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Comment on "On the linearity of the generalized Lorentz transformation"

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In the article by Verheest (Am. J. Phys. 90, 425429 (2022)), the author presents an original derivation of the one-dimensional generalized Lorentz transformations. We fill in some minor gaps in his derivation and complement his enlightening formulation with a few relevant observations regarding some fundamental concepts.

I. INTRODUCTION

In [1], Verheest presents an original proof for the linearity of the one-dimensional Lorentz transformations. We fill in some minor gaps left in his derivation to facilitate the student a full grasp of Verheest's enlightening approach.

The author also expresses a long-standing concern that some physicists have pointed out about the central role that the speed of light seems to play in the principles of relativity theory. That uneasiness is justified since relativity constitutes a central pillar in the theories of modern physics. Notwithstanding the importance of electromagnetic theory, it seems odd that a particular type of phenomenon should play such a central role. We explain that the crucial role that light purportedly plays in Einstein's principles is only apparent and is owed to historical and practical reasons.

II. THE LIGHT PRINCIPLE

As observed in [1], Einstein based his special theory of relativity on two principles (i) the laws of physics are invariant in all inertial frames of reference, and (ii) the speed of light in vacuum is the same for all inertial observers.

Principle (i) is an extension of the equivalence of inertial reference frames from mechanics to all physical phenomena, while (ii) is also known as the light principle.

In 1905 only two fundamental interactions were known, gravitational and electromagnetic. Newtonian gravity is described by an action at a distance, i.e., instantaneous interaction. On the other hand, light was known to be an electromagnetic phenomenon and possessed a finite speed. That historic prospect explains why Einstein gave light such a dominant role, notwithstanding his first principle encompasses all physical laws.

The tradition of teaching relativity through the light principle continues to this day. As Verheest has observed, from a conceptual viewpoint, it is more compelling to derive the Lorentz transformations without mentioning the speed of light at all. The first to do that was Vladimir Ignatowski, as early as 1910 [2].

If only for didactic purposes, we add the following:

(a) The light principle can be replaced by the more general principle, (ii') the principle of finiteness of the speed of propagation of interactions [3].

(b) As a corollary of (i) and (ii'), we obtained that interactions must propagate with the same speed in all inertial systems. That speed must therefore be a universal constant that establishes the speed of all possible fundamental interactions.

Thus (i) and (ii') can lead us to Lorentz transformations through the usual derivations but replacing light speed with a finite universal speed limit.

Also, as done by Verheest, we can hold only to principle (i), which leaves open the possibility of instantaneous (infinite speed) interactions and Galilean transformations. Then, we would obtain Lorentz transformations only if we assume a finite speed of interactions.

Finally, we point out that instantaneous interactions and Newton's absolute time are inextricably related when we assume (i). Indeed, if an object A at $x = x_a$ causes an instantaneous effect at time $t = t_1$ through a fundamental interaction on a distant object B at $x = x_b$ in frame $\mathcal{O}$, then in another inertial frame $\mathcal{O}'$ that effect must also occur at the same time on A and B, say $t' = t'_1$. The time coordinate transformation between $\mathcal{O}$ and $\mathcal{O}'$ is

$$t'_1 = G(x_1, t_1; v) \tag{1}$$

$$t'_1 = G(x_2, t_1; v) \tag{2}$$

Since x_1 and x_2 are arbitrary, the time coordinate must be independent of the spacial coordinate $t' = G(t; v)$. Homogeneity of time requires that the ratio dt'/dt be independent of time then $t' = a(v)t + b(v)$. We can take $b(v) = 0$ by adequate initial conditions, for instance, by choosing $t' = 0$ when $t = 0$, then, by symmetry we have the following relations

$$t' = a(v)t \tag{3}$$

$$t = a(-v)t' \tag{4}$$

$$\frac{dt'}{dt} = \frac{dt}{dt'} \rightarrow a(v) = a(-v) \tag{5}$$

Combining the former equations lead us to $a(v)^2 = 1 \rightarrow a(v) = \pm 1$, then conserving the time direction we are left with $t' = t$. Thus we have no escape from absolute time when interactions are instantaneous.

On the other hand, elementary considerations between inertial observers in relative motion prove that Newton's absolute time has to be abandoned if a finite speed remains invariant in all inertial frames.

The former considerations about absolute time and its abandonment are related only to principle (i) and the existence of an infinite or finite universal speed for all inertial observers without mentioning light or electromagnetism.

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III. LINEARITY

Here we address two minor issues that were not sufficiently clarified in section **B** of [1]. We reference equations in [1] with an asterisk. Equations (7*) and (8*) are

$$\frac{\partial F}{\partial t} + v\frac{\partial G}{\partial t} \; = \; 0 \tag{6}$$

$$v\frac{\partial F}{\partial x} + \frac{\partial F}{\partial t} \; = \; 0 \tag{7}$$

The first issue arises after equation (7) [(8*)]. There Verheest asserts, "This implies that F is a function of the combined argument $x - vt$ as well as of v." without further explanation.

It is clear that if F has the functional form $F(x-vt;v)$, (7) [(8*)] is satisfied. However, the former argument only proves a sufficient condition, and Verheest's derivation requires F to have necessarily that functional form. Luckily that has an elegant solution.

As observed in [1], from (6) [(7*)] and (7) [(8*)] we obtain [(9*)]

$$\frac{\partial G}{\partial t} = \frac{\partial F}{\partial x} \tag{8}$$

Taking derivatives with respect to t in (6) [(7*)] and (8) [(9*)] then with respect to x in (7) [(8*)], we can eliminate $\partial^2 G/\partial t^2$ from (6) [(7*)] obtaining

$$\frac{\partial^2 F}{\partial t^2} + v\frac{\partial^2 F}{\partial t \partial x} \; = \; 0 \tag{9}$$

$$v\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial x \partial t} \; = \; 0 \tag{10}$$

Eliminating the cross derivatives in (9) and (10)

$$\frac{\partial^2 F}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 F}{\partial t^2} = 0 \tag{11}$$

The general solution of the wave equation (11) is well-known to be

$$F(x,t;v) = f_v(x - vt) + g_v(x + vt) \tag{12}$$

The g_v part of the solution does not satisfy (7) so we must have $g_v = 0$ and we obtain the necessary solution

$$F(x,t;v) = f_v(x - vt) \tag{13}$$

The second issue arises when solving the following homogeneous linear system

$$\frac{\partial^2 F}{\partial t \partial x} + v\frac{\partial^2 G}{\partial t \partial x} \; = \; 0 \tag{14}$$

$$C(v)\frac{\partial^2 F}{\partial t \partial x} - \frac{\partial^2 G}{\partial t \partial x} \; = \; 0 \tag{15}$$

In [1], Verheest assumes a nonzero determinant of the coefficients, $[1 + vC(v)]$. It is also necessary to study the case when $1 + vC(v) = 0 \to C(v) = -1/v$. When this happens, we cannot assume that both cross derivatives vanish. In this case the system (14) and (15) reduces to a single equation

$$\frac{\partial^2 G}{\partial t \partial x} = -\frac{1}{v}\frac{\partial^2 F}{\partial t \partial x} \tag{16}$$

For this case, from (6) and (11*)

$$\frac{\partial G}{\partial t} = -\frac{1}{v}\frac{\partial F}{\partial t} \tag{17}$$

$$\frac{\partial G}{\partial x} = -\frac{1}{v}\frac{\partial F}{\partial x} \tag{18}$$

Replacing (13) in the former two equations

$$\frac{\partial G}{\partial t} \; = \; f'_v \tag{19}$$

$$\frac{\partial G}{\partial x} \; = \; -\frac{1}{v}f'_v \tag{20}$$

By integration we have

$$G(x,t;v) \; = \; -\frac{1}{v}f_v + h(x) \tag{21}$$

$$G(x,t;v) \; = \; -\frac{1}{v}f_v + l(t) \tag{22}$$

Therefore $h(x) = l(t) = k = const.$ and we are left with the following spacetime transformation

$$t' \; = \; -\frac{1}{v}f_v(x - vt) + k \tag{23}$$

$$x' \; = \; f_v(x - vt) \tag{24}$$

However,this transformation is inadmissible because it does not have an inverse. Really, when $t' \neq -(1/v)x' + k$ it does not have solution in (x,t).

[1] F. Verheest, American Journal of Physics **90**, 425 (2022), https://pubs.aip.org/aapt/ajp/article-pdf/90/6/425/16190483/425_1_online.pdf.

[2] Wikisource, Translation: Some general remarks on the relativity principle — wikisource, (2021), [Online; accessed 25-June-2023].

[3] L. Landau and E. Lifshitz, *Course of Theoretical Physics, Volume 2, The Classical Theory of Fields*, 4th ed. (ELSEVIER, Oxford, 1975).