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Representation of physical quantities: From scalars, vectors, tensors and spinors to multivectors

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Abstract

Mathematical representations of physical variables and operators are of primary importance in developing a theory – the relationship among different relevant quantities of any physical process. A thorough account of the representations of different classes of physical variables is drawn up with a brief discussion of various related mathematical systems including quaternion and spinor. The present study is intended to facilitate a comprehensive introduction to the 'geometric algebra', which provides an immensely productive unification of these systems and promises more.

Keywords: Vector, tensor, quaternion, spinor, exterior algebra, bivector, pseudoscalar and geometric algebra.

Scalars, vectors and tensors (of second and higher ranks) formed the basic mathematical representation for the description of physical quantities independent of the choice of reference frame. However, the universe of the elementary particles requires the introduction of another class of mathematical objects called *spinor*. Truly speaking, it does not belong to the family of tensors. Spin-zero particles like π mesons, spin-1 particles (like deuterons) and spin-2 particles (like gravitons) are adequately described by scalars, vectors and tensors respectively. Interestingly, the most common particles like electrons, protons, and neutrons, all with one-half intrinsic angular momentum, are missing from this list. These particles are properly described by spinors which differ from vectors or tensors with respect to the properties under coordinate transformation [1]. *Twistor* is another mathematical representation, introduced by Penrose, which also attracts recent attention of string theorists.

Use of matrices as transformation operators also has limitations. Quaternions, introduced earlier by Hamilton [2], eventually turn out to be more useful in describing rotation about an arbitrary axis in three dimensions (3-D). In fact, Hamilton's work on quaternions has provided a sort of unification of complex algebra and 3-D vector algebra (VA). The *wedge product* ($\mathbf{u} \wedge \mathbf{v}$) of two vectors $\mathbf{u}$ and $\mathbf{v}$, defined in the following year by Grassmann in his *Algebra of Extension* (or simply, the *exterior algebra*), on the other hand, is by far an inclusive definition and aptly generalises the cross product of 3-D vector algebra to higher dimensions [3]. Two, three or any number of linearly independent vectors in a given dimension are 'wedged' together to produce *bivector*, *trivector* or even higher grade elements of the algebra. Clifford [4] has finally unified the works of Hamilton and Grassmann in 1876 to form the foundation for a new algebra of mathematical physics with several distinctive features. The multiplication rule of *geometric product* (introduced by Clifford) combines both *dot* and *wedge* products of vectors and allows any linear combination of scalar(s), vector(s), bivector(s) and higher grade *multivectors* to be a member of this algebra. A seamless extension to any dimension and the further development to its modern version *geometric algebra* (GA) by Hestenes [5], usher in a rapid advancement of this proficient replacement for the vector and matrix algebras.

Hestenes has elaborated Clifford's work to show how it unites "vectors, spinors, and complex numbers into a single mathematical system with a comprehensive geometric significance" [6]. It provides an approach to a coordinate-free geometry where the geometric objects (points, lines, planes etc.) are represented by members of an algebra, rather than by equations relating coordinates. The geometric operations (rotation, translation etc.) are then implemented by algebraic operations on these (geometric) objects. The advantages of GA over traditional approaches of vector, tensor and spinor algebras "are similar to those of elementary algebra over arithmetic." The geometric approach emphasises and explores the geometric properties of the basic ingredients vectors, bivectors etc., independent of any basis [7, 8]. Increasing number of scientific groups are now applying GA to

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a range of problems in varied research fields.

We intend to provide an introductory exposure of this rapidly growing powerful apparatus of theoretical physics for advanced undergraduate students in two parts. In this part, after an initial recapitulation of multiplication rules in conventional 3-D vector algebra, we will discuss the 'wedge product' of vectors introduced by Grassmann and also the quaternion and spinor algebras which form the basis of our introductory discussion on GA in sec.4. Starting from the usual definition of geometric product between two vectors, procedure for obtaining *the product of three or more vectors* are clearily described using the *proper order of multiplication* for taking the three products – dot, wedge and geometric of two vectors. With the help of appropriate multiple dot/inner products, introduced earlier in Grassmann algebra, geometric product between *multivectors of higher grades* are also derived in some detailfor any vector space of arbitrary dimension. We will also learn in the following that, the very notion of geometric product endows the basic vector space with an algebraic structure that embraces the vector, complex and the spin algebras in a single formalism and sets apart geometric algebra from others.

For advanced students, well conversant with vector and tensor algebras, sec.1 up to 1.2 might appear familiar. The dot, cross and triple products of standard vector algebra each admits both geometric and algebraic interpretations. But, whereas the cross and triple products of VA are definable only in 3-D, the wedge product allows for similar interpretations even in two and all higher dimensions. To start with, in the following discussions we will assume the addition/subtraction rules of VA and proceed with the different multiplication rules.

1. *Cross and dyadic products – products of the vector and tensor algebras*:

Using Einstein's sign convention and the definition of cross product ($\hat{e}_j \times \hat{e}_k = \epsilon_{ijk}\hat{e}^i$) for orthogonal basis vectors $\hat{e}_i$, the cross product of two 3-D vectors $\mathbf{u}$ and $\mathbf{v}$ in VA is given by:

$$\mathbf{w} = \mathbf{u} \times \mathbf{v} = \hat{e}^i \epsilon_{ijk} u^j v^k = \hat{e}^i T_{ij} u^j = \hat{e}^i w_i, \tag{1}$$

where ϵ_{ijk} is the Levi-Civita symbol – each element of which is given by the scalar triple product of unit orthogonal coordinate basis vectors $\hat{e}_i$ in 3-D, as:

$$\epsilon_{ijk} = \hat{e}_i.(\hat{e}_j \times \hat{e}_k)$$

and collectively ϵ_{ijk} form an antisymmetric (on each pair of indices) covariant tensor of rank three. Also, T_{ij} is a second rank covariant tensor obtained by contracting[1] ϵ_{ijk} with the second vector $\mathbf{v}$,

[1] In tensor analysis, the generalisation of the dot or scalar product of VA is known as contraction or inner product. The dot product of two vectors in VA is an operation which associates a real number (scalar) to each pair of vectors in a vector space i.e. $\mathbf{u}.\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$. In terms of components of the vectors (in arbitrary coordinate system): $\mathbf{u}.\mathbf{v} = \hat{e}_i u^i.\hat{e}_j v^j$ and the scalar product is defined by interpreting $\hat{e}_i.\hat{e}_j$ $(= g_{ij})$ as the $(ij)_{th}$ element of the fundamental metric tensor or simply the metric of the (vector) space, whose components transforms like a second rank covariant symmetric tensor, since $g_{ij} = g_{ji}$. Actually, it maps a vector $\{u^i\}$ to its covector $\{u_j\}$ $(= \{g_{ij}u^i\})$, in a dual vector space (or just dual space in short), to produce a scalar quantity, its squared norm (length), i.e. $\mathbf{u}.\mathbf{v} = g_{ij}u^i v^j = u_j v^j$. Likewise, the inverse of g_{ij} defines the contravariant metric $g^{ij} = \hat{e}^i.\hat{e}^j$ $(g^{ik}g_{kj} = g_{jk}g^{ki} = \delta^i_j)$ which may be used to raise the indices, converting a covariant vector to its contravariant dual. Any vector space has a corresponding dual space having the same dimension as the original space. Elements of the dual space are also called associate vectors. A basis corresponds (one to one) to its dual basis if and only if, it is an orthonormal basis. If the coordinate system is orthogonal, the metric g_{ij} is diagonal. For flat Euclidean space the metric is a unit tensor. Also, for orthogonal coordinate systems, the reciprocal contravariant and covariant basis vectors ($\hat{e}_i$ and $\hat{e}^i$) become identical sets and no distinction is made between contravariant and covariant components of a vector.

When two indices, one covariant and the other contravariant, are set equal to each other, then the implied summation over the repeated index induces the contraction. For example, one can contract a second rank mixed tensor T^i_j to a scalar and verify it from the following transformation equation:

$$\bar{T}^i_j = \frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial x^l}{\partial \bar{x}^j} T^k_l,$$

$$\text{Now, with } i = j, \text{ we get, } \bar{T}^i_i = \frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial x^l}{\partial \bar{x}^i} T^k_l = \frac{\partial x^l}{\partial x^k} T^k_l = \delta^l_k T^k_l = T^k_k.$$

So, the contracted second rank mixed tensor remains invariant under coordinate transformation $x^k \to \bar{x}^i$ and, therefore, is a scalar. However, it may be noted that, the inner product or contraction of tensor algebra is not necessarily scalar-valued and thus differs from the scalar product in VA.

$\mathbf{w}$ being the product vector. Furthermore,

$$\begin{aligned}\mathbf{u}\times\mathbf{v} = -\mathbf{v}\times\mathbf{u} &= -\hat{e}^i\epsilon_{ijl}v^ju^l = -\hat{e}^i\epsilon_{lij}v^ju^l \\ &= -\hat{e}^iT_{li}u^l = -\hat{e}^iT_{ji}u^j \Rightarrow T_{ij} = -T_{ji}.\end{aligned} \tag{2}$$

So, T_{ij} is an antisymmetric tensor of rank two and the cross product of two contravariant vectors may be expressed as a contraction (inner product) of an antisymmetric covariant second rank tensor with one of the (contravariant) vectors. In mathematics, a tensor is introduced as a linear operator and a second rank tensor transforms one vector into another. In physical theories we have the examples of quantities like the (electric) polarisability tensor converting the applied field into the induced polarisation (vector) in an anisotropic medium, the moment of inertia tensor transforming the angular velocity into angular momentum of a rigid body. It may be noted here that, for orthogonal (curvilinear) coordinate systems[2] the distinction between contravariant and covariant disappears. However, in the present discussion we retain the distinct notations to make the process of contraction apparent.

The product vector of eq.(1) is defined along the normal to plane formed by the two vectors $\mathbf{u}$ and $\mathbf{v}$ and its magnitude is given by $|\mathbf{w}| = |\mathbf{u}\times\mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta$, θ being the angle between the two vectors. The magnitude is, therefore, equal to the area of the parallelogram determined by the two vectors. The cross product of two vectors can also be expressed in the form of a determinant of a special 3×3 matrix formed with the 3 orthogonal unit basis vectors and the components of the two vectors as the three respective rows. Thus, the cofactors along the first row gives the components of the resulting vector. However, it is to be noted that the absolute direction of the product vector is fixed according to some convention – right- or left-handed. Physical quantities like linear displacement, velocity, acceleration and force, having an absolute direction in space, are represented by ordinary or polar vectors. But the representation of the angular velocity of a rotating body as a vector along the axis of rotation in VA, lacks an absolute direction and the direction is fixed according to the convention. Similarly, the direction of the magnetic field lines, produced by moving charged particles, forming concentric circles around the current carrying conductor (the length of the wire) is again convention (the right/left hand grip rule) dependent. According to VA, such quantities are expressed in physical theories by the cross product of two polar vectors and called axial or pseudo vectors. They transform like ordinary vectors except for improper rotation such as a reflection or inversion which produces an additional sign flip in them. The transformation rules for polar vectors and pseudovectors can be compactly stated as:

$$\begin{aligned}\bar{\mathbf{v}} &= R\mathbf{v};\ (\mathbf{v},\ \text{a polar vector}) \\ \bar{\mathbf{w}} &= \det(R)(R\mathbf{w});\ (\mathbf{w},\ \text{a pseudovector})\end{aligned} \tag{3}$$

where the rotation matrix R can be either proper or improper. The symbol 'det' denotes determinant – the Jacobian of the transformation. This formula works because the determinant (the Jacobian) of proper and improper rotation matrices are $+1$ and -1, respectively. Furthermore, it is evident from eq.(1) that, the cross product of two polar vectors transforms like a covariant vector under proper rotations[3]. It may also be noted that depending on the types of the two vectors to be multiplied, the product vector may be either polar or pseudo. Cross product of two like vectors yield pseudovectors, whereas for two unlike vectors (one polar another pseudo), the product is an ordinary or polar vector.

The scalar product of an ordinary and a pseudovector (or equivalently the scalar triple product of polar vectors) gives a pseudoscalar which, unlike a true scalar, changes sign under improper rotation. Geometrically, it is the volume of the parallelepiped formed by the three vectors and algebraically – the determinant of the matrix formed with the three column vectors. Similarly, the divergence of an axial vector field represents a pseudoscalar. Also, multiplication of an ordinary vector by a

[2]For arbitrary set of nonorthogonal basis vectors $\hat{e}_1$, $\hat{e}_2$ and $\hat{e}_3$ the metric g is nondiagonal. Also, the cross product, say $\hat{e}_2\times\hat{e}_3$, is not simply equal to $\hat{e}^1$ and in general given by: $\hat{e}_j\times\hat{e}_k = \sqrt{|g|}\,\epsilon_{ijk}\,\hat{e}^i$. This can be extended for two arbitrary vectors, $\mathbf{u}$ and $\mathbf{v}$ (say) as:
$\mathbf{u}\times\mathbf{v} = \sqrt{|g|}\,\epsilon_{ijk}\,u^j\,v^k\hat{e}^i$.

[3]The components of the product vector $\mathbf{w}$ of eq.(1) transform as:

$$\bar{w}_i = \bar{T}_{ij}\bar{u}^j = \frac{\partial x^k}{\partial\bar{x}^i}\frac{\partial x^l}{\partial\bar{x}^j}T_{kl}\frac{\partial\bar{x}^j}{\partial x^k}u^k = \frac{\partial x^l}{\partial\bar{x}^i}T_{kl}u^k = \frac{\partial x^l}{\partial\bar{x}^i}w_l.$$

pseudoscalar produces a pseudovector. There are number of quantities in physics which behave as pseudoscalars[4].

In usual vector equations like $\mathbf{u} = \mathbf{v}$, both $\mathbf{u}$ and $\mathbf{v}$ are either polar or axial vectors. Similar restrictions apply to scalars and pseudoscalars and, in general, to the tensors and pseudotensors, to be considered subsequently. Also, if the nature does not distinguish between a right-handed or a left-handed coordinate system, adding or mixing an axial vector with a polar vector is not needed then. However, a pronounced exception to this occur in the case of beta decay involving weak interactions, where the physical universe distinguishes between right- and left-handed systems, and a proper description of the process calls for a V-AV (vector minus axial vector) Lagrangian for the weak interaction as proposed by Sudarshan and Marshak [9].

Finally we note the difficulties with the definition of cross product in VA. The product vector lacks an absolute direction and the definition introduces unwarranted notion of 'handedness' in certain physical theories. More importantly, it can be defined only in 3-D. In 2-D there is no space for the product vector, whereas in higher dimensions, the concept of a vector orthogonal to a pair of vectors is not unique. Since it is not possible to define the cross product except for 3-D space, the notion of pseudovector based upon the cross product cannot be extended. In fact, an appropriate generalisation of it is indeed possible in terms of the wedge product, which allows for more efficient geometric and algebraic interpretations and elucidates the notion of pseudovectors. A brief outline of Grassmann's exterior algebra will be presented subsequently.

1.1 *The dyadic or tensor product of two vectors*:

In an n-dimensional Euclidean space, the dyadic product of two vectors $\mathbf{u}$ $(= \hat{e}_i u^i)$ and $\mathbf{v}$ $(= \hat{e}_j v^j)$ is given by:

$$\mathbf{u} \otimes \mathbf{v} = \hat{e}_i \hat{e}_j u^i v^j = \hat{e}_i \hat{e}_j D^{ij} = \mathbf{D}. \tag{4}$$

Since each component ($u^i v^j$ or D^{ij}) is associated to a pair of unit vectors, it is called a dyad (the word 'dyad' means pair). A dyad represents a quantity that has magnitude and two associated directions and transforms like a tensor of rank two. Also, it acts as a linear operator that transforms one vector into another and therefore, is a tensor of rank two. The dyad or any second-rank tensor is also represented by a square array (matrix), with the ij-th element given by $u^i v^j$. If $u^i v^j = \delta_{ij}$, it is the unit dyad, similar to the identity matrix with the defining property: $I\mathbf{v} = \mathbf{v}I = \mathbf{v}$. The dyadic product is also called the tensor product of two vectors. The dyadic product of a vector and a pseudovector gives a pseudotensor.

In tensor calculus, the dyad $\nabla \otimes \mathbf{v}$ is defined as the gradient of a vector $\mathbf{v}$:

$$\nabla \otimes \mathbf{v} = D_i^j \; \hat{e}^i \hat{e}_j,$$
$$\text{where } \; D_i^j = \frac{\partial v^j}{\partial x^i}$$

are the components of a second-rank mixed tensor. If we contract over the indices i and j we get a scalar $\frac{\partial v^i}{\partial x^i} = \nabla \cdot \mathbf{v}$ – the divergence of $\mathbf{v}$. Similarly, one can define the action of the vector differential operator ∇ on a dyad. The gradient of a dyad is a tensor of rank three, the divergence gives a vector and the curl of a dyad is also a dyad.

Although a dyad represents a tensor of rank two, not all second rank tensors can be constructed using dyadic product of two vectors, i.e. all tensors are not dyads. The sum of two dyads or any linear combination of dyads represents a tensor of rank two but not necessarily a dyad because it

[4] The volume of a parallelepiped denoted by the scalar triple product of three polar vectors provides a simple example of a pseudoscalar in 3-D. Each element of the Levi-Civita tensor, given by a scalar triple product of the unit orthogonal cartesian bases, is a pseudoscalar. Other examples of pseudoscalar include (i) the magnetic flux - it is the result of a dot product between a vector (the unit surface normal) and the magnetic field pseudovector, (ii) the helicity is the projection (dot product) of spin angular momentum pseudovector onto the direction of momentum (a polar vector) and (iii) pseudoscalar particles like pseudoscalar mesons, i.e. particles with spin 0 and odd parity (whose statefunction changes sign under parity inversion). In the following, we will discuss the appropriate generalisation of the notion of pseudovectors of conventional vector algebra with the introduction of the concept of bivectors and also the generalisation of the definition of the pseudoscalar in exterior algebra.

may not be written as a dyadic product of two vectors (unlike the sum of two vectors which is a vector). It may be also noted that the nine components of dyadic product actually involve six distinct scalars (the components of the two vectors) only, whereas a general second rank tensor has nine independent components which are not related to each other in any way. Dyads, therefore, represent a sub class of second rank tensors.

1.2 *Properties of dyads*:

• The dyadic product is distributive over vector addition:
$\mathbf{u} \otimes (\mathbf{v} + \mathbf{w}) = \mathbf{u} \otimes \mathbf{v} + \mathbf{u} \otimes \mathbf{w}$

• If $\mathbf{D}$ is subtracted from itself, we get a null dyad $\mathbf{O}$, each elements/components of which is zero. Now, it is evident that $\mathbf{u} \otimes \mathbf{v} - \mathbf{v} \otimes \mathbf{u} \neq \mathbf{O} \implies$ dyadic products are not commutative ($\mathbf{u} \otimes \mathbf{v} \neq \mathbf{v} \otimes \mathbf{u}$).

• Multiplication with a scalar is associative:
$\alpha(\mathbf{u} \otimes \mathbf{v}) = (\alpha\mathbf{u}) \otimes \mathbf{v} = \mathbf{u} \otimes (\alpha\mathbf{v})$, for any scalar α.

• The operations of a dyad/tensor on a vector, defined by the pre and post dot and cross products of the vector with the dyad may be appended as follows:
(i) $\mathbf{w}.\mathbf{u} \otimes \mathbf{v} = (\mathbf{w}.\mathbf{u})\mathbf{v}$,
(ii) $\mathbf{u} \otimes \mathbf{v}.\mathbf{w} = \mathbf{u}(\mathbf{v}.\mathbf{w})$,
(iii) $\mathbf{w} \times \mathbf{u} \otimes \mathbf{v} = (\mathbf{w} \times \mathbf{u}) \otimes \mathbf{v}$
and (iv) $\mathbf{u} \otimes \mathbf{v} \times \mathbf{w} = \mathbf{u} \otimes (\mathbf{v} \times \mathbf{w})$
where, (i) and (ii) represent vectors (with a null vector $\mathbf{o}$, $\mathbf{D} \cdot \mathbf{o} = \mathbf{o}$), (iii) and (iv) gives dyads representing pseudotensors.

Here, the dot and cross multiplications are to be carried out first. Note that, the dot (inner) product of two dyads: $\mathbf{u} \otimes \mathbf{v}.\mathbf{w} \otimes \mathbf{x} = \mathbf{u}(\mathbf{v}.\mathbf{w}) \otimes \mathbf{x} = (\mathbf{v}.\mathbf{w})\mathbf{u} \otimes \mathbf{x}$ is a dyad and not a scalar-valued function. One can also define the double dot product of two dyads as: $\mathbf{u} \otimes \mathbf{v} : \mathbf{w} \otimes \mathbf{x} = (\mathbf{u}.\mathbf{x})(\mathbf{v}.\mathbf{w})$, which gives a scalar. Similarly, one can define double cross and dot cross mixed products of two dyads. Tensor product of a dyad with a vector gives triad – a tensor of rank 3: $\mathbf{D} \otimes \mathbf{w} = \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w}$, in which there are now $3 \times 3 \times 3$ or 27 elements (components). Tensor product of two dyads gives a tensor of rank 4 and so on.

Since $\mathbf{u} \otimes \mathbf{v}.\mathbf{w}$ always has to be parallel to $\mathbf{u}$, the representation provided by the dyadic product, therefore, cannot map a vector onto an arbitrary vector. Also, any linear combination of dyads represents a second rank tensor. It follows that not all second rank tensors can be represented as a dyad as noted earlier.

Again, since $\mathbf{u} \otimes \mathbf{v}.\mathbf{w}$ is a vector parallel to $\mathbf{u}$ whereas, $\mathbf{v} \otimes \mathbf{u}.\mathbf{w}$ is a vector always parallel to $\mathbf{v}$, it also implies that dyadic products are in general noncommutative. Swapping of vectors results in the conjugate or transposed or adjoint dyad. In the special case when it is commutative, the dyad is called symmetric. A symmetric dyad $\hat{u} \otimes \hat{u}$ obtained from the unit vector $\hat{u}$ projects an arbitrary vector $\mathbf{v}$ parallel to $\hat{u}$. These projection operators ($\hat{P} = \hat{u} \otimes \hat{u}$) are idempotent ($\hat{P}\hat{P} = \hat{P}$).

Tensor relationships of the form $\mathbf{y} = \mathbf{T}\mathbf{x}$ that take the vector (quantity) $\mathbf{x}$ to a new vector $\mathbf{y}$, and allow each component of $\mathbf{x}$ to influence each component of $\mathbf{y}$, arise in many physical contexts, such as polarization or current flow in an anisotropic medium, or wave propagation in a plasma. Each element of the tensor $\mathbf{T}$ relates the influence of a component of the stimulus to one component of the response; for example, T_{ij} accounts for the contribution to the i-th component of the response field $\mathbf{y}$ due to the j-th component of the applied field $\mathbf{x}$.

Under coordinate transformations, components of a tensor transform in a specified manner that conforms with the golden rule of physics: physical observables must not depend on the choice of coordinate frames. Tensors furnish a concise mathematical framework for solving problems in physics, especially in the theory of elasticity, solid and fluid mechanics, electromagnetism and general relativity.

A tensor $\mathbf{T}$ is said to be reducible, if it can be decomposed into parts of lower rank tensors. Consider a general second rank Cartesian tensor T_{ij}. One can always separate it into its symmetric and antisymmetric parts as:

$$T_{ij} = \frac{1}{2}(T_{ij} + T_{ji}) + \frac{1}{2}(T_{ij} - T_{ji})$$

The antisymmetric part A_{ij} ($= \frac{1}{2}(T_{ij} - T_{ji})$) has only three independent components and acts like a pseudovector $\mathbf{w}$ in 3D, with $w_k = A_{ij}$ in cyclic permutation of i, j, k. As we will see in the following section, an antisymmetric second rank tensor is actually equivalent to a bivector – the wedge product of two vectors.

Also, the trace of T_{ij}, $\tau = T_{ii}$, is a scalar quantity and subtracting the scalar τ and the antisymmetric part A_{ij} or the 'vector' $\mathbf{w}$ from the original tensor, one can have an *irreducible*, symmetric, zero-trace second-rank tensor S_{ij}, given by:

$$S_{ij} = T_{ij} - A_{ij} - \frac{1}{3}\tau\delta_{ij} = \frac{1}{2}(T_{ij} + T_{ji}) - \frac{1}{3}\tau\delta_{ij}$$

with only five independent components. So, the original Cartesian tensor T_{ij} is composed of the three quantities τ, w_k and S_{ij} which have the same transformation properties as the spherical harmonics Y_L^M for $L = 0$, 1 and 2 and are assumed to represent spherical tensors of rank 0, 1, and 2, respectively.

Similarly, one can have a unique decomposition of a higher rank tensor. Thus, it is possible to associate angular momentum quantum numbers with the irreducible tensors (operators). Exploitation of these properties leads to the Wigner-Eckart theorem which offers a quick determination of the selection rules that follow from rotational invariance.

Defined on a vector space, tensors describe linear relations between scalars, vectors and other tensors. On the other hand, the dot and cross products of VA are formally defined by the metric and the Levi-Civita tensors respectively. Tensors thus generalise and extend VA by including scalars and vectors in the 'tensor family'. However, the inadequacies of VA remains in the general formulation. Moreover, the standard tensor algebra is still treated as an add-on in the physics curriculum and the language of Einstein's theory of relativity so differs from the ordinary vector algebra that it amounts to a new language for students to learn. In this context, it is important to probe the assertion of Hestenes that, in applications, GA is more versatile than tensor algebra [10].

2. *The wedge product and Grassmann algebra*:

The Wedge product, also known as the exterior product or the progressive product of vectors, was defined by Grassmann in 1844 in his new algebra – a theory of scalars, vectors and multivector *blades* (bivectors, trivectors etc.)[5]. It operates on both scalars and vectors – with scalars, it is the simple scalar multiplication. The wedge product of a vector $\mathbf{v}$ with itself is always zero – as in the case of cross product (defined only in 3D). Using the basis vectors $\hat{e}_i$, wedge product is precisely defined as $\hat{e}_i \wedge \hat{e}_j = -\hat{e}_j \wedge \hat{e}_i$. This is clearly more accommodating than the defining equation of cross product of VA and it is this distinction which enables the wedge product to be defined in any dimension starting from 2-D. Using the above defining equation, the wedge product between two distinct vectors $\mathbf{u}$ and $\mathbf{v}$ in 2-D can be easily expressed in terms of their components as:

$$\begin{aligned}\mathbf{u} \wedge \mathbf{v} &= u_i v_j \, \hat{e}_i \wedge \hat{e}_j, \quad i \neq j, \\ &= u_1 v_2 \, \hat{e}_1 \wedge \hat{e}_2 + u_2 v_1 \, \hat{e}_2 \wedge \hat{e}_1 \\ &= (u_1 v_2 - u_2 v_1) \, \hat{e}_1 \wedge \hat{e}_2\end{aligned}$$

and the single-component 'bivector' in 2-D represents another new mathematical entity – 'pseudoscalar', distinct from both scalar and vector. Also, $I_2 = \hat{e}_1 \wedge \hat{e}_2$ represents the unit pseudoscalar $(I_2^2 = (\hat{e}_1 \wedge \hat{e}_2)^2 = \hat{e}_1 \wedge \hat{e}_2 : \hat{e}_1 \wedge \hat{e}_2 = -1)$.

For two distinct vectors $\mathbf{u}$ and $\mathbf{v}$ in 3-D, one gets similarly:

$$\begin{aligned}\mathbf{u} \wedge \mathbf{v} &= u_i v_j \, \hat{e}_i \wedge \hat{e}_j, \quad i \neq j, \\ &= u_1 v_2 \, \hat{e}_1 \wedge \hat{e}_2 + u_1 v_3 \, \hat{e}_1 \wedge \hat{e}_3 + u_2 v_3 \, \hat{e}_2 \wedge \hat{e}_3 + u_2 v_1 \, \hat{e}_2 \wedge \hat{e}_1 + u_3 v_1 \, \hat{e}_3 \wedge \hat{e}_1 + u_3 v_2 \, \hat{e}_3 \wedge \hat{e}_2,\end{aligned}$$

a 3-D bivector. Since $\hat{e}_i \wedge \hat{e}_j = -\hat{e}_j \wedge \hat{e}_i$, a bivector has 3 $(= \binom{n}{2}$, n being the dimensionality of space) independent components, but each component has 2 $(= 2!)$ independent coefficients ($u_i v_j$ and $u_j v_i$)

[5] In fact, a 'blade' is defined to represent any scalar, vector, or the wedge product of any number of vectors. It is also important to note the distinction between a trivector and a 3-vector or more generally between a k-blade and a k-vector. A trivector/k-blade is visualized as the 3-dimensional/k-dimensional region spanned by three vectors/k vectors. This region may be embedded in a space that is 3-dimensional/k-dimensional or higher. In contrast, a 3-vector/k-vector is a single vector that lives in a space with exactly 3-dimensions/k-dimensions.

associated with it. Hence, we finally have,

$$\begin{aligned}\mathbf{u}\wedge\mathbf{v} &= (u_2v_3-u_3v_2)\,\hat{e}_2\wedge\hat{e}_3+(u_3v_1-u_1v_3)\,\hat{e}_3\wedge\hat{e}_1+(u_1v_2-u_2v_1)\,\hat{e}_1\wedge\hat{e}_2 \\ &= \begin{vmatrix}\hat{e}_2\wedge\hat{e}_3 & \hat{e}_3\wedge\hat{e}_1 & \hat{e}_1\wedge\hat{e}_2 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3\end{vmatrix}.\end{aligned} \tag{5}$$

The bivector represents an oriented plane segment (containing $\mathbf{u}$ and $\mathbf{v}$) in contrast to the directed line segment representation of a vector. The orientation is specified by circulation around the edge of the bivector, which is completely geometrical and devoid of any convention for handedness. For the bivector $\mathbf{u}\wedge\mathbf{v}$ the 'direction of circulation' is traced by moving along the $\mathbf{u}$ direction first and then along the $\mathbf{v}$ direction. The wedge product anticommutes ($\mathbf{u}\wedge\mathbf{v}=-\mathbf{v}\wedge\mathbf{u}$) which implies that both have the same magnitude, but opposite orientation. The components of the bivector in 3-D so defined are identical to the components of the cross product vector (pseudo) of VA. However, only in three dimensions the number of independent components of a vector is same as that of a bivector, and one can write $\mathbf{u}\wedge\mathbf{v}=u_iv_j\,\hat{e}_i\wedge\hat{e}_j=B_{ij}\,\bar{e}_{ij},\;\; i\neq j$ with $B_{ij}=(u_iv_j-u_jv_i)$ representing the three components of the bivector and $\bar{e}_{ij}=\hat{e}_i\wedge\hat{e}_j$ as the unit bivector bases, i,j,k in cyclic order. The two appear to be the same object! But this is not true for higher dimensions ($n>3$) and in the following, we will see how the difference between the two becomes apparent. In 3-D also, it may be noted that the squared 'norm' of the three distinct basis bivectors ($\bar{e}_{ij}$) is $\sqrt{-1}$. A bivector may also be expressed in terms of the antisymmetric part of the dyadic or tensor products as:

$$\mathbf{u}\wedge\mathbf{v}=\frac{1}{2}(\mathbf{u}\otimes\mathbf{v}-\mathbf{v}\otimes\mathbf{u}). \tag{6}$$

With the vector differential operator ∇, one can similarly define $\nabla\wedge\mathbf{f}$ for a vector field $\mathbf{f}$ (in cartesian system):

$$\nabla\wedge\mathbf{f}=\begin{vmatrix}\hat{e}_2\wedge\hat{e}_3 & \hat{e}_3\wedge\hat{e}_1 & \hat{e}_1\wedge\hat{e}_2 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3\end{vmatrix}.$$

The wedge product thus modifies and extends the notion of cross product of VA, with the following stipulations:

(i) $\mathbf{v}_1\wedge\mathbf{v}_2=-\mathbf{v}_2\wedge\mathbf{v}_1$ for any two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in $\mathcal{V}^n$ (n-dimensional vector space over the real numbers) $\Longrightarrow \mathbf{v}\wedge\mathbf{v}=0$ for any vector $\mathbf{v}$ in $\mathcal{V}^n$;

(ii) The wedge product is, by definition, *associative* in the sense: $\mathbf{v}_1\wedge(\mathbf{v}_2\wedge\mathbf{v}_3)=(\mathbf{v}_1\wedge\mathbf{v}_2)\wedge\mathbf{v}_3$ unlike the cross product;

(iii) The wedge product also incorporates a kind of closure property. For example, a bivector $\mathbf{v}_1\wedge\mathbf{v}_2$ in two dimensions and a trivector $\mathbf{v}_1\wedge\mathbf{v}_2\wedge\mathbf{v}_3$ in three dimensions, both having only one component each that flips sign under reflection, represent pseudoscalars of the respective dimensions. But a trivector $\mathbf{v}_1\wedge\mathbf{v}_2\wedge\mathbf{v}_3$ in 2-D and a quadrivector $\mathbf{v}_1\wedge\mathbf{v}_2\wedge\mathbf{v}_3\wedge\mathbf{v}_4$ in 3-D collapse back down to scalar zero so as to prevent construction of any element of grade higher than the dimensionality of the space. Also, With the exceptions of (i) - (iii) all algebraic rules which apply to ordinary multiplication, also apply to the '$\wedge$' product.

In Grassmann algebra, the dot or inner product of two vectors is similarly defined as in VA, but always to be carried out first in a sequence. Moreover, the inner product or contraction between any arbitrary bivector and a vector is naturally not scalar-valued and produces a new vector. Like a second rank tensor, the bivector, therefore, defines a vector transformation equation in the form:

$$\begin{aligned}\mathbf{B}.\mathbf{v} &= B_{ij}\,\hat{e}_i\wedge\hat{e}_j.\,v_k\,\hat{e}_k=B_{ij}\,v_j\,\hat{e}_i-B_{ij}\,v_i\,\hat{e}_j \\ &= (B_{ij}-B_{ji})\,v_j\hat{e}_i,\;\; i\neq j, \\ &= \mathbf{v}'\;\text{say},\end{aligned} \tag{7}$$

– a transformed vector from $\mathbf{v}$ by the bivector operator $\mathbf{B}$. Unless otherwise specified, we use Euclidean metric which yields a positive-definite quadratic form. It is also evident from eq.(6) that a bivector is equivalent to an antisymmetric tensor of rank two. Also note that, the dot product between vector and a bivector anticommutes, i.e. $\mathbf{v}.\mathbf{B}=-\mathbf{B}.\mathbf{v}$.

Another important aspect of Grassmann algebra is the provision for appropriate multiple inner products or contractions between two of its elements. For example, between two bivectors, both single and double contractions are possible. These are obtained respectively by permuting the 'wedged' basis vectors only for distinct terms:

$$\begin{aligned}
\mathbf{A}.\mathbf{B} &= A_{ij}\,\hat{e}_i \wedge \hat{e}_j .\, B_{kl}\,\hat{e}_k \wedge \hat{e}_l \\
&= A_{ij}\,B_{jl}\,\hat{e}_i \wedge \hat{e}_l - A_{ij}\,B_{kj}\,\hat{e}_i \wedge \hat{e}_k - A_{ij}\,B_{il}\,\hat{e}_j \wedge \hat{e}_l + A_{ij}\,B_{ki}\,\hat{e}_j \wedge \hat{e}_k \\
&= A_{ij}\{(B_{jk} - B_{kj})\hat{e}_i \wedge \hat{e}_k + (B_{ki} - B_{ik})\hat{e}_j \wedge \hat{e}_k\}, \;\; i \neq j \neq k, \qquad (8)\\
&= -\mathbf{B}.\mathbf{A},
\end{aligned}$$

giving a bivector, and

$$\begin{aligned}
\mathbf{A} : \mathbf{B} &= A_{ij}\,\hat{e}_i \wedge \hat{e}_j : B_{kl}\,\hat{e}_k \wedge \hat{e}_l = A_{ij}(B_{ji} - B_{ij}), \;\; i \neq j \qquad (9)\\
&= \mathbf{B} : \mathbf{A},
\end{aligned}$$

producing a scalar. Also, $\mathbf{B} : \mathbf{B}$ gives the squared norm of the bivector $\mathbf{B}$.

The Wedge product of a vector and a bivector is a trivector given by:

$$\mathbf{v} \wedge \mathbf{B} = v_i\,\hat{e}_i \wedge B_{jk}\,\hat{e}_j \wedge \hat{e}_k = v_i\,B_{jk}\,\hat{e}_i \wedge \hat{e}_j \wedge \hat{e}_k; \;\; i \neq j \neq k,$$

and the product commutes i.e., $\mathbf{v} \wedge \mathbf{B} = \mathbf{B} \wedge \mathbf{v}$. In three dimensions, the three basis elements $\hat{e}_1 \wedge \hat{e}_2 \wedge \hat{e}_3$, $\hat{e}_2 \wedge \hat{e}_3 \wedge \hat{e}_1$ and $\hat{e}_3 \wedge \hat{e}_1 \wedge \hat{e}_2$ of a trivector are equivalent. Hence, a trivector $(\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \mathbf{v}_3)$ in 3-D has only one component (given by eq.(12) with $n = 3$), which is identical with the scalar triple product representing volume in VA – a pseudoscalar. The wedge product of the three basis vectors $\hat{e}_1 \wedge \hat{e}_2 \wedge \hat{e}_3$ is the unit pseudoscalar in 3D. Interestingly, the double contraction of the bivector of eq.(5) with the unit pseudoscalar $I_3 = \hat{e}_1 \wedge \hat{e}_2 \wedge \hat{e}_3$ gives:

$$\begin{aligned}
-I_3 : \mathbf{u} \wedge \mathbf{v} &= -\hat{e}_1 \wedge \hat{e}_2 \wedge \hat{e}_3 : \{(u_2 v_3 - u_3 v_2)\,\hat{e}_2 \wedge \hat{e}_3 + (u_3 v_1 - u_1 v_3)\,\hat{e}_3 \wedge \hat{e}_1 + (u_1 v_2 \\
&- u_2 v_1)\,\hat{e}_1 \wedge \hat{e}_2\} = (u_2 v_3 - u_3 v_2)\,\hat{e}_1 + (u_3 v_1 - u_1 v_3)\,\hat{e}_2 + (u_1 v_2 - u_2 v_1)\,\hat{e}_3 \\
&\equiv \mathbf{u} \times \mathbf{v} \qquad (10)
\end{aligned}$$

– the cross product of eq.(1). With nonorthogonal basis vectors $\hat{e}_i$, however, $-I_3 : \mathbf{u} \wedge \mathbf{v} = \sqrt{|g|}\,\mathbf{u} \times \mathbf{v}$. Similarly, $-I_3 : \nabla \wedge \mathbf{f} \equiv \nabla \times \mathbf{f}$ and one gets accordingly the curl of a vector field $\mathbf{f}$ of Gibbs-Heaviside VA. Moreover, we note that:

$$\begin{aligned}
-I_3 : \{(-I_3 : \mathbf{u} \wedge \mathbf{v}) \wedge \mathbf{w}\} &= -I_3 : [\{(u_2 v_3 - u_3 v_2)\hat{e}_1 + (u_3 v_1 - u_1 v_3)\hat{e}_2 + (u_1 v_2 - u_2 v_1)\hat{e}_3\} \wedge w_i \hat{e}_i] \\
&= -\hat{e}_1 \wedge \hat{e}_2 \wedge \hat{e}_3 : \{(u_2 v_3 - u_3 v_2)(w_2 \hat{e}_1 \wedge \hat{e}_2 + w_3 \hat{e}_1 \wedge \hat{e}_3) + (u_3 v_1 - u_1 v_3) \\
&\quad (w_1 \hat{e}_2 \wedge \hat{e}_1 + w_3 \hat{e}_2 \wedge \hat{e}_3) + (u_1 v_2 - u_2 v_1)(w_1 \hat{e}_3 \wedge \hat{e}_1 + w_2 \hat{e}_3 \wedge \hat{e}_2)\} \\
&= (u_2 v_3 - u_3 v_2)(w_2 \hat{e}_3 - w_3 \hat{e}_2) + (u_3 v_1 - u_1 v_3)(-w_1 \hat{e}_3 + w_3 \hat{e}_1) \\
&+ (u_1 v_2 - u_2 v_1)(w_1 \hat{e}_2 - w_2 \hat{e}_1) \\
&= (u_2 w_2 v_1 - v_2 w_2 u_1 + u_3 w_3 v_1 - v_3 w_3 u_1)\hat{e}_1 + (...)\hat{e}_2 + (...)\hat{e}_3 \\
&= (u_1 w_1 v_1 + u_2 w_2 v_1 + u_3 w_3 v_1 - v_1 w_1 u_1 - v_2 w_2 u_1 - v_3 w_3 u_1)\hat{e}_1 \\
&+ (...)\hat{e}_2 + (...)\hat{e}_3 = (\mathbf{u}.\mathbf{w})\mathbf{v} - (\mathbf{v}.\mathbf{w})\mathbf{u} \equiv (\mathbf{u} \times \mathbf{v}) \times \mathbf{w},
\end{aligned}$$

by adding and subtracting appropriate $u_i v_i w_i$ to each component. Similarly, we have: $-I_3 : \mathbf{u} \wedge \{-I_3 : (\mathbf{v} \wedge \mathbf{w})\} = (\mathbf{u}.\mathbf{w})\mathbf{v} - (\mathbf{u}.\mathbf{v})\mathbf{w} \equiv \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$. Although the wedge product is associative, the above result shows that, by mixing inner and exterior products, Grassmann algebra can manifest the nonassociativity of the triple cross product of VA. We will encounter similar results for geometric product in section 4.

In higher dimensions, one can define a trivector or a quadrivector by taking wedge product among three or four vectors and in general, with k ($< n$) vectors a 'k-blade' may be formed. Thus, each dimension is accordingly represented in exterior algebra. In the reduced form any k-fold ($k < n$) wedge product can be expressed as:

$$\begin{aligned}
\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \,...\, \wedge \mathbf{v}_k &= \sum_{i_1=1}^{n} v_{1i_1}\,\hat{e}_{i_1} \wedge \sum_{i_2=1}^{n} v_{2i_2}\,\hat{e}_{i_2} \,...\, \wedge \sum_{i_k=1}^{n} v_{ki_k}\,\hat{e}_{i_k} \\
&= \sum_{i_1 < i_2 \,...\, < i_k} (-1)^{\gamma} v_{1i_1}\, v_{2i_2} \,...\, v_{ki_k}\,\hat{e}_{i_1} \wedge \hat{e}_{i_2} \,...\, \wedge \hat{e}_{i_k}, \qquad (11)
\end{aligned}$$

where γ denotes the number of transpositions required to obtain $i_1 < i_2 ... < i_k$.

The wedge product of any number k ($< n$) of independent vectors is usually called a 'blade' or 'form' of grade k or simply, a k-blade or k-form. It lives in a geometrical space known as the k-th exterior power. The magnitude of the resulting k-blade is the volume of the k-dimensional parallelotope (a generalisation of the parallelepiped in higher dimensions). According to this terminology for the elements of the algebra, a scalar is of grade zero, a vector has grade 1, bivector has grade 2 and a trivector is assigned grade 3 etc. In n dimensions, both vectors and $(n-1)$-blades have n components and each $(n-1)$-blade basis involve a combination of all except one basis vector, and hence get the name *antivectors* or *pseudovectors*. Now if we consider the wedge product between a vector and a bivector in 3-D, for example: $v_i\hat{e}_i \wedge B_{jk}\hat{e}_j \wedge \hat{e}_k = v_i B_{jk}(\hat{e}_i \wedge \hat{e}_j \wedge \hat{e}_k)$, the single component (of the pseudoscalar) is similar to the result of dot product between two unlike vectors (or scalar triple product of like vectors) in VA. Actually in any dimension, the wedge product between a vector and an antivector is a pseudoscalar. This result clarifies and extends the notion of dual space of VA – the dual of a vector is an antivector. In 3-D, for example, the dual of a vector is a bivector, the wedge product of a pair of vectors that are orthogonal to it, and the area of the bivector is equal to the magnitude of its dual vector. Consequently, the dual of a bivector is its normal vector in 3-D. This important notion will be clarified further in the following discussions.

Consequently, this algebra has a total number of 8 *multivector* basis elements in 3-D: 1 scalar element, 3 vector elements, 3 bivector elements, 1 trivector (pseudoscalar) element and no higher-grade elements. Similarly, in four dimensions the total number of basis elements is 16 and whereas the number of components of a vector is 4, it is $\binom{4}{2} = 6$, for a bivector. Thus the distinction between a vector and a bivector becomes apparent.

For a k-blade and an $(n-k)$-blade, the number of basis elements, respectively given by the binomial coefficients $\binom{n}{k}$ and $\binom{n}{n-k}$, are the same. This produces an exact symmetry and the total number of 'multivector' basis elements is always

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Since the wedge product between a k- and an $(n-k)$-blade is a pseudoscalar (in n dimensional space) and the number of basis elements of those blades being same, they constitute *dual form.* Full Contractions with unit pseudoscalar I_n establishes a one-to-one mapping from the k-blade basis space to the $(n-k)$-blade basis space and vice-versa.

The number of k-blade basis elements and those of $(n-k)$-blade (in n dimensional space) are same, the wedge product between a k- and an $(n-k)$-blade can be defined and the product is a pseudoscalar. Also with k-tuple contractions (inner products), a k-blade basis reduces I_n into a basis of $(n-k)$-blade and vice-versa. Contractions with unit pseudoscalar I_n establishes a one-to-one mapping from the space of the k-blade to the space of $(n-k)$-blade and vice-versa. In fact, it provides an implementation of the Hodge duality operation of differential geometry [11]. The image of a k-blade under this mapping is called the (Hodge) dual of the k-blade, in the sense that when applied twice, the mappings result in an identity operation up to a sign factor which also depends on the metric of the vector space.

Finally following eq.(11), the n-fold wedge product ($k = n$) in $\mathcal{V}^n$ is given by:
$\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge ... \wedge \mathbf{v}_n = \Delta\, \hat{e}_1 \wedge \hat{e}_2 ... \wedge \hat{e}_n$, where,

$$\Delta = \begin{vmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ v_{n1} & v_{n2} & \cdots & v_{nn} \end{vmatrix} \tag{12}$$

is the determinant obtained from the n components of each of the n vectors $\mathbf{v}_i$.

The highest-grade (n) element of this algebra has the lone component (Δ), and evidently from its expression (eq.12), it flips sign under reflection. Hence follows the name – pseudoscalar (or antiscalar), for the single component n-blade. A pseudoscalar can be interpreted as the volume of an n-parallelotope in an n-dimensional vector space and the exterior algebra thus provides an appropriate generalisation of the notion of the pseudoscalar.

An important concept of Grassmann algebra, as mentioned earlier, is the introduction of the dual form – each element of the algebra has its dual. Pseudoscalar, the highest-grade element of the algebra is the dual of the lowest grade element, scalar and vice-versa. The scalar has no spatial extent while its dual pseudoscalar, representing volume of the n-parallelotope, has all the spatial extent. The study of dual spaces states that the pseudoscalar plays the role that the scalar does in normal space.

The computation of the square of the pseudoscalar I_n is given by: $I_n^2 = (\hat{e}_1 \wedge \hat{e}_2 \wedge \cdots \wedge \hat{e}_n)(\hat{e}_1 \wedge \hat{e}_2 \wedge \cdots \wedge \hat{e}_n)$, one can either reverse the order of the second group or apply a perfect shuffle, both require $(n-1)n/2$ swaps and yielding the sign factor $(-1)^{n(n-1)/2}$, which is 4-periodic, and combined with $\hat{e}_i \cdot \hat{e}_i$ the square is given by $I_n^2 = \pm 1$. Note that the inverse $I_n^{-1} = \hat{e}_n \wedge \hat{e}_{n-1} \wedge \cdots \wedge \hat{e}_1 = (-1)^{(n-1)n/2} I_n = \pm I_n$.

On the other hand, physical quantities represented by pseudovectors of classical Gibbs-Heaviside VA are aptly replaced by bivectors in Grassman algebra. For a point-like object with mass m and linear momentum $\mathbf{p}$ $(= m\mathbf{v})$, both the angular velocity $(\boldsymbol{\Omega})$ and angular momentum $(\mathbf{L})$ are represented by bivectors as: $mr^2\boldsymbol{\Omega} = \mathbf{L} = \mathbf{r} \wedge \mathbf{p}$ where $\mathbf{r}$ is the object's position/radius vector and $r = |\mathbf{r}|$. The linear velocity $\mathbf{v}$ in this algebra[6] is given by the inner product $\mathbf{r}.\boldsymbol{\Omega}$. Also, the centripetal and the Coriolis accelerations are expressed by the angular velocity bivector $\boldsymbol{\Omega}$ as $\boldsymbol{\Omega}.(\boldsymbol{\Omega}.\mathbf{r})$ and $2\boldsymbol{\Omega}.\mathbf{v}$ respectively. In a recent formulation of the elastic theory of shells using GA, Gregory et al [12] have discussed the advantages of using the bivector representation of angular velocity which allows for a much more physical representation of the governing laws and clarified the confusion created by the usual conventions of VA. Similarly the torque $\boldsymbol{\Upsilon}$ $(= \mathbf{r} \wedge \mathbf{f})$ in a force field $\mathbf{f}$ is described by a bivector and so also the magnetic field $\mathbf{B}$ of electrodynamics. According to Biot-Savart law, the magnetic induction $\mathbf{B}$ at a point due to a current-element is conventionally given by the cross-product of the vectorial current-element with the position vector of the point.

VA also introduces in its definition of cross product, a handedness (chirality), even where there is no chirality in the entity being modelled. The apparent chirality in electromagnetism, i.e. Fleming's right-hand rule for electric generators, and the left-hand rule for electric motors, turns out to be actually a mathematical artifact used by VA to describe the physical process and not a part of the reality itself. In this context the resolution of the 'Pierre's puzzle' [13] regarding the symmetry (or lack of it) in the magnetic field of a magnetic needle (a permanent magnet) may be mentioned. It turns out that electromagnetism has no chirality, as revealed in geometric algebra which adopts exterior product of Grassman. However, the Hodge dual mentioned earlier, introduces a notion of handedness and can be used as an optional feature added to the basic geometric algebra package to handle *genuine chirality which appear due to spin of elementary particles in certain weak nuclear processes.*

3. *Quaternions and quaternion algebra*:

If a given complex number z is multiplied by $e^{i\phi}$, the product $e^{i\phi}z$ has the same length/magnitude as z, but gets a rotation through an angle ϕ. The complex multiplication can, therefore, be used to produce a geometric operation – rotation[7]. While investigating a higher dimensional generalisation of the complex numbers, Hamilton in 1843 has developed the first noncommutative algebra – the quaternion algebra. In analogy with the imaginary i of two component complex number, he introduced the basic unit triplet q_1, q_2, q_3[8] – all square roots of -1, representing the set (S^3) of unit quaternions, to define the 4-component (a_0, a_1, a_2, a_3) quaternion a as:

$$a = a_0 + q_k a_k,\ k = 1, 2, 3 \tag{13}$$

and enunciated the fundamental equation (multiplication rule) of quaternion algebra:

$$q_k q_l = -\delta_{kl} + \epsilon_{klm} q_m \tag{14}$$

If the real part (a_0) is zero, then a is a 'pure quaternion'.

[6]Since $\mathbf{r} \wedge \boldsymbol{\Omega} = 0$, $\mathbf{v}$ is finally given by $\mathbf{r}\boldsymbol{\Omega}$ – the geometric product (eq.21) between a vector and a bivector in GA.

[7]In complex algebra multiplication by a complex number, in general, produces both rotation and dilation.

[8]This notation relates to traditional Hamilton's notation [2] as $q_1 = i$, $q_2 = j$, $q_3 = k$, where the fundamental quaternion equation reads: $i^2 = j^2 = k^2 = ijk = -1$.

3.1 Quaternion algebra:
Just like the addition (subtraction) of vectors, addition (subtraction) of quaternions acts component-wise. More specifically, consider the quaternion a defined above and another quaternion $b = b_0 + q_k b_k$. Then we have $a \pm b = (a_0 \pm b_0) + q_k(a_k \pm b_k)$.

Multiplication of quaternion with a scalar follows ordinary (scalar) multiplication rules. The complex conjugate of a is defined as $a^* = a_0 - q_k a_k$ and from this definition we immediately have $(a^*)^* = a_0 - (-q_k a_k) = a$; $a_0 = (a + a^*)/2$, $q_k a_k = (a - a^*)/2$. Also, using the fundamental multiplication rule (eq.14), we get:
$a^*a = (a_0 - q_k a_k)(a_0 + q_l a_l) = a_0^2 + a_1^2 + a_2^2 + a_3^2 = aa^*$. The norm of a quaternion a, is the scalar denoted by $|a| = \sqrt{a^*a}$. A quaternion is called a unit quaternion if its norm is 1. The only quaternion with norm zero is zero, and every nonzero quaternion has a unique inverse. It implies that the quaternions form a division algebra. An algebra $\mathcal{A}$ is a division algebra if given $a, b \in \mathcal{A}$ with $ab = 0$, then either $a = 0$ or $b = 0$. Equivalently, $\mathcal{A}$ is a division algebra if the operations of left and right multiplication by any nonzero element are invertible[9]. The inverse a^{-1} of a quaternion a is defined as: $a^{-1}a = aa^{-1} = 1 \Longrightarrow a^{-1} = a^*|a|^{-2}$. For a unit quaternion α, the inverse is its conjugate α^*.

One can define the product of two arbitrary quaternions a and b using eq.(14) as:

$$\begin{aligned} ab = (a_0 + q_k a_k)(b_0 + q_l b_l) = a_0 b_0 - (a_1 b_1 + a_2 b_2 + a_3 b_3) + a_0(q_1 b_1 + q_2 b_2 + q_3 b_3) \\ + b_0(q_1 a_1 + q_2 a_2 + q_3 a_3) + q_1(a_2 b_3 - a_3 b_2) + q_2(a_3 b_1 - a_1 b_3) + q_3(a_1 b_2 - a_2 b_1) \end{aligned} \tag{15}$$

Thus the product ab is in general not equal to ba and it follows that $(ab)^* = b^*a^*$. Also, the norm of the product of two quaternions a and b is equal to the product of the individual norms. This is easily verified in the following:
$|ab|^2 = (ab)(ab)^* = ab\, b^*a^* = a|b|^2 a^* = aa^*|b|^2 = |a|^2|b|^2$.

Expanding the result in terms of components, one gets the Euler's four-square identity,

$$\begin{aligned} (a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3)^2 + (a_0 b_1 + a_1 b_0 + a_2 b_3 - a_3 b_2)^2 + (a_0 b_2 + a_2 b_0 + a_3 b_1 \\ - a_1 b_3)^2 + (a_0 b_3 + a_1 b_2 + a_3 b_0 - a_2 b_1)^2 = (a_0^2 + a_1^2 + a_2^2 + a_3^2)(b_0^2 + b_1^2 + b_2^2 + b_3^2). \end{aligned}$$

This identity states that the product of two numbers, each of which is a sum of four squares, is itself a sum of four squares and is not quite as obvious as the 2-squares rule (derived from ordinary complex algebra).

In modern mathematics, the quaternions are a number system and its algebra extends both the complex and vector algebras. In fact, the algebra of quaternions belongs to a subalgebra (even) of GA − the algebra consisting of scalars, bivectors and quadrivectors etc. Quaternion is easily identified with the geometric product (eq.17) of two 3-D vectors in GA. Expressed as the sum of a scalar a_0 (the scalar product part) and the pure quaternion part $q_k a_k$, for which the wedge product (bivector $\mathbf{A}$, say) provides an appropriate representation. In fact, the similar algebraic properties of the pure imaginary i and the 'unit' bivector bases allow an algebraic isomorphism. In the following section, it will be shown that this representation of the quaternion in GA is consistent with the quaternion algebra and correctly reproduce the product of two arbitrary quaternions (eq.15). Most literatures, however, erroneously represent the pure quaternion part with a 'vector' and express this equation in terms of the scalar and cross products of two 'vectors'.

3.2 Euler-type formula for a quaternion and rotation about any arbitrary axis:
Making use of the Taylor expansion, one can define the exponential of a quaternion i.e. $\exp(a)$. The inverse operation, logarithm of a, may also be defined accordingly. Now, from the definition of unit quaternion $\alpha = a|a|^{-1} = (a_0 + q_k a_k)|a|^{-1}$ and noting that $|a|^{-2}(a_0^2 + a_k a_k) = 1$, one can write $a_0|a|^{-1} = \cos\theta$ and $\sqrt{a_k a_k}\,|a|^{-1} = \sin\theta \Rightarrow \alpha = \cos\theta + (q_1 a_1 + q_2 a_2 + q_3 a_3)\frac{\sin\theta}{\sqrt{a_1^2 + a_2^2 + a_3^2}} = \cos\theta + \bar{a}\sin\theta = \exp(\bar{a}\theta)$, where the factor $\frac{q_1 a_1 + q_2 a_2 + q_3 a_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}}$ represents any unit pure quaternion (unit 3-D bivector in GA) $\bar{a}$. From this one gets an Euler-type expression for a quaternion: $a = |a|\alpha =$

[9] Of the four possible normed division algebras (real, complex, quaternion and octonion), GA provide a way to generalize the first three. The octonions being nonassociative [14, 15], the family of GA appears to diverge at the point of the octonions. Nonassociative algebras are also being used in some recent string theoretic models and in quantum systems with magnetic charges. Proper accomodation of this algebra in physical theory requires introduction of nonassociative star products [16].

$|a|\exp(\bar{a}\theta)$. It also allows to define the power of a as $a^r = |a|^r \exp(\bar{a}r\theta) = |a|^r (cos(r\theta) + \bar{a}\, sin(r\theta))$, r being a real number.

Now with α as a unit quaternion and $\mathbf{v}$ a vector in 3D, consider the transformation:

$$\mathbf{v}' = \alpha\mathbf{v}\alpha^{-1} = e^{\bar{a}\theta}\mathbf{v}e^{-\bar{a}\theta} \tag{16}$$

The inverse of this operation is simply $\mathbf{v} = \alpha^{-1}\mathbf{v}'\alpha$. In literatures, this transformation (eq.16) is usually described by treating the vector $\mathbf{v}$ as a pure quaternion (a quaternion with zero real part). This is not proper – *a pure quaternion should be represented by a bivector* in 3-D. Hestenes [6] has provided an appropriate account to show that eq.(16) describes pure rotation of $\mathbf{v}$ through an angle 2θ where the unit bivector represents both the plane and the sense of rotation[10]. Thus, Euler's expression for a unit quaternion consists of a unit bivector representing the plane of rotation and half the angle of rotation i.e. θ (representing the scalar part). It may also be noted here that, it is always better to define a rotation by the plane of rotation rather than the axis of rotation which is only definable in 3D. The occurrence of half-angles in eq.(16) for a rotation is due to the bilinear form of the transformation. It follows that a unit bivector, geometrically representing a directed (unit) plane, is also a generator of rotation and, multiplication by it rotates any vector in that plane (of rotation) through 90^0. It is also shown there that in two dimensions, since all the vectors lie entirely in the plane of rotation, the bilinear form is not needed and this formula reduces to the single sided operation, analogous to the conventional formula for the rotation of ordinary complex number in the Argand plane. With a simple computer programme, the action of the quaternion rotation operator may be visually demonstrated [17].

Note that both α and $-\alpha$ represent the same rotation, since $-\alpha\mathbf{v}(-\alpha^{-1}) = \alpha\mathbf{v}\alpha^{-1}$. Furthermore with α^2, the rotation is twice the angle and in general, for α^n it is n times the angle along the same plane of rotation as α. This can be extended to arbitrary real n, allowing for smooth interpolation between spatial orientations. Alternatively, from $\alpha(\beta\mathbf{v}\beta^{-1})\alpha^{-1} = (\alpha\beta)\mathbf{v}(\beta^{-1}\alpha^{-1}) = (\alpha\beta)\mathbf{v}(\alpha\beta)^{-1}$, we see that composition of rotations simply corresponds to multiplication of quaternions. While a unit quaternion represent pure rotation in 3-D, it should be noted that in addition to rotation a quaternion produces a dilation, since $a\mathbf{v}a^{-1} = |a|^2\alpha\mathbf{v}\alpha^{-1}$ – similar to the way of a complex number representing rotation and dilation in 2-D.

Using the quternion multiplication rule (eq.14), one can also rewrite the quaternion $a = a_0 + q_1a_1 + q_2a_2 + q_3a_3$ in terms of a pair of complex numbers $c_1 = a_0 + q_1a_1$ and $c_2 = a_2 - q_1a_3$ as $a = a_0 + q_1a_1 + q_2a_2 + q_3a_3 = (a_0 + q_1a_1) + q_2(a_2 - q_1a_3) = c_1 + q_2c_2$ – also called a complex 2-vector. In the literature, quaternions embrace following equivalent definitions:

(i) a 4-component 'hyper' complex number with three 'imaginary' components,

(ii) a complex 2-vector, which, together with the basic eq.(14), describe the quaternion algebra, and

(iii) a scalar plus a 3-D bivector in GA and the equivalent Euler's definition of a unit quaternion in terms of an angle and a unit bivector.

3.3 Applications and advantages of quaternion algebra:

(i) In many applications the quaternion rotation procedure is found to be more effective than the conventional rotation matrix. Quaternions encode rotations by four real numbers only, whereas the linear representation of these transformations as 3×3 matrices requires nine. Moreover, Hamilton's algebraic system guides intuition and facilitates implementation in every detail with its explicit geometrical import. In a sequence of rotations, interpolation with quaternionic representation is more convenient than that with the familiar Euler angles. Quaternions are frequently used in computer graphics programming.

(ii) The present day understanding of spinors, which describe the spin states of electrons and other spin-half elementary particles, is closely linked to Hamilton's quaternions.

(iii) A variety of fractals can be explored using hypercomplex numbers. For example, interesting 3-D fractals have been generated with quaternions [18].

[10] It is also shown here that, similar bilinear mapping with a unit vector $\hat{u}_1$ can reverse the direction of any vector collinear with $\hat{u}_1$ and leaves any vector orthogonal to it unaffected, and represents a reflection in the plane orthogonal to $\hat{u}_1$. In fact if the first mapping is followed by a mapping with a second unit vector $\hat{u}_2$, the resulting composite mapping is equivalent to the transformation represented by eq.(16) and is simply a rotation of $\mathbf{v}$ through an angle twice the angle between $\hat{u}_1$ and $\hat{u}_2$. Note that this result is derived using the rules of geometric product and not only that the composition of two reflections is equivalent to a rotation, but every rotation can be expressed as a product of two reflections.

(iv) The quaternion product is invertible and the product of two quaternion norms immediately gives the four-square identity of Euler. Quaternions are also used in one of the proofs of Lagrange's four-square theorem in number theory, which states that sum of four integer squares is required to obtain any nonnegative integer.

Despite so many advantages of quaternion algebra, the original formulation by Hamilton was plagued by some serious problems which upset its further developments. For instance, the status of the pure quaternion introduced as vector was not clear. Also, even the proponents felt that the full quaternion product was of little use and preferred to keep the scalar and the 'vector' parts separate. This approach missed a major advantage of the fact that *the quaternion product is invertible!* Hamilton was also wrong in describing rotation with quaternions. Instead of using the double-sided transformation rule (eq.16) he used a single-sided transformation rule $\mathbf{v}' = \alpha\mathbf{v}$ [3].

Maxwell in his 'Treatise' on Electromagnetic theory (of 1873), has modified his original equations of 1865 and also presented them in terms of twenty quaternion equations [19]. But actually, he kept the scalars separate from the pure quaternion parts in his calculations. During the formative period of the Electromagnetic theory, the quaternion notation was often used without using the whole benefit of quaternion algebra[11].

The vector algebra also, was not fully developed at that point of time and after several years during mid 1880's, Gibbs [20] and Heaviside [21] have independently developed it in a seemingly direct and easily applicable form for 3-D vectors which transformed Maxwell's equations into a compact form. However, this has created a prolonged and acrimonious debate [22] among many scientists and Maxwell himself was convinced that the quaternions, and not the vectors, can only provide a correct description of electromagnetism. In fact, after the introduction of Special Relativity, the four-vector representation close to the quaternion formulation of Maxwell's equations became more favoured.

Birkhoff and von Neumann [23] have proposed that quantum mechanics can be formulated on vector space defined over quaternion field instead of the usual complex field. Constructing the quaternionic Hilbert space, quaternionic generalisation of standard quantum mechanics, was worked out extensively by Finkelstein et al. [24]. Adler [25] has discussed at length, the motivation, both mathematical and physical, and the importance of this generalisation.

3.4 Spinors and quaternions:

A mathematical entity $\psi(\theta)$ is called 'spinor' if it changes sign under rotation of 2π, i.e.

$$\psi(\theta + 2\pi) = -\psi(\theta)$$

Spinors thus differ from vectors or tensors under rotational transformation. Compared to rather simple visualisations of vectors and tensors, the pictorial depiction of a spinor is more subtle. However, a spinor may be visualized as a vector pointing along the Möbius band, exhibiting a sign inversion when it is rotated through a full turn of 2π.

In the process of classifying all possible linear representations of rotation group, Cartan [26] first formalized spinor as a mathematical object, closely related to Hamilton's quaternion in the form of a two-component complex vector. The term 'spinor' was actually coined by Ehrenfest in his work on quantum physics, and there are various equivalent ways to introduce spinors.

Following the discovery of the intrinsic angular momentum called 'spin' associated to electrons and other elementary particles, Pauli in 1927, has introduced spinors in the theory of physics as the basic representation (two component spin statefunctions) of his 2×2 (complex) matrices (σ_k) for the quantum mechanical spin observables s_x, s_y and s_z corresponding to the components of the equivalent spin pseudovector. It turns out that spin-1/2 particles have statefunctions that change sign under rotation of 2π and, therefore, can be represented by spinors. Unlike a vector, rotating a spinor by 2π does not bring it back to the same state but to the state of opposite phase. To

[11] It is argued that, if the founders of electrodynamics would have used the full quaternion notation, "they would have discovered relativity much before Voight, Lorentz and Einstein". Electromagnetic theory is inherently a relativistic theory and according to the special relativity, a pure quaternion in one inertial frame of reference is not a pure quaternion in a different one. New set of Maxwell's equations, developed in recent literatures [27] using quaternions/biquaternions, transform to usual vector equations when the Lorentz gauge is applied. Quaternionic formulations of electromagnetic duality and the Maxwell's equations in the presence of both electric and magnetic charges (dyons, with a zero electric charge is usually referred to as a magnetic monopole) have also been developed recently.

understand this, we note that the three Pauli matrices together with the unit matrix form the complete basis of an algebra, the Pauli algebra [28]. This is identical with the quaternionic algebra if $-i\sigma_k$ is used instead of σ_k as the basis elements. Now, consider the operator $R_\phi = e^{\frac{-is_z\phi}{\hbar}}$ which transforms (rotates) the spin state of the system from $|\chi>$ to $|\chi>_\phi$, i.e. $|\chi>_\phi= R_\phi|\chi>$. It can be easily shown that this transformation changes the expectation values $< s_x >$ and $< s_y >$ ($< s_x >=< \chi|s_x|\chi >$ etc.) of the operators s_x and s_y to

$$\begin{aligned} < s_x >_\phi &= < s_x > \cos\phi - < s_y > \sin\phi, \quad \text{and} \\ < s_y >_\phi &= < s_y > \cos\phi + < s_x > \sin\phi \end{aligned}$$

respectively, leaving $< s_z >$ unchanged. R_ϕ, therefore, produces ccw rotation of the system about the z-axis through an angle ϕ. Now, if we take any arbitrary spin-half ket (spinor) $|\chi >= c_1|+ > +c_2|- >$:

$$|\chi >_\phi = e^{\frac{-is_z\phi}{\hbar}}|\chi >= e^{\frac{-i\phi}{2}}c_1|+ > +e^{\frac{i\phi}{2}}c_2|- > .$$

For $\phi = 2\pi$ we have, $|\chi >_{\phi=2\pi}= -c_1|+ > -c_2|- >= -|\chi >$ and it requires a 4π spatial rotation to get back the original state vector.

Also, since a spinor represents probability amplitude of the spin state, its modulous is a positive-definite scalar. So, it can always be written as $\chi = |\chi|R$ and can be regarded as an instruction for rotation and dilation. Therefore, both Pauli spinors and the spin matrices can be represented by quaternions, and equivalently by arbitrary even elements (scalars plus bivectors) of 3-D GA.

In the following year 1928, Dirac has developed the fully relativistic theory of electron spin by showing the connection between spinors and the Lorentz group. Both tensors and spinors are connected with the rotation group representations and are defined in terms of their transformation properties. Actually, spinors allow a more general treatment of the notion of invariance under rotation and Lorentz boosts. They can be used without reference to relativity, but arise naturally in the discussions of Lorentz group. Further insightful developments by Dirac, Weyl and Majorana will be discussed in the next part of our study.

It may be noted that the quaternions and spinors have equivalent algebraic properties as well as the same geometric significance [6, 29]. In fact, a generalisation of the rotation-dilation produced by quaternions in 3-D to n-space is also possible with the spinors. Like quaternions, the spinors form a subalgebra of GA – the even subalgebra formed by the vectors of an n-dimensional vector space $\mathcal{V}^n$ and generate rotation and dilation. Hestenes [30] has also carried out a reformulation of Dirac theory of spinors using this concept.

Apart from the crucial application in the study of matter and fields in quantum mechanics, spinors are also important in the study of the spacetime geometry [31]. Moreover, some closely related ideas were already present in the study of rotation of rigid bodies. The appellation 'spinors', as suggested by Ehrenfest, was formally introduced by van der Waerden in 1929 to all finite order of the Lorentz Group representations, long after Cartan found them in 1913 in his study of representations of simple groups. Even earlier in 1897, Klein [32] has introduced similar objects – the so-called 'Cayley-Klein parameters' in rotational dynamics, in terms of which one can define two conjugate spinor kets or column vectors [33].

The 'Twistor' theory, proposed by Penrose [34] in 1967 with the intention of unifying general relativity and quantum mechanics into a theory of Quantum Gravity, also requires an understanding of spinors. It may be viewed as an extension of spinor algebra which uses the subset of 'pure spinors'. In 2003, Witten [35] has proposed a connection between the string theory and the twistor geometry that he called 'twistor string theory'. String theorists are presently showing renewed interest in twistors.

Spinors are used in a wide range of fields, from the quantum physics of fermions, general relativity, pulse synthesis and analysis, image processing, computer vision and recognition, aeronautics and robotics to fairly abstract areas of algebra and geometry. Also, Hestenes [6] has clearly demonstrated the advantages of using the spinor formulation in the study of dynamics of rigid bodies in classical mechanics. Both quaternions and spinors are also used in the formulation of various technological developments in the fields of computer science, image processing, edge detection to name a few.

A number of new mathematical techniques and languages were introduced by some great mathematicians during the second half of the 19th centuries. The search for a unifying mathematical language, which began with the works of Hamilton, Grassmann, Cartan and Clifford, generated considerable interest among contemporaries. This, however, was largely eclipsed with the development of a more straightforward and easily applicable 'vector algebra' by Gibbs and Heaviside. Subsequently discrete algebraic systems, such as matrix, tensor and spinor algebras etc. are adapted and created as and when required. In this context it may be pointed out that, the first formulation of quantum mechanical spinor algebra by Pauli signalled, though not being acknowledged, a resurgence of Clifford algebra in the form of geometric algebra. Hestenes has revealed the geometric meaning of Clifford, Pauli and Dirac algebras and pioneered GA in the mid-1960s. While providing an immensely powerful mathematical framework in which the most advanced concepts of quantum mechanics, relativity, electromagnetism, etc. can be expressed, it is also claimed that GA is straightforward and simple enough to be taught to school children! It has taken so many years since, to acknowledge the claim of Hestenes "that GA is the universal language for physics and mathematics" [36].

4. *Rudiments of Clifford's Geometric Algebra:*

Inspired by Hamilton's work on quaternions, Clifford searched for a higher dimensional algebra [4] and ultimately found unification of quaternions with Grassmann algebra. The distinctive feature of this algebra is incorporated through its novel multiplication rule. The geometric product $\mathbf{uv}$ of any two vectors $\mathbf{u}$ and $\mathbf{v}$ is designed to contain all the information about the relative directions of vectors $\mathbf{u}$ and $\mathbf{v}$. This is done first by separating the symmetric and the antisymmetric parts of $\mathbf{uv}$. The symmetric part is then identified with the projection of one vector on to the other i.e. the dot product $\mathbf{u}\cdot\mathbf{v}$ and the antisymmetric part is represented by the wedge product $\mathbf{u}\wedge\mathbf{v}$. The geometric product is, therefore, given by:

$$\mathbf{uv} = \mathbf{u}.\mathbf{v} + \mathbf{u}\wedge\mathbf{v}, \tag{17}$$

$$\text{with } \mathbf{u}.\mathbf{v} = (\mathbf{uv} + \mathbf{vu})/2 \tag{18}$$

$$\text{and } \mathbf{u}\wedge\mathbf{v} = (\mathbf{uv} - \mathbf{vu})/2 \tag{19}$$

Note that a scalar quantity ($\mathbf{u}.\mathbf{v}$) is added here to a bivector and $\mathbf{uv}$ contains all the n^2 elements $v_1^i v_2^j$ of the dyad $\mathbf{u}\otimes\mathbf{v}$. In fact, the even subalgebra generated by the geometric product $\mathbf{uv}$ contains only scalars and bivectors and in two dimensions the even subalgebra is isomorphic to the complex numbers, while in three it is isomorphic to the quaternions. The algebra thus introduces a linear and invertible product – the fundamental product of the algebra, since other products i.e. the inner and the exterior products, can be derived from it. Also, a fact should be mentioned here that Grassmann, in his later years, combined the inner and exterior products to form a new product [37] very similar to eq.(17). Thus, Grassmann has also discovered the key idea of geometric product independently of Clifford and evidently somewhat before him. Many historians of mathematics have overlooked this important later work of Grassmann.

Clifford algebra incorporates both the inner and wedge products with the defining equation: $\hat{e}_j\hat{e}_k + \hat{e}_k\hat{e}_j = 2\delta_{jk},\ \ j,k = 1,2,...,n$. The rule makes the manipulation of orthogonal basis vectors quite simple. Given a product $e_{i_1}e_{i_2}...e_{i_k}$ of distinct orthogonal basis vectors of $\mathcal{V}^n$, one can put them into standard order while including an overall sign determined by the number of pairwise swaps needed to do so. Orthogonal Clifford algebra - the most familiar one, is necessarily Riemannian and the symplectic[12] Clifford algebra is referred to as Weyl algebra.

The product $\mathbf{uv}$ (eq.17) acquires a geometrical significance from the interpretations given to $\mathbf{u}\cdot\mathbf{v}$ and $\mathbf{u}\wedge\mathbf{v}$ and gives a direct measure of the relative directions of vectors $\mathbf{u}$ and $\mathbf{v}$. Thus, if the two vectors are collinear, then they commute, i.e., $\mathbf{uv} = \mathbf{vu}$, and if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, they anticommute: $\mathbf{uv} = -\mathbf{vu}$. In general, $\mathbf{uv}$ describes the 'degree of commutativity' somewhere between these two extremes. The two products $\mathbf{u}\cdot\mathbf{v}$ and $\mathbf{u}\wedge\mathbf{v}$ together, i.e. the full product $\mathbf{uv}$,

[12]Symplectic geometry is not directly related to the properties of space and time. It is rather intimately related to the description of the dynamics of physical systems. The symplectic transformation of the $2n$ dynamical variables/operators preserves the basic algebraic bracket relations (Poisson or commutator) of the relevant mechanics. It is, therefore, implicit that symplectic geometry is defined on a smooth even dimensional differentiable manifold. The symplectic 'form' in symplectic geometry allows for the measurement of sizes of two dimensional objects in the space and plays a role analogous to that of the metric (tensor) in Riemannian geometry. Whereas the metric 'measures' lengths and angles, the symplectic form measures areas. The symplectic geometry has been one of the most rapidly advancing areas of mathematics over the past years.

provides the complete geometrical relation between $\mathbf{u}$ and $\mathbf{v}$. Since $\mathbf{v} \wedge \mathbf{v} = 0$, the geometric product of $\mathbf{v}$ with itself is identical with $\mathbf{v}.\mathbf{v}$, i.e. represents the squared norm of the vector $\mathbf{v}$.

Apart from the comprehensive geometrical significance gained from the geometric product defined as a sum of a scalar and a bivector, combinations of scalar, vector, bivector, and pseudoscalar turn out to be significant in many applications in physics [38]. Any general element of this algebra is called a 'multivector', 'clif' or 'geometric'. A multivector that has a definite grade is called homogeneous and is necessarily either a blade or the sum of a number of blades, all of same grade. Scalars, vectors, bivectors, trivectors etc., therefore, represent elements of different grades of the GA. We are familiar with the idea of adding scalars to scalars, vectors to vectors and so on i.e. addition of elements of same grade only. But while a comparison of elements of different grades is not possible, GA admits addition of elements of different grades, like a complex number having an imaginary part added to the real part. For example, it can have an element C such that: $\mathrm{C} = s + \mathbf{v} + \mathbf{B}$, where s is a scalar, $\mathbf{v}$ is a vector, and $\mathbf{B}$ is a bivector[13] in the extended *Clifford space* $\mathcal{G}^n$. Hestenes [6] has shown that the electromagnetic field is best represented by a clif consisting of a vector and a bivector. On the other hand, the algebra of ordinary complex numbers, quaternions or spinors are even subalgebra of multivectors formed by the geometric product of two vectors.

Dirac was first to recognize Clifford algebra as a superior expression for his theory of electrons, although it was Hestenes who appreciated its wider significance and applicability. Hestenes argues that it is not just another algebra, but a fundamental discovery of the geometrical roots of all algebras, and prefers to call it 'geometric algebra'. It was actually Clifford's own choice, because 'Clifford algebra' sounds much like 'just another algebra' rather than what it really is. One may generate a finite-dimensional GA by choosing a unit pseudoscalar (I_n). The set of all vectors $\{\mathbf{v}\}$ that satisfy $\mathbf{v} \wedge I_n = 0$ constitutes a vector space. The geometric product of the vectors in this vector space then defines the GA, of which I_n is a member. Since every finite-dimensional GA has a unique I_n, one can define or characterize the GA by it. With the pseudoscalar, the dual of a clif is also easy to calculate – it is the geometric product of the clif and the pseudoscalar.

In geometric calculus, the vector differential operator ∇ is equivalent to the usual gradient operator when operates on a scalar field. Here the gradient of a vector, or of any clif, is also a well defined quantity. Just like the geometric product of eq.(17), for a vector field $\mathbf{f}$,

$$\nabla \mathbf{f} = \nabla.\mathbf{f} + \nabla \wedge \mathbf{f} \tag{20}$$

and contains both the divergence and curl ($I_3(\nabla \times \mathbf{f})$, to be precise) of VA and describes the complete rate of change of $\mathbf{f}$ across a surface.

Using the multiplication rules of exterior algebra, one can similarly write the geometric product of a vector $\mathbf{v}$ and bivector $\mathbf{B}$ as:

$$\mathbf{vB} = \mathbf{v}.\mathbf{B} + \mathbf{v} \wedge \mathbf{B} \tag{21}$$

with, $\mathbf{v}.\mathbf{B} = (\mathbf{vB} - \mathbf{Bv})/2$, and $\mathbf{v} \wedge \mathbf{B} = (\mathbf{vB} + \mathbf{Bv})/2$. Note here the differences with the equations (18) and (19) – with the change of grade, there is a change of sign in the commutation relation. Also note that, in $\mathcal{V}^n$, the geometric product of two blades of grade r and of grade s, is defined to contain blade of all grades from $|r - s|$ to $r + s$ ($\leq n$) in steps of +2. For example, the product between two bivectors $\mathbf{A}$ and $\mathbf{B}$ is given by:

$$\mathbf{AB} = \mathbf{A} : \mathbf{B} + \mathbf{A}.\mathbf{B} + \mathbf{A} \wedge \mathbf{B} = < \mathbf{AB} >_0 + < \mathbf{AB} >_2 + < \mathbf{AB} >_4, \tag{22}$$

and it contains a scalar (grade 0), a bivector (grade 2) and a quadrivector (grade 4) for $n > 3$. Following the multiplication rules of exterior algebra, all these product terms between arbitrary multivectors of different grades are derived explicitly in ref. [39]. For a simple bivector[14] ($\mathbf{B}$), since both $\mathbf{B}.\mathbf{B}$ and $\mathbf{B} \wedge \mathbf{B}$ are identically zero, the geometric product $\mathbf{BB}$ is equal to $\mathbf{B} : \mathbf{B}$, and gives the squared magnitude of the bivector. In 3-D, the final term i.e. the wedge product term on r.h.s. of eq.(22) vanishes and the product contains a scalar and a bivector only, representing a

[13] In geometric Algebra, it is traditional not to distinguish between scalars, vectors, bivectors etc. using boldface or other decorations. It treats all multivectors on pretty much the same footing. Multivectors can be scalars, vectors, bivectors et., pseudoscalars or linear combinations of the above. However, we retain here the distinctions we used in the introductory section of vector algebra.

[14] Multivectors of definite grade k (like bivectors, trivectors etc.) that can be written as the wedge product of k independent vectors are called a simple k-blade.

quaternion. Similarly, the geometric product of two trivectors can be defined with the introduction of triple dot product in addition to the single and double dot products. It may be noted that the geometric product of two (nonzero) elements of the same grade always yield a quaternion in 3-D and in higher dimensions, elements of even subalgebra, containing only even grade components. We also mention in this context that since $I_3.\mathbf{u}\wedge\mathbf{v}$ and $I_3\wedge\mathbf{u}\wedge\mathbf{v}$ both are zero, eq.(10) can be rewritten as: $-I_3\mathbf{u}\wedge\mathbf{v}=\mathbf{u}\times\mathbf{v}$ – the l.h.s. being identical with the geometric product between the pseudoscalar and a wedge product of two vectors.

The scalar plus bivector representation of quaternions in 3-D and with its appropriate multiplication rules, GA reproduces the product:

$$ab=(a_0+\mathbf{A})(b_0+\mathbf{B})=a_0b_0+a_0\mathbf{B}+b_0\mathbf{A}+\mathbf{AB}=a_0b_0+a_0\mathbf{B}+b_0\mathbf{A}+\mathbf{A}:\mathbf{B}+\mathbf{A}.\mathbf{B}, \qquad (23)$$

as $\mathbf{A}\wedge\mathbf{B}=0$ in 3-D. By taking $a_1=A_{12}-A_{21},\ a_2=A_{31}-A_{13},\ a_3=A_{23}-A_{32}$ and $q_1=\hat{e}_1\wedge\hat{e}_2,\ q_2=\hat{e}_3\wedge\hat{e}_1,\ q_3=\hat{e}_2\wedge\hat{e}_3$ (which satisfy the fundamental multiplication rule of quaternion algebra (eq.14)), we get from eq.(8) and eq.(9):

$$\begin{aligned}\mathbf{A}:\mathbf{B} &= A_{ij}(B_{ji}-B_{ij}),\ \ i\neq j\\ &= -a_i\,b_i,\end{aligned}$$

and

$$\begin{aligned}\mathbf{A}.\mathbf{B} &= A_{ij}\{(B_{jk}-B_{kj})\,\hat{e}_i\wedge\hat{e}_k+(B_{ki}-B_{ik})\,\hat{e}_j\wedge\hat{e}_k\},\ \ i\neq j\neq k,\\ &= q_1(a_2b_3-a_3b_2)+q_2(a_3b_1-a_1b_3)+q_3(a_1b_2-a_2b_1),\end{aligned}$$

Thus eq.(23) is identical with eq.(15).

The inner and exterior products of a vector $\mathbf{u}$ with the geometric product $\mathbf{vw}$ is defined as: $\mathbf{u}.\mathbf{v}\wedge\mathbf{w}=(\mathbf{u}.\mathbf{v})\mathbf{w}\neq\mathbf{u}.(\mathbf{v}\wedge\mathbf{w}),\ \mathbf{u}.\mathbf{vw}=(\mathbf{u}.\mathbf{v})\mathbf{w}\neq\mathbf{u}.(\mathbf{vw})$ and $\mathbf{u}\wedge\mathbf{vw}=(\mathbf{u}\wedge\mathbf{v})\mathbf{w}\neq\mathbf{u}\wedge(\mathbf{vw})$, i.e. multiplications indicated by symbols are to be carried out first, starting with the inner product. Maintaining this order of multiplications, one can now readily define the geometric product of more than two vectors and easily verify that the product is *associative*. For example, with three vectors we get:

$$\begin{aligned}(\mathbf{uv})\mathbf{w} &= (\mathbf{u}.\mathbf{v}+\mathbf{u}\wedge\mathbf{v})\mathbf{w}=(\mathbf{u}.\mathbf{v})\mathbf{w}+(\mathbf{u}\wedge\mathbf{v}).\mathbf{w}+(\mathbf{u}\wedge\mathbf{v})\wedge\mathbf{w}\\ &= (\mathbf{u}.\mathbf{v})\mathbf{w}+(\mathbf{v}.\mathbf{w})\mathbf{u}-(\mathbf{u}.\mathbf{w})\mathbf{v}+\mathbf{u}\wedge\mathbf{v}\wedge\mathbf{w}\\ &= \mathbf{u}(\mathbf{v}.\mathbf{w})+(\mathbf{u}.\mathbf{v})\mathbf{w}-(\mathbf{u}.\mathbf{w})\mathbf{v}+\mathbf{u}\wedge\mathbf{v}\wedge\mathbf{w}\\ &= \mathbf{u}(\mathbf{v}.\mathbf{w}+\mathbf{v}\wedge\mathbf{w})\equiv\mathbf{u}(\mathbf{vw})\,.\end{aligned}$$

The associativity also follows even with nonsimple multivectors. For example, using equations like (7), (8), (9) etc., it can be easily shown that for arbitrary bivectors $\mathbf{A}$ and $\mathbf{B}$: $(\mathbf{uv})\mathbf{B}=\mathbf{u}(\mathbf{vB})$ and $(\mathbf{vA})\mathbf{B}=\mathbf{v}(\mathbf{AB})$ and so on [39]. The wedge product also has the associative property and with three vectors, eq.(19) takes the form: $\mathbf{u}\wedge\mathbf{v}\wedge\mathbf{w}=\frac{1}{3!}(\mathbf{uvw}+\mathbf{vwu}+\mathbf{wuv}-\mathbf{vuw}-\mathbf{uwv}-\mathbf{wvu})$. Fully general expression for a pure k-blade can accordingly be written as the sum over all possible $k!$ permutations of geometric product of k linearly independent vectors ($\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_k$), with a sign factor defined to be $+1$ for even permutations and -1 for odd permutations.

Geometric product equips the vector space with an algebraic structure that provide a very powerful tool to unify and describe in a single formalism the structures of the vector, complex and the spin algebras in the realm of GA. Clifs are in general represented as the sum of different pure grade elements and spinors form even subalgebra. Tensors, on the other hand, are of definite rank, having fixed number of multilinear vector arguments, cannot represent spinors. In this sense GA is more general than tensor algebra. However, pure grade multivectors can represent only antisymmetric tensors of rank equal to or less than the dimension of the basic vector space. Unlike the grade of a multivector, the rank of a tensor is not restricted by the dimension of the vector space. Hestenes has proposed to introduce tensor as multilinear functions defined on geometric algebras [5].

Physical theories are more conveniently and economically described with GA. Furthermore, noncommutative compositions and nonassociative structures are consistently introduced using various *star product* procedures in algebraic formulation of some recent advanced physical theories [16, 40]. We also intend to discuss these and other important issues in the next part of our work.

The strength of GA, compared to other mathematical tools, may also be argued using 'Occams Razor' as it provides a simpler and economic model, naturally extending from one to two, to higher dimensions. The effectiveness of this algebra is amply demonstrated as it encapsulates the usual four Maxwell's equations of electromagnetism describing the electromagnetic field for the charge density ρ and current density $\mathbf{j}$ as sources in a *single, compact equation* [41]:

$$(c^{-1}\frac{\partial}{\partial t} + \nabla)\mathrm{F} = \rho - \frac{\mathbf{j}}{c}, \tag{24}$$

where c is the velocity of light. The electromagnetic field F is described as the sum of the electric field vector and the magnetic field bivector, i.e. represented by a clif and $\nabla\mathrm{F}$ is similarly defined like eq.(20). The four geometrically distinct parts of eq.(24) – its scalar, vector, bivector, and pseudoscalar parts, are respectively equivalent to the standard set of four equations. The formalism allows descriptions of electrodynamics, fluid mechanics and special relativity by extending the algebra of space to the algebra of spacetime (e.g. GA on Minkowski space, with replacement of the Euclidean metric by the Minkowski metric) [3, 5, 42].

The advantages of using this geometric formulation in the algebra of spacetime and in Dirac theory are discussed with examples in several studies [5, 10, 40, 43]. Specifically we note here that the unification of the separate equations for divergence and curl in electromagnetism in a single equation is nontrivial – the unified equation can be inverted directly to determine the field. Also this equation is invariant under coordinate transformations, rather than covariant like the tensor form. The field bivector is the same for all observers; there is no question about how it transforms under a change of reference system. However, it is easily related to a description of electric and magnetic field in a given inertial system.

Secondly, the gauge invariance of electromagnetic field bivector can be demonstrated easily. Regarding gravity, Doran has noted in his thesis [44]: "The gauge theory of gravity developed from the Dirac equation has a number of interesting and surprising features." Lasenby et al [45] have presented a theory of gravity in terms of gauge fields, rather than spacetime geometry. The field equations are then derived from an action principle and the requirement that the gravitational action should be consistent with the Dirac equation leads to a unique choice for the action integral (up to the possible inclusion of a cosmological constant). It is claimed that the physical and mathematical content of the theory is best expressed in the language of 'geometric algebra' and "reproduces the predictions of general relativity for a wide range of phenomena, including all present experimental tests".

Hestenes has formulated standard problems on particle and rigid body dynamics using GA and developed the *spinor theory* of rotations and rotational dynamics [46]. Calculations with spinors are demonstrably more efficient compared to the conventional matrix theory and provides new insights into the treatment of the topics discussed. Specially in rotational dynamics and celestial mechanics, this unique treatment has both practical as well as theoretical advantages. Hestenes [47] has also given an invariant formulation of the Hamiltonian mechanics in terms of 'geometric calculus' – a generalization of the calculus of differential forms according to GA. He also envisaged an extension of the invariant formulation for systems of linked rigid bodies in phase space to have important applications in robotics.

The structures of the Pauli's σ- and Dirac's γ-matrices correspond to the structures of geometric algebra [48], rendering an implicit geometric interpretation for quantum mechanics. Consistent formulations of both classical and quantum mechanics (QM) with GA facilitates the introduction of spin as a physical observable and thereby removes the first conceptual barrier for unification. Moreover, Hestenes [49] has defined "the 'geometrically purified' version of the Dirac algebra" by eliminating 'irrelevant features' and finally identified the formulation with the geometric algebra on four-dimensional Minkowski ($M4$) metric over the real number field and christened it as the spacetime algebra (STA). In this formulation the spin is revealed as a dynamical property of the electron motion. It is emphasized that the association of the spin with a local circulatory motion 'zitterbewegung', first proposed by Schrödinger, is actually characterized by the complex phase factor, which is the main feature, the Dirac wave function shares with its nonrelativistic limit.

Developing the nonrelativistic QM as a statistical theory over phase space, the Weyl-Wigner-Moyal (WWM) formalism [50] represents the observabes in terms of the corresponding phase space functions (c-number) instead of the Hilbert space operators of standard QM. Replacing the conventional product of functions, the star product regime (also called the Moyal product or Weyl-Groenewold product) [51] used in this formalism, produce noncommutative composition of the phase space functions (the so-called deformation quantization). The star product encodes the quantum mechanical action and the formalism thus accommodates the uncertainty principle in systematic analogy with the noncommuting Hilbert space operators.

Bayen *et al* [52] have elaborated and used this formalism in the derivations of harmonic oscillator, angular momentum and hydrogen atom spectra. Hirshfeld and Henselder [53] have shown that the deformation quantization through its star product formalism leads to 'Cliffordization' [40] of phase space variables i.e., allows a consistent algebraic formulation of quantum mechanics (of both the bosonic and fermionic systems) on phase space with GA. Phase space description being the traditional framework of classical dynamics, the WWM star-deformation formalism imparts proper understanding of the classical-quantum interface. Treating both coordinate and momentum spaces on equal footing, this formulation of QM in GA manifestly reveals closer connections with classical mechanics and offers better insights into the problem of classical limit of QM.

Starting from the early acquaintances of vector algebra, we have gone through some elaborate discussion of various multiplication rules (cross, dyadic and wedge products) of vector and exterior algebras – up to the geometric product of GA. The resurgence of Clifford algebra as GA and its increasing use in some most advanced areas of physics are indicated. In fact, eliminating the passive transformations and retaining only active transformations, GA dispenses with the unnecessary excess baggage and irrelevant features of vector and tensor algebras in extending and exploring new horizons of mathematical physics. We conclude the present discussion which forms the basis of our intended further discussion on these issues in the next part of our study.

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