On bundles of varieties V_2^3 in PG(4, q) and their codes

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Abstract

In this note we use the spatial representation in $\Sigma = PG(4, q)$ of the projective plane $\Pi = PG(2, q^2)$, by fixing a hyperplane Σ' with a regular spread S of lines. We consider a bundle X of varieties V_2^3 of Σ having in common the $q + 1$ points of a conic C^2 of a plane π_0, $\pi_0 \cap \Sigma' = l_0 \in$ S, thus representing an affine line of Π, and a further affine point $O \notin \pi_0$. This subset X of Σ represents a bundle of non-affine Baer subplanes of Π, each of them having one point at infinity (corresponding to a line of S), having in common a subline of affine points of Π and a further affine point. Then X is considered as a projective system of Σ and, by using such a representation of Π, we can calculate the ground parameters of the code C_X arising from it.

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1. Introduction

It is known that a projective translation plane Π of order $n = q^2$ of dimension 2 over its kernel $F = GF(q)$ can be represented by a 4-dimensional projective space $\Sigma = PG(4, q)$ over F, fixing a hyperplane $\Sigma' = PG(3, q)$ and a spread S of lines of Σ'. The points of Π are represented by (i) the points of $\Sigma \setminus \Sigma'$ and (ii) the lines of S. The lines of Π are represented by (i) the planes α of $\Sigma \setminus \Sigma'$ such that $\alpha \cap \Sigma'$ belongs to S and by (ii) the spread S. The translation line l of Π is represented by S (cf. [1]).

A Baer subplane B of Π has order q and it is *dense* in the sense that a line of Π either is a line of B (that is, meets B in a *subline* of $q+1$ points, such a subplane is *affine*) or it meets B in one point (such a subplane is *non-affine*).

The *affine* Baer subplanes B of Π are represented by the *transversal* planes β to S, that is, the planes of $\Sigma \setminus \Sigma'$ such that the line $\beta \cap \Sigma' \notin$ S meets $q+1$ lines of S. In such a way l is a line of B (cf. [2], pp. 68--72). Of course all that holds also in case Π is the Desarguesian plane $PG(2,q^2)$ when S is a regular spread (cf. [3], [2]).

A variety V_2^3 of Σ with a line l_∞ in S as the minimum (linear) order directrix, a conic C^2 as a 2*nd* order directrix with $C^2 \subset \pi_0$, $\pi_0 \cap \Sigma' = l_0 \in S \setminus l_\infty$ and $C^2 \cap l_0 = \varnothing$, represents a non-affine Baer subplane of Π having one point on the translation line l and the *subline* C^2 of the *line* π_0 (cf. [3]).

In this paper we consider bundles of $q+1$ varieties V_2^3 of $\Sigma = PG(4,q)$ with the linear directrix in S and having in common a same conic C^2 as a 2*nd* order directrix and one further affine point. By using the spatial representation of $\Pi = PG(2,q^2)$ in $PG(4,q)$, we can characterize such a bundle X from the intersection point of view, construct a linear code C_X arising from it and show that its ground parameters allow C_X to correct an enough large number of errors.

2. Preliminary Notes

Let $F = GF(q)$ be a finite field, $q = p^s$, p prime. Denote F^{r+1} the $(r+1)$-dimensional vector space over F,

$P^r = PrF^{r+1} = PG(r,q)$ the r-dimensional projective space contraction of F^{r+1} over F. Let $\bar{F}$ be the algebraic closure of the field $F = GF(q)$.

Denote S_t with $t \geq 2$ a subspace of P^r of dimension t. A hyperplane S_{r-1} will be denoted also by H, a plane by π.

The geometry P^r is considered a sub-geometry of $\bar{P}^r$, the projective geometry over $\bar{F}$. We refer to the points of P^r as the *rational points* of $\bar{P}^r$.

Definition 2.1. *A variety V_u^v of dimension u and of order v of P^r is the set of the rational points of a projective variety $\bar{V}_u^v$ of $\bar{P}^r$ defined by a finite set of polynomials with coefficients in the field F.*

From [4], p.290, 7.- for $r \geq 4$ follows

Lemma 2.2. *The ruled variety V_2^{r-1} of $PG(r,q)$ is generated by the lines connecting the corresponding points of two birationally (or, projectively) equivalent curves in two complementary subspaces, of order m and $r-1-m$, respectively. It has order the sum of the orders of the curves as there are no fixed points.*

Let P^4 be the projective geometry $PG(4,q)$.

Lemma 2.3. A variety V_2^3 of $PG(4,q)$ is obtained by joining the corresponding points of a directrix line l and a directrix conic C in a plane π, l and C being projectively equivalent and with $l \cap \pi = \varnothing$.

Proof. See [5] p. 90.

Choose a coordinate system in P^4 so that it is a coordinate system for $\bar{P}^4$ too, denote a point

$P \approx (x_1, x_2, y_1, y_2, t) := \bar{F}^* (x_1, x_2, y_1, y_2, t)$, $\bar{F}^* = \bar{F} \setminus \{0\}$.

P is a *rational point* if there exists $(x_1, x_2, y_1, y_2, t) \in F^5$ such that $P \approx (x_1, x_2, y_1, y_2, t)$. A variety V of P^4 is the set of the rational points of $\bar{P}^4$ solutions of a finite set of polynomials of $F[x_1, x_2, y_1, y_2, t]$.

Lemma 2.4. *The variety V_2^3 can be represented as the definite intersection of two quadrics of $PG(4, q)$, that is, the cone of planes $Q_1: sx_2^2 - x_1^2 - sx_2 t = 0$ (where s is a non square of $GF(q)$) and the cone of planes $Q_2: x_1 y_1 - x_2 y_2 = 0$. The plane $\pi': x_1 = 0, x_2 = 0$ is contained in both quadrics so that, by Bezout, the order of the intersection variety is $4 - 1 = 3$.*

Proof. See [3] Theorem 1.1, [5] p. 92.

Let $\Pi = PG(2, q^2)$ be the Desarguesian plane over $GF(q^2)$. Denote l the line at infinity of Π. In the spatial representation of Π in $P^4 = PG(4, q)$ fix a hyperplane $\Sigma' = PG(3, q)$ and a regular spread S of lines of Σ', where $|S| = q^2 + 1$.

Lemma 2.5. *The points of Π are represented by (i) the points of $\Sigma \setminus \Sigma'$ (the affine points of Π) and by (ii) the lines of S (the points at infinity of Π). The lines of Π are represented by (i) the planes α of $\Sigma \setminus \Sigma'$ such that $\alpha \cap \Sigma'$ belongs to S and by (ii) the spread S, representing the line at infinity l.*

Proof. See [1] the Bruck and Bose representation and [2], p. 775.

Definition 2.6. *A Baer subplane of $\Pi = PG(2, q^2)$ is an affine subplane if it meets the line at infinity l of Π in a subline l_1, it is a non-affine subplane if it meets the line l in one point.*

Lemma 2.7.

(i) *Two affine Baer subplanes of Π having in common the subline l_1 can meet in at most one further point.*

(ii) *The Baer subplanes having in common only a subline l_1 are q^2.*

(iii) *The Baer subplanes having in common a subline l_1 and one further point are $q + 1$.*

Proof. (i) Two Baer subplanes having in common a subline l_1 and two further points coincide, because they have in common at least four *reference* (three by three non collinear) points.

Without loosing generality, we can consider two affine Baer subplanes B and B' of Π having in common a subline l_1 of l. In the spatial representation of Π, they are represented by two planes B and B' of P^4, respectively, such that the lines $B \cap \Sigma' = r$ and $B' \cap \Sigma' = r'$ are transversal lines of the same regulus $R \subset S$. Denote R' the opposite regulus to R.

There are two cases:

(ii) If $r = r'$, the planes B and B' have in common the line r meeting the regulus R in its $q + 1$ lines so that the subplanes B and B' have in common the *subline* l_1 (represented by R) of the *line* l (represented by S) and no further (affine) points.

Such planes are $\frac{q^4}{q^2} = q^2$ and represent q^2 affine Baer subplanes of Π having in common only the subset l_1 of $q+1$ points of the line at infinity l.

(iii) If $r \neq r'$, the planes B and B' have in common an affine point $O \in \Sigma \setminus \Sigma'$ so that the two subplanes B and B$'$ meet along the subline l_1 represented by R and in the affine point O. The regulus R has $q+1$ transversal lines $\{t_i | i = 1, \dots, q+1\}$ belonging to R$'$. Each space $O \oplus t_i$ is a transversal plane τ_i, so that $\{\tau_i | i = 1, \dots, q+1\}$ represent the $q+1$ affine Baer subplanes of Π having in common l_1 and the affine point O.

Choose and fix a line l_∞ of the (regular) spread S, a plane π_0 such that $\pi_0 \cap \Sigma' = l_0 \in S \setminus l_\infty$ and a non-degenerate conic $C^2 \subset \pi_0 \setminus l_0$. Let Λ be a projectivity between l_∞ and C^2. Denote V_2^3 the variety arising by connecting corresponding points of l_∞ and C^2 via Λ (cf. [5], p. 90).

Lemma 2.8. *The variety V_2^3 represents a non-affine Baer subplane of Π meeting the line at infinity l in the point l_∞ and containing the subline C^2 of the line represented by π_0.*

Proof. See [3] and [2].

Let F^n be the n-dimensional vector space over $F = GF(q)$.

Definition 2.9. *A linear $[n, k]_q$-code C of length n is a k-dimensional subspace of the vector space F^n.*

Definition 2.10. *An $[n, k]_q$-projective system X is a set of n non necessarily distinct points of the projective geometry $PrF^k = PG(k-1, q)$. It is non-degenerate if these points are not contained in a hyperplane (cf.[6], p. 2).*

Assume that X consists of n distinct points having maximum rank.

Codes and projective systems are linked by a strict connection one can read in[6], so that from subsets X of a projective geometry linear codes C_X can be generated. More precisely, for each point of X choose a generating vector. Denote M the matrix having as rows such n vectors and let C_X be the linear code having M^t as a generator matrix. The code C_X *is the k-dimensional subspace of F^n* which is the *image of the mapping from the dual k-dimensional space* $(F^k)^*$ *onto F^n that calculates every linear form over the points of* X. Hence the length n of codeword of C_X is the cardinality of X, the dimension of C_X being just k (cf. [6], p. 3).

Denote H the set of all hyperplanes of $P^{k-1} = PrF^k$.

There exists a natural 1-1 correspondence between the equivalence classes of a non-degenerate $[n, k]_q$*-projective system* X and a non-degenerate $[n, k]_q$*-code* C_X such that if X is an $[n, k]_q$-projective system and C_X is a corresponding code, then the non-zero codewords of C_X correspond to hyperplanes $H \in$ H, up to a non-zero factor. The correspondence preserves the ground parameters.

The weight of a codeword c corresponding to the hyperplane H_c is the number of points of $X \setminus H_c$, thus the minimum weight (or, the minimum distance) d of the code C_X is $d = |X| - max\{|X \cap H| \mid H \in \mathrm{H}\}$. Therefore in order to find the minimum distance of the code C_X it needs to calculate the maximum intersection of X with the hyperplanes of H.

A linear code with length n, dimension k and minimum distance d over the field $F = GF(q)$ can be denoted also as an $[n, k, d]_q$-code.

If C is an $[n, k, d]_q$-code, then C is an s-error-correcting code for all $s \leq \lfloor \frac{d-1}{2} \rfloor$. We call $t = \lfloor \frac{d-1}{2} \rfloor$ the *error-correcting capability of C* (cf.[6], p.3).

3. Main Results

With the notations of the previous section, choose and fix the line $l_0 \in S$, the plane π_0 such that $\pi_0 \cap \Sigma' = l_0 \in S$ and the non-degenerate conic $C^2 \subset \pi_0 \setminus l_0$.

Denote Σ'' a hyperplane of $\Sigma = PG(4, q)$ containing the plane π_0. Let $\pi = \Sigma'' \cap \Sigma'$. The plane π contains the line l_0 and each of the q^2 points of $\pi \setminus l_0$ belongs to one of the q^2 lines of $S \setminus \{l_0\}$. Let O be a point, $O \in \Sigma'' \setminus \{\pi_0 \cup \pi\}$. Denote Q the quadric cone having vertex the point O and directrix the conic C^2. Let $C'^2 = Q \cap \pi$. Obviously C'^2 is a non-degenerate conic with $C'^2 \cap l_0 = \varnothing$.

Let $\{R_i | i = 1, \ldots, q+1\}$ be the set of the $q+1$ points of C^2, $\{r_i | i = 1, \ldots, q+1\}$ the $q+1$ lines of the cone Q with $R_i \in r_i$, $\{R'_i = r_i \cap C'^2 | i = 1, \ldots, q+1\}$ the *corresponding* set of $q+1$ points of C'^2 with $R'_i \in r_i$, $\{s_i | i = 1, \ldots, q+1\}$ the $q+1$ lines of S with $\{R'_i \in s_i | i = 1, \ldots, q+1\}$.

For each line s_i let λ_i be a projectivity between s_i and C^2 such that $\lambda_i(R'_i) = R_i$

Denote S_i the point at infinity of the plane Π represented by the line $s_i \in S$, p_0 the line of Π represented by the plane π_0 and c_2 the subline of p_0 corresponding to C^2.

Let V_i be the variety V_2^3 having the conic C^2 and the line s_i as directrices constructed via λ_i. Note that, by construction, the line r_i is one of the $q+1$ generatrix lines of V_i.

From Lemma 2.8 follows that each of the $q+1$ variety V_i is a non-affine Baer subplane of Π meeting the line l in the point S_i, containing $c_2 \subset p_0$ and the point O.

Define $V := \bigcup_i V_i$ the union of the points of all varieties V_i for all $i = 1, \ldots, q+1$.

Lemma 3.1. V *represents the bundle of the full set of* $q+1$ *non-affine Baer subplanes having in common the subline* c_2 *and the point* O.

Proof. See (*iii*) of Lemma 2.7 and [3].

Proposition 3.2. $\Sigma'' \cap V = Q$.

Proof. By construction the hyperplane Σ'' contains Q. As for any variety V_i, $\Sigma'' \cap V_i$ cannot contain the directrix line s_i (otherwise $\Sigma'' = \Sigma'$), then Σ'' meets V_i at most in a cubic curve $C^2 \cup r_i$ (cf. [5], (*ii*), p. 93).

Assume $\Sigma'' \cap V$ contains $C^2 \cup r_i \subset V_i$ and a further point $P_j \in V_j$ with $j \neq i$. Hence Σ'' contains the line $r = P_jR_j \in V_j$ with $R_j \in C^2$. If $r \neq r_j$, then Σ'' should meet V_j in $C^2 \cup r_j \cup r$ where r_j and r are two generatrix lines of V_j, then the line s_j should belong to Σ'', a contradiction (cf. [5], (*ii*), p. 93). Hence $\Sigma'' \cap V = Q$.

Denote $V_{aff} = V \setminus \Sigma'$.

Proposition 3.3.

(*i*) A hyperplane of Σ having maximum intersection with V is Σ', and $\Sigma' \cap V$ consists of the points of the lines $\{s_i | i = 1, \ldots, q+1\} \subset S$.

(*ii*) A hyperplane of Σ having maximum intersection with V_{aff} is Σ'' and $\Sigma'' \cap V_{aff}$ consists of the points of $Q \setminus C'^2$.

Proof. (*i*) Let $H \in \mathrm{H}$ a hyperplane. If $H = \Sigma'$ then $H \cap V$ is the set of the $(q+1)^2$ points of $\{s_i | i = 1, \ldots, q+1\} \subset S$. If $H = \Sigma''$ then $H \cap V$ is the set of the $q^2 + q + 1$ points of Q.

Let $H \neq \Sigma', \Sigma''$.

Denote $H \cap \Sigma' = \pi'$, $H \cap \Sigma'' = \pi''$.

For H there are two possibilities: 1) H contains π_0, 2) H does not contain π_0.

1) It is $\pi'' = \pi_0$ so that it contains C^2. Moreover $\pi' \neq \pi$ otherwise $H = \Sigma''$. The plane π' forms bundle with axis the line l_0 with π_0 and π. Each point of π' belongs to one line of $S \setminus l_0$ then it meets the $q+1$ points $\{P_i = \pi' \cap s_i | i = 1, \ldots, q+1\}$. Therefore $H \cap V$ contains at least the $q+1$ points P_i and the points of C^2. Then $|H \cap V| \geq 2(q+1)$. The maximum intersection is reached if each line P_iR_i coincides with one generatrix line of the variety V_i for every i, In such a case $|H \cap V| = (q+1)^2$.

2) Let $\pi'' \cap \Sigma' = l$. Then l is a line of π' too.

Let $l = l_0$. The plane π'' contains no generatrix line of the varieties V_i otherwise l_0 would meet some line s_i, it meets V in at most a conic C_Q of Q. Set $\{P_i \in C_Q | i = 1, \ldots, q+1\}$.

If $\pi' = \pi$, then $\pi' \cap V = C'^2$. If $\pi' \neq \pi$, then it contains no line s_i (otherwise $l_0 \cap s_i \neq \varnothing$), it can meet at most $q+1$ lines s_i in points T_i. In both cases the maximum intersection is reached if the $q+1$ lines P_iR_i', or P_iT_i, respectively, coincide with the generatrix lines of the varieties V_i. Hence $|H \cap V| \leq (q+1)^2$.

Let $l \neq l_0$. Denote $r' = \pi'' \cap \pi_0$. Then $l = s_i$ for some i or l meets at most $q+1$ lines s_i.

If $\pi' = \pi$, it contains the $q+1$ points of C'^2 and according to r' is secant, tangent or external to the conic C^2, $|H \cap V|$ is less or equal to $(q+1) + 2q = 3q+1$, $(q+1) + q = 2q+1$ or $q+1$, respectively.

Assume $\pi' \neq \pi$. The plane π' must contain one line t of S and the q^2 points of the remaining lines of S. Then the plane π'

contains the $q+1$ points of $t = s_i$ for some i, or the $q+1$ points of the set $\{s_i \cap \pi' \mid i = 1, \dots q+1\} \subset \mathrm{V}$.
According to r' is secant, tangent or external to the conic C^2, H meets V in 2 generatrix lines, in 1 generatrix line or in no generatrix line. Therefore $|H \cap \mathrm{V}|$ is less or equal to $(q+1) + 2q = 3q+1$, $(q+1) + q = 2q+1$ or $q+1$.

Hence the maximum intersection a hyperplane can have with V consists of $(q+1)^2$ points. Σ' is one of such hyperplanes.

(ii) Let H be a hyperplane, $H \neq \Sigma'$. From [7], Lemma 11, it is known the maximum intersection a hyperplane of Σ has with a variety V_2^3 consists of two generatrix lines and the directrix line. Of course H cannot meet two different varieties in such a way otherwise H, containing two lines of S would coincides with Σ'. Therefore H can meet at least q varieties along the conic C^2 and one generatrix line for each variety, then q points of the conic C'^2. In any case H contains O then the cone Q . Therefore $H = \Sigma''$. Hence the maximum intersection a hyperplane can have with V_{aff} is $\mathrm{Q} \setminus \mathrm{C}'^2$ with $|\mathrm{Q} \setminus \mathrm{C}'^2| = q^2$.

Denote $\mathrm{X} := \mathrm{V}$ the projective system defined by V, C_{X} the linear code arising from X, $\mathrm{X}_{aff} := \mathrm{V}_{aff}$ the projective system defined by V_{aff}, $C_{\mathrm{X}_{aff}}$ the linear code arising from X_{aff}.

Theorem 3.4.
(i) *C_{X} is an $[n,k,d]_q$-code with $n = q^3 + 2q^2 + q + 1$, $k = 5$, $d = q^3 + q^2 - q$.*
(ii) *$C_{\mathrm{X}_{aff}}$ is an $[n', k, d']_q$-code with $n' = q^3 + q^2 - q$, $k = 5$, $d' = q^3 - q$.*

Proof. (i) Each variety V_i consists of $q+1$ skew lines, hence it has $(q+1)^2$ points. Every two varieties V_i and V_j have in common the conic C^2 and the point O so that for each variety remain $q^2 + 2q + 1 - (q+1) - 1 = q^2 + q - 1$ points. The varieties are $q+1$ so that the cardinality of X is $(q^2+q-1)(q+1) = q^3 + 2q^2 - 1$ plus the point O and the $(q+1)$ points of the conic C^2. Hence $|\mathrm{X}| = q^3 + 2q^2 + q + 1$. The length of the code C_{X} is therefore $n = q^3 + 2q^2 + q + 1$.

The dimension of C_{X} is obviously 5, that is, the vector dimension of Σ.

From Proposition 3.3, (i), follows the distance of C_{X} is $d = n - |\{P \in s_i \mid i = 1, \dots, q+1\}|$ that is,
$d = q^3 + 2q^2 + q + 1 - (q^2 + 2q + 1) = q^3 + q^2 - q$.

(ii) The length of the code $C_{\mathrm{X}_{aff}}$ equals
$n' = |\mathrm{X}| - |\{P \in s_i \mid i = 1, \dots, q+1\}| = q^3 + 2q^2 + q + 1 - (q^2 + 2q + 1) = q^3 + q^2 - q$. Its dimension is $k = 5$. From Proposition 3.3, (ii), follows the distance is $d' = n' - |\mathrm{Q} \setminus \mathrm{C}'^2|$ that is, $d' = q^3 + q^2 - q - q^2 = q^3 - q$.

Examples

For $q = 2$, C_{X} is a $[19,5,10]_2$-code and it can correct at most $\lfloor \frac{10-1}{2} \rfloor = 4$ errors. For $q = 3$, C_{X} is a $[49,5,33]_3$-code and it can correct at most $\lfloor \frac{33-1}{2} \rfloor = 16$ errors.

For $q = 2$, $C_{\mathrm{X}_{aff}}$ is a $[10,5,6]_2$-code and it can correct at most $\lfloor \frac{6-1}{2} \rfloor = 2$ errors. For $q = 3$, $C_{\mathrm{X}_{aff}}$ is a $[33,5,24]_3$-code and it can correct at most $\lfloor \frac{24-1}{2} \rfloor = 11$ errors.

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