/2 1/9 1 1/3 1/9 1/9 1/9 1/9 1/5 3 1 1/3 1/4 1/3 1/9 1/2 9 3 1 1/2 1 1/3 1 9 4 2 1 2 1/2 1 9 3 1 1/2 1 1/3 2 9 9 3 2 3 1	77 75 71 76 74 73 72 57 55 51 56 54 53 52 17 15 11 16 14 13 12 67 65 61 66 64 63 62 7 45 41 46 44 43 42 37 35 31 36 34 33 32 27 25 21 26 24 23 22	1 2 2 3 3 9 9 1/2 1 1 2 2 4 9 1/2 1 1 1 2 5 9 1/3 1/2 1 1 1 3 9 1/3 1/2 1/2 1 1 3 9 1/9 1/4 1/5 1/3 1/3 1 3 1/9 1/9 1/9 1/9 1/9 1/3 1
BWM	DEMATEL, *B*=	SWARA
Best-to-Other, (1st row of P1) A_B =(1 2 2 3 3 9 9) Other-to-Worst, (n^{th} col of P1) A_W =(9 9 9 9 9 3 1)	0 1 1 2 2 8 8 0 0 0 1 1 3 8 0 0 0 0 1 4 8 0 0 0 0 0 2 8 0 0 0 0 0 2 8 0 0 0 0 0 0 2 0 0 0 0 0 0 0	*q*=(1 2 1 1 1 3 3) w_1 =1, $w_k = w_{k-1} / q_k,\ k = 2, \ldots, n.$ *w*=(1 0,5 0,5 0,5 0,5 1/6 1/6)

Table 4. Weighing result and relative error

Weighing method	w_1	w_2	w_3	w_4	w_5	w_6	w_7
AHP EVM (*CR*=0,022)	0,327	0,190	0,177	0,129	0,116	0,042	0,019
AHP MSG	0,349	0,189	0,169	0,125	0,111	0,040	0,019
standart deviation, *s*	0,046	0,033	0,031	0,026	0,022	0,008	0,009
BWM	0,297	0,191	0,191	0,127	0,127	0,042	0,024
BWM MSG	0,338	0,185	0,176	0,127	0,116	0,041	0,017
standart deviation, *s*	0,062	0,035	0,033	0,026	0,023	0,007	0,007
DEMATEL	0,309	0,180	0,175	0,135	0,135	0,044	0,022
DEMATEL MSG	0,355	0,177	0,177	0,126	0,105	0,039	0,020
standart deviation, s	0,020	0,010	0,010	0,030	0,020	0,006	0,009
SWARA	0,3	0,15	0,15	0,15	0,15	0,05	0,05

Relative error	(x-y)/x·100%						
BWM vs AHP EVM, %	9,2	0,5	7,9	1,6	9,5	0,0	26,3
DEMATEL vs AHP EVM, %	5,5	5,3	1,1	4,7	16,4	4,8	15,8
SWARA vs AHP EVM, %	8,3	21,1	15,3	16,3	29,3	19,0	163,2
AHP MSG vs AHP EVM, %	6,7	0,5	4,5	3,1	4,3	4,8	0,0
BWM MSG vs AHP EVM, %	3,4	2,6	0,6	1,6	0,0	2,4	10,5
DEMATEL MSG vs AHP EVM, %	8,6	6,8	0,0	2,3	9,5	7,1	5,3

The distributions of the values of the weight coefficients of the first object (after resorting it is Water), calculated on the basis of the MSG method for the matrix of pairwise comparisons according to the Table 1 are shown in Figure 8.

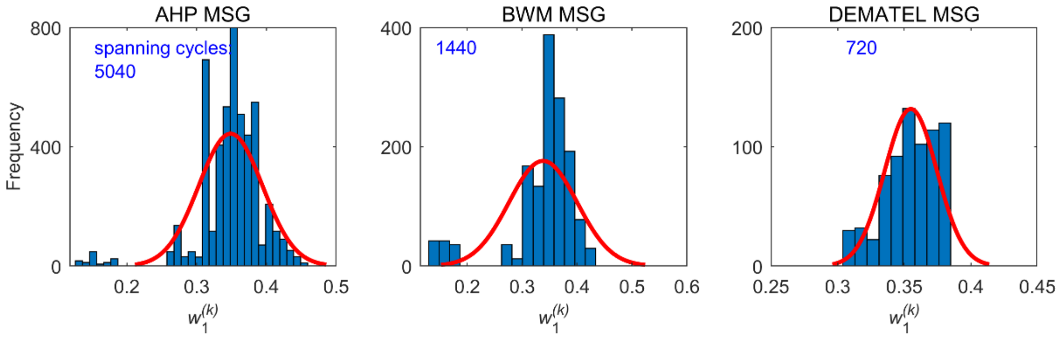

Fig. 8. The distributions of the values of the weight coefficients of the first object calculated on the basis of the MSG method for the matrix of pairwise comparisons according to Table 1.

4 Discussion

In the discussion, we will try to answer the question about the effectiveness of the MSG method.

According to Table 3, the results of all methods (except SWARA) are in good agreement, the discrepancy is about 5% on average. The strong discrepancy for the 7th object is due to the small value of the weight and low variability.

The MSG weighting result is in line with AHP EVM and its peers, but its advantage cannot be determined due to the lack of performance criteria.

The MSG approach is useful because it reveals the causal relationship between the pairwise comparison matrix and the decision through the distribution of weights computed on the set of spanning graphs. Now the degree of inconsistency of judgments has a natural interpretation in the form of the amount of dispersion in the distribution of weight coefficients.

For the AHP, BWM, DEMATEL methods, the amount of information for estimating the weights is redundant. Therefore, these methods belong to evaluation methods and use "soft calculations" when finding a solution, based on minimization with an excessive number of connection equations. In the SWARA-method, the initial ratios w_i/w_j are quantitative, and mean how many times the i-th criterion is preferable to the j-th one.

Unlike AHP, only $2·(n–1)$ pairwise comparisons are performed in the BWM-method, which makes the task of the decision maker easier. However, it is not clear whether the reduction in information content is sufficient for the correctness of the results. As in the previous case, the advantage of the method cannot be determined due to the lack of performance criteria.

DEMATEL for evaluating the weights of the criteria, according to the author, is not justified due to the fact that the intensity of the impact of the *i*-th object on the *j*-th object and the preference of the *i*-th object over the *j*-th object are two different categories. Although there is no explicit reverse effect in the matrix, in the weighting method, the reverse effect takes place in the formula (16): $T=B\cdot(I-B)^{-1}$. The information content of the direct influence matrix in DEMATEL is significantly lower than the information content of the PCs matrix in AHP. A reasonable question is why apply the worst version of the method.

The SWARA method is a "direct" method of assigning weights based on quantitative priority in an ordered sequence. Obviously, any system of (n-1) independent ratios wi/wj and the normalization condition are equivalent to the SWARA procedure, but give a different weighting result. Setting a quantitative priority uniquely determines the weights, and vice versa. It follows from this that it is possible to derive a priority vector using objective weighting methods using the information contained in the decision matrix (Entropy, CRITIC, SD methods [32]). The author repeatedly carried out such experiments:

1) order the criteria, set the priority vector, determine the weights (SWARA),

2) we determine the decision matrix, determine the weights (for example, CRITIC) and recalculate the priority vector using SWARA.

Result: the discrepancy is catastrophic.

Why not use a deterministic preference-based method (eg SWARA) to evaluate criteria weights? Because people's preferences remain inconsistent. Any row (column) of the pairwise comparison matrix in AHP, in combination with the normalization condition, represents a closed system and is sufficient to directly determine the weights of the criteria. However, it is easy to see that these rows (columns) are only partially consistent, although they represent preferences between the same criteria, but in a different sequence.

5 Conclusions

The MSG approach reveals the causal relationship between the pairwise comparison matrix and the decision through the distribution of weight coefficients calculated on the set of spanning graphs. The degree of inconsistency of judgments has a natural interpretation in the form of a scattering value in the distribution of weight coefficients. MSG needs a qualitative theoretical justification, which may constitute a direction for further research. According to the author, the solution to the problem of comparing weighting methods lies in the field of information theory.

Appendix A. MSG-method. MatLab code

```
%-- MSG procedure
%-- by Prof. Irik Z. Mukhametzyanov
%-- Dec.21 2022, Ufa, USPTU

clear all
%-- Pairwise Comparison Matrix PCs:
 P=[1  1/4  1/6 1/4 1/8
    4  1    1/3 3   1/7
    6  3    1   4   1/2
    4  1/3  1/4 1   1/7
    8  7    2   7   1];
n=size(P,2);
in1=ones(n,n-1);
in2=repmat([1:n-1],n,1);
for k=2:n
    for j=1:k-1
        in1(k,j)=1+j;
        in2(k,j)=k;
    end
end
n1=in1'; n2=in2';

%- only for n=5 (for other n similarly)
%  who can write a recursive function?
ij=0;
if n==5
 for k=1:n
  R(k)=10*k+k;
  p=setdiff([1:n],[k]);
  for i1=n1(1,k):n2(1,k)
     for i2=n1(2,k):n2(2,k)
       for i3=n1(3,k):n2(3,k)
         for i4=n1(4,k):n2(4,k)
             R(p(1))=10*i1+p(1);
             R(p(2))=10*i2+p(2);
             R(p(3))=10*i3+p(3);
             R(p(4))=10*i4+p(4);
             ij=ij+1;
             iXY(ij,:)=R;
         end
       end
     end
  end
 end
end
%------------------------------------
m=size(iXY,1);
iX=floor(iXY/10);
iY=iXY-iX*10;
for ii=1:m
  for i=1:n
    G(ii,i)=P(iX(ii,i),iY(ii,i));
  end
end
```

```
% Calculation of weights for each
% Spanning Graph:

K=factorial(n-1);
for k=1:n
  q(k)=1;
  for ii=K*(k-1)+1:K*k    %--rows G
%-- left of diagonal element to element
    in 1 column
   for j=k-1: -1 : 1
       q(j)=q(iX(ii,j))/G(ii,j);
   end
%-- right on diagonal element
    to the element in the n column
   for j=k+1: n
       q(j)=q(iX(ii,j))/G(ii,j);
   end
   w(ii,:)= q/sum(q);
   fprintf('\n%2d: ',ii)
   fprintf('%d, ', iXY(ii,1:n))
   fprintf('%5.3f,', w(ii,1:n))
end
fprintf('\n')


>>
 # Pij elements    Weight
-- ------------ -----------------------
```

1: 11, 12, 13, 14, 15, 0.043, 0.174, 0.261, 0.174, 0.348,
2: 11, 12, 13, 14, 25, 0.023, 0.093, 0.140, 0.093, 0.651,
3: 11, 12, 13, 14, 35, 0.037, 0.148, 0.222, 0.148, 0.444,
…
23: 11, 12, 23, 34, 35, 0.023, 0.091, 0.273, 0.068, 0.545,
24: 11, 12, 23, 34, 45, 0.024, 0.098, 0.293, 0.073, 0.512,
25: 21, 22, 13, 14, 15, 0.043, 0.174, 0.261, 0.174, 0.348,
26: 21, 22, 13, 14, 25, 0.023, 0.093, 0.140, 0.093, 0.651,
27: 21, 22, 13, 14, 35, 0.037, 0.148, 0.222, 0.148, 0.444,
…
71: 31, 32, 33, 34, 35, 0.044, 0.089, 0.267, 0.067, 0.533,
72: 31, 32, 33, 34, 45, 0.048, 0.095, 0.286, 0.071, 0.500,
73: 21, 32, 43, 44, 15, 0.036, 0.143, 0.429, 0.107, 0.286
…
118: 51, 42, 53, 54, 55, 0.057, 0.195, 0.228, 0.065, 0.455,
119: 51, 52, 43, 54, 55, 0.063, 0.072, 0.288, 0.072, 0.505,
120: 51, 52, 53, 54, 55, 0.065, 0.075, 0.262, 0.075, 0.523,
Mean: 0.034 0.123 0.276 0.074 0.493

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