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THE VLASOV-POISSON EQUATION AND ASTROPHYSICAL COHERENT STRUCTURES

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Abstract: The possibility of formation of periodic structures in a system of gravitating particles (described by the Vlasov-Poisson system of equations) has been established. The branching conditions for solutions of a nonlinear equation for a potential lead to a criterion for the emergence of 1D and 2D systems of particles.

Keywords: gravitational instability, Vlasov–Poisson equation, Bernstein–Greene–Kruskal waves, strates, quasi–periodical structures

1. Introduction

In modern publications on cosmology and astrophysics, it is assumed that the main reason for the quasi-ordered structures origin in the Universe is the development of gravitational instabilities from small perturbations of the characteristics of a homogeneous medium. In this case, the type of disturbances can belong to the main classes: adiabatic or acoustic, vortex and entropy (pre- and post-recombination), as well as their various combinations [1]-[4]. Moreover, to describe the forms of genesis and evolution of disturbances in the literature the hydrodynamic approximation [5]-[9] was used, as well as the formalism based on the Vlasov - Poisson equation [10]-[11] (mainly in the form of direct statistical modeling of the Liouville equation and by passing to an expansion in harmonics for the linearized case). The vast majority of the problems considered by modern researchers are based on fairly simple general considerations.

It is known from astronomical observations that in the expanding Universe the amplitude of density perturbations reaches the value of the order of unity at the stage when the linear dimensions of the inhomogeneities are much less than ct (and also much less than the radius of curvature R, involved in the Friedmann–Lemaitre–Robertson–Walker metric). Thus, the growth theory of inhomogeneities ceases to be linear on stage when relativistic effects are insignificant and Newtonian physics is quite applicable. Large-scale structure of the Universe (galaxies, galaxy clusters, voids) arises from small

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perturbations density. When density contrast becomes of order $\delta\rho/\rho \simeq 1$, perturbation ceases to participate in the cosmological expansion and can to form gravitationally bound system. In accordance with the methodology proposed in the works [13]–[15], an approximate solution of the equations of cosmological dynamics in the Milne–McCree form, based on on the use of Lagrangian variables and extrapolation of linear theory; at sufficiently large gradients of peculiar velocities the stage of growth of perturbations leads, based on general considerations, to a predominant compression along one of the directions (much less likely, along two directions) with the formation of flattened structures (the so-called Zeldovich's "pancakes"). Crossing, "pancakes" create a honeycomb structure, and clusters of galaxies are formed in places of the highest density. Latest Observational Data Received, also allow the existence of one–dimensional structures on a cosmological scale (previously the presence of "threadlike formations" was allowed, in particular, when modeling the Laniakea supercluster shape.

Despite the outwardly extremely simple, practically evaluative, form of reasoning leading to the fact of the appearance of "pancakes", according to most authors of publications on the topics under discussion, the corresponding mathematical formalism is quite adequate describes the whole range of problems, associated with the emergence of large–scale astrophysical 2–dimensional structures. At the same time, in the logic of this reasoning has place is a rather disturbing factor associated with the indeterminism of the structure formation. Namely, for the functioning of this mechanism, it is necessary a priori presence of perturbations distributed in space density, and there should be some space–time coherence (connectivity) of these perturbations. The origin of these disturbances can be explained on the basis of the hydrodynamic approach as, for example, a consequence of the given specific Cauchy conditions for an evolutionary problem or the effects of periodic "overturning" of a propagating density wave (the origin of which is caused by external cosmological processes). However, as a generating disturbance at the hydrodynamic level, the mechanism is quite sufficient consider the properties of a kinetic system with a self-consistent gravitational interaction between the particles of the system. In this paper, the authors propose a method for describing the processes of realization on a cosmological scale within the framework of the original Milne model pseudoperiodic 2–dimensional formations in the form of layers of increased density and the formation of secondary substructures in the vicinity each of the mentioned layers; this technique is based on taking into account the properties of localized pseudoperiodic solutions the Vlasov–Poisson equations, considered as adiabatic stable in time, and their possible ramification in the configuration space depending on the degree of dispersion of the particles constituting the solution in terms of velocities and effective sizes of 2–dimensional layers.

2. The conditions for the existence of stationary solutions of the Vlasov–Poisson system of equations for particles with masses

To study the behavior of a system of gravitating massive particles in astrophysics problems, it is necessary to analyze solutions of the Vlasov–Einstein equation system. However, in the general case, this is an extremely time–consuming problem, therefore it is supposed to be reasonable at first to significantly limit the range of problems to be solved, and to conduct a detailed study properties of solutions to a simplified problem (with the aim, in particular, in the future to take the analysis methodology more complex tasks). Further, we restrict ourselves to the case of the dynamics of massive particles in cosmological conditions in the modern era. This means that instead of the Vlasov - Einstein system of equations, we will consider the Vlasov–Poisson system, assuming for This is that the stationary solution of the Liouville equation can be represented as a function of the integrals of motion of the particles. We will be interested in can, under such assumptions, arise in the system (when changing in the region of some critical point) periodic structures a certain dimension (which can be compared with objects observed on large scales in the Universe and quasi–regular voids).

Let $f_i(\boldsymbol{x},\boldsymbol{p})$ is a distribution function of i–th type of particles by momenta $\boldsymbol{p}\in R^3$ and by coordinates $\boldsymbol{x}\in R^3$. We assume $f_i(\boldsymbol{x},\boldsymbol{p})$ are solutions of Vlasov–Poisson equation system:

$$\frac{\boldsymbol{p}}{m_i}\frac{\partial f_i}{\partial \boldsymbol{x}} - m_i\frac{\partial\varphi}{\partial \boldsymbol{x}}\frac{\partial f_i}{\partial \boldsymbol{p}} = 0, \quad \Delta\varphi = \sum_i 4\pi G m_i \int f_i(\boldsymbol{x},\boldsymbol{p})d\boldsymbol{p}, \tag{1}$$

where m_i are masses of i–th particles types, $\varphi(\boldsymbol{x})$ is gravitational potential in a point $\boldsymbol{x}$. We interest solutions of system (1) in the region $\boldsymbol{x}\in D$, where D may be a 3–cube (with periodic boundary conditions) or an arbitrary area with a smooth boundary. Let distribution functions f_i depend on the energy integral only: $f_i = g_i(\varepsilon)|_{\varepsilon=\boldsymbol{p}^2/(2m_i)+m_i\varphi(\boldsymbol{x})}$, where $g_i(\varepsilon)(\geq 0)\in C^1(\Omega_\varepsilon)$. In this case the Liouville equation of system (1) (1) the equation is satisfied (so left–hatd side of Liouville equation is a consequense of Poisson bracket), and Poisson equation for gravitation potential has following form:

$$\Delta\varphi = F(\varphi), \quad F(\varphi) \equiv \sum_i 4\pi G m_i \int g_i(\boldsymbol{p}^2/(2m_i) + m_i\varphi(\boldsymbol{x}))\, \boldsymbol{dp}. \tag{2}$$

Lemma. Let in definitin of right–hand side functional $F(\varphi)$ the momenta $\boldsymbol{p}\in R^d$, integral–kernel function $g_i(\varepsilon)(\geq 0)\in C^1(D)$ are satisfied the conditions $g_i(\varepsilon)\cdot\varepsilon^{d/2+1}\to 0$ iff $\varepsilon\to\infty$ (for $d=1{,}2$, and for $d=1$ functions $g_i(\varepsilon)$ are decrease monotonically). Then for $d\leq 2$ the inequality $dF/d\varphi\geq 0$ is satisfied.

Proof. The correctness of the boundary Dirichlet problem for nonlinear elliptial equation (2) depends on sign of $dF/d\varphi$: for $F'(\varphi)\geq 0$ this problem is correct formulated. For cases $S_2 = 2\pi$, $S_1 = 1$:

$$\frac{dF}{d\varphi} = 4\pi S_d \sum_i \int g_{i'}(\frac{\boldsymbol{p}^2}{2m_i} + m_i\varphi)|p|^{d-1}d|\boldsymbol{p}| =$$

$$= 4\pi S_d \sum_i \int g_{i'}(\frac{\boldsymbol{p}^2}{2m_i} + m_i\varphi)|\boldsymbol{p}|^{d-2}(2m_i)d(\frac{\boldsymbol{p}^2}{2m_i}) =$$

$$= 4\pi S_d \sum_i (2m_i)\int_0^\infty |\boldsymbol{p}|^{d-2}dg_i =$$

$$= 4\pi S_d \sum_i (2m_i)(|\boldsymbol{p}|^{d-2}g_i|_0^\infty - \int_0^\infty g_i d|\boldsymbol{p}|^{d-2}) =$$

$$= 8\pi S_d \sum_i m_i \times \begin{pmatrix} g_i(m_i\varphi), & if\ d = 2, \\ (2-d)\int_0^\infty g_i|\boldsymbol{p}|^{d-3}d|\boldsymbol{p}|, & \text{if } d = 1, \end{pmatrix}$$

as by the condition of Lemma functions g_i decrease fast enough for the argument tending to infinity, and $d \leq 2$. In this case, for the one–dimensional case, the divergence at the lower limit is formal, since in the case of physical realization of the considered model, the momenta of the particles of the system have statistical (thermal) dispersion, so that in fact the lower limit integral is determined by the local hydrodynamic (macro)characteristics of the system of gravitating particles.

The Lemma implies the uniqueness of the solution to equation (2) in the class $C^2(D)$. Let φ_1 and φ_2 are two solutions of equation $\Delta\varphi = F(\varphi)$. According to Green's formula

$$\int_D (F(\varphi_1) - F(\varphi_2))(\varphi_1 - \varphi_2)dx = -\int_D [\nabla(\varphi_1) - \nabla(\varphi_2)](x)[\nabla(\varphi_1) -$$

$$-\nabla(\varphi_2)](x)dx + \int_{\partial D} (\varphi_1 - \varphi_2)\frac{\partial(\varphi_1 - \varphi_2)}{\partial n}ds.$$

The left-hand side of the last equality is non-negative due to the inequality proved in Lemma. The right-hand side is non-positive since the second term is zero for periodic boundary conditions or for a given function on the boundary in the problem Dirichlet. Hence, in the last equality, both sides are zeros, $\varphi_1 = \varphi_2 + const$.

In the case of anisotropy of the distribution function (if, in particular, as stationary solutions of the Vlasov system not energy functions of Maxwellian type are considered, but distributions with dimensionally reduced momentum space), that is in the case of its dependence on two integrals (energy and momentum):

$$f_i = g_i\left(\frac{\boldsymbol{p}_{II}^2}{2m_i} + m_i\varphi(\boldsymbol{x});\ p_3\right), \quad \boldsymbol{x} = (x_1, x_2), \quad \boldsymbol{p}_{II}^2 = (p_1)_{II}^2 + (p_2)_{II}^2,$$

we have the following form of the right-hand side in formula (2): $F(\varphi) = \sum_i 4\pi G m_i \int g_i(\boldsymbol{p}_{II}^2/(2m_i) +$

$m_i\varphi, p_3)dp_1dp_2dp_3$. From the above Lemma the monotonic increase of the function $F(\varphi)$ follows, and we have the same situation as for the case of the dependence of the distribution function on energy.

Let the potential of $varphi$ depend on only one variable: $\varphi = \varphi(x_1)$. Then there is periodic solution of equation (2) with distribution function depending only on energy

$$F(\varphi) = \sum 4\pi m_i \int g_i(\varepsilon)\delta(\varepsilon - \varepsilon_i)|_{\varepsilon_i=[p^2/(2m)]_i+m_i\varphi} d\boldsymbol{p}.$$

Let us consider (as the most physically realistic case) the representation of the stationary distribution function in terms of Maxwell distribution in the external gravitational field $g_i(\varphi) = \exp(-\boldsymbol{p}^2/(2m_i\theta) + m_i\varphi/\theta)$ (let's take for simplicity $m_1 = m_2 = m$). In this case, for $d = 1$ we have:

$$\frac{d^2\varphi}{dx_1^2} + A(\theta)\exp(\lambda\varphi) = 0, A(\theta) \equiv 4\pi Gm \int \exp(-\boldsymbol{p}^2/(2m\theta))d\boldsymbol{p}, \lambda \equiv m/\theta.$$

Linearization of this equation (of the Bratu type) leads to an equation close in form to the differential equation of the oscillator:

$$\exp(\lambda\varphi) \approx 1 + \lambda\varphi, \quad {\varphi_{x_1x_1}}'' + \lambda A(\theta)\varphi + A(\theta) = 0.$$

The criterion for the appearance of periodic solutions in this case is obvious:
$\lambda A(\theta) > 0$.

We can also consider, in a sense, the limiting case of Maxwellian, which actually corresponds to $\theta \to 0$: $g_i(\varepsilon) = c_i\delta(\varepsilon - \varepsilon_i)$ (in this case, however, problems arise with the convergence of the integral in Lemma, as noted above; however, for the case $d = 2$ this is not essential). Integrating (2), we get:

$$\frac{1}{2}(\frac{d\varphi}{dx_1})^2 = \zeta(\varphi) + C_0, \quad \frac{d\zeta}{d\varphi} = F(\varphi), \quad \zeta(\varphi) = \sum_i \zeta_i.$$

We have $\zeta_i = 4\pi c_i m_i\sqrt{2m_i(\varepsilon_i - m_i\varphi)}\eta(\varepsilon_i - m_i\varphi)$. The function $\zeta(\varphi)$ при $\sum_i \varepsilon_i > 0$ has a minimum, therefore, periodic solutions are possible of the above nonlinear differential equation.

A consequence of the periodicity of solutions to one–dimensional equation (2) is the possibility of translational composite solutions of Vlasov equation with potential $\varphi = \varphi_1(x_1) + \varphi_2(x_2) + \varphi_3(x_3)$ (where functions $\varphi_k(x_k)$ are periodical by arguments x_k):

$$f_i = g_{i1}(p_1^2/(2m_i) + m_i\varphi_1(x_1);\ p_2, p_3) +$$
$$+g_{i2}(p_2^2/(2m_i) + m_i\varphi_2(x_2);\ p_1, p_3) + g_{i3}(p_3^2/(2m_i) + m_i\varphi_3(x_3);\ p_1, p_2).$$

The results obtained indicate that in the case of gravitational interaction one can consider an analogue of the Bernstein–Green–Kruskal waves, but the dimension of the velocity space in our case will differ from these waves. Formation of structures in a self–consistent field for a system of massive particles in the one–dimensional case is possible in the presence of temperature dispersion of particles.

3. The branching of solutions of the Vlasov–Poisson equations and criteria for the emergence of macrostructures

Let us consider the possibility of the emergence of structures in a system of gravitating particles under cosmological conditions. It is natural to assume that the factors determining these processes are the properties of the dynamics of a system of particles on a large scale in the modern cosmological era. The main model, quite successful in terms of descriptions of such evolutionary dynamics (with the conditions of uniformity of the density of distribution of particles in space and proportionality of their velocities coordinates) can be called the nonrelativistic Friedmann model (otherwise, the Milne–McCree model). It is naturally obtained from the Vlasov - Poisson system of equations, and the gravitational potential has a quadratic form $\varphi_2 \propto |x|^2$ as a solution of the Poisson equation with constant right side of the equation. This corresponds to the results of V. Gurzadyan, generalizing Newton's theorem on the identity of the gravitational field of a sphere and a point with the mass of the sphere:

$$\Phi(r) = Gmr^{-1} - \frac{c^2\Lambda}{6}r^2. \quad (3)$$

The $U(r)$ function allows one to describe dark matter in galaxies and clusters of galaxies, also allows you to eliminate the mystery of the constant Hubble (the discrepancy between the values of the Hubble constant obtained, on the one hand, with the data of the Planck satellite on the relic radiation, on the other, through the Hubble diagram for galaxies in the vicinity of the Local Group). Thus, the above–described approach to the self–similar problem has observational justification in connection with the description of dark matter.

Thus, it seems appropriate to further analyze the emergence of astrophysical structures use potential (3) as the main interparticle potential (for two massive particles far from the main system). The stationary nonlinear equation for the self–consistent potential (the existence of which mathematically justifies the applicability of the Vlasov equation) can be written as follows (from now on we will assume $m_i = m$):

$$\varphi(\boldsymbol{x}) = \int \text{ù}\Phi(|\boldsymbol{x} - \boldsymbol{x}'|)\exp(-\varphi(\boldsymbol{x}')/\theta)d\boldsymbol{x}', \quad (4)$$

where the constant ù is determined from the conditions for the existence of a spatially homogeneous distribution (that is, if the distribution function particles depends only on velocities, for example, it has the form of a Maxwellian distribution): $\text{ù} = \rho_0\exp(\varphi_0/\theta)$, $\rho_0, \varphi_0 = const_{1,2}$. We will solve the problem of determining the end points of a spatially homogeneous solutions and emergence of spatially periodic solutions of a nonlinear integral equation for a self-consistent potential (we we still assume that the system of particles is located in the region of the physical space $L^3 = D$ with the corresponding boundary conditions periodicity). Equation (4) can be rewritten in the form

$$\psi(\boldsymbol{x}) = \mu \int \Phi(|\boldsymbol{x}-\boldsymbol{x}'|)\exp\big(\psi(\boldsymbol{x}')\big)d\boldsymbol{x}', \quad \mu = -\frac{ù}{\theta}, \quad \psi(x) = -\varphi(x)/\theta. \quad (5)$$

The critical point of the (possible) emergence of a new solution to the equation will be (ψ_c, μ_c), where μ_c to be determined,

$$\psi_c = \mu_0 \exp(\psi_c)\cdot\sigma(0), \sigma(0) = \int_D \Phi(|\boldsymbol{x}-\boldsymbol{x}'|)d\boldsymbol{x}' = 4\pi\int_0^L \Phi(y)y^2dy.$$

If we introduce the notations $\mu = \mu_c + \delta\mu$, $\psi = \psi_c + \delta\psi(\boldsymbol{x})$, then we have from the original equation (4) the equation, equivalent to it in the vicinity of the critical point:

$$\psi_0 + \delta\psi = (\mu_c + \delta\mu)\int_D \Phi(|\boldsymbol{x}-\boldsymbol{x}'|)\exp(\psi_0 + \delta\psi(\boldsymbol{x}'))d\boldsymbol{x}'.$$

We expand the integrand in a power series in $\delta\psi$:

$$\delta\mu\exp(\psi_c)\sigma(0) + \delta\mu\exp(\psi_c)\int_D \Phi(|\boldsymbol{x}-\boldsymbol{x}'|)\delta\psi(\boldsymbol{x}')d\boldsymbol{x}' +$$

$$+(\mu_c\exp(\psi_c) + \delta\mu\exp(\psi_c))\cdot\frac{1}{2}\int_D \Phi(|\boldsymbol{x}-\boldsymbol{x}'|)\delta\psi^2(\boldsymbol{x}')d\boldsymbol{x}' + \ldots =$$

$$= \delta\psi(\boldsymbol{x} - \mu_c\exp(\psi_c)\int_D \Phi(|\boldsymbol{x}-\boldsymbol{x}'|)\delta\psi(\boldsymbol{x}')d\boldsymbol{x}'. \quad (6)$$

Let's select a linear equation:

$$\delta\psi(\boldsymbol{x}) = \mu_c\exp(\psi_0)\int_0 \Phi(|\boldsymbol{x}-\boldsymbol{x}'|)\delta\psi(\boldsymbol{x}')d\boldsymbol{x}',$$

and represent its solution in the form of a periodic function $\delta\psi(x) = \Theta\exp(ikx)$. Substitution of this solution into a linear equation gives a criterion for the existence of a post–homogeneous periodic solution:

$$4\pi\mu_0\exp(\psi_0)\int_0^L y\Phi(y)\frac{sin(ky)}{k}dy = 1, \Phi(y) = Gmy^{-1} - \frac{c^2}{6}\Lambda y^2.$$

The spatial period in this case $\delta\ell = 2\pi/k_c$, k_c: $\frac{d}{dk}4\pi\int_D yk^{-1}\Psi(y)sin(ky)dy|_{k=k_c} = 0$.

If we consider the second (quadratic in φ) term (and higher) in expansion (6), then we can get "fine structure" of the periodic system: change in the period of neighboring clusters, density distribution between two–dimensional walls in the system, the influence of the neighboring thread on the shape of the current thread in the one–dimensional case.

In the previous section, we obtained the conditions for the existence of solutions to the equation for the gravitational potential associated with the dimension of the velocity space in the system. It is clear that this dependence is also valid for the coordinate space, that is related to the divergence of the integral $\int L^3\Psi(y)y^2dy$ with increasing scales of the system $L \to infty$, in fact, it can be stated that for the gravitational potential (especially with the repulsive term $\sim c^2\Lambda r^2$) the mass of a particle cluster will grow indefinitely. However, for the case $d = 1{,}2$ and a finite domain of integration D, the bounded solutions of the nonlinear elliptic equations for the potential and the nonlinear integral equation for the potential exist and form solutions of the Vlasov–Poisson system of equations. Obviously, equation (5) connects with the problem of finding the eigenvalues μ of the Hammerstein equation (in the appropriate dimension), to which it is necessary to add the finiteness requirement the mass of the system per unit cross-sectional area perpendicular to the direction in which the system is limited (or per unit length systems for the one-dimensional case). Thus, the periodic solutions $\sim sin(\boldsymbol{kx})$ form either thread–like structures or periodic layers with void spaces between them. From a physical point of view, the formation of a two–dimensional periodic structure begins with a local spontaneous or external action-induced violation of the isotropy of the primary particle flux (at release by an increase in density or mass flow rate or the energy of the allocated direction of propagation particles) conditions for the formation of structures arise (the boundedness of the local action ensures the convergence of the integrals in item 2); in this case, structure formation does not occur in less energetically loaded directions, and, moreover, under certain conditions (for example, in the vicinity of the ordered distribution of the subsystem of more massive particles, the formation of a secondary flow of more light particles from the vicinity).

4. Conclusion

The emergence of two–dimensional structures of the "Zeldovich pancake" type in the literature is most often associated with density perturbations, the growth of which is described by using classical or (weakly) relativistic hydrodynamics. However, the question arises as to the degree of randomness leading to such perturbations. Basically, they can be regarded as secondary, and the primary cause can be considered spatially separated maxima of the velocity (or momentum) distribution of gravitating particles of the original large system. Thus, it is quite natural to turn to the kinetic description of evolution a multiparticle ensemble in a quasi–stationary state (such as a Hubble flow in the expanding Friedmann universe). In this case, the presence of interparticle interaction is extremely important (since we are considering cosmological scales, the natural apparatus for studying the many-particle dynamics becomes the formalism of the Vlasov equation, since Boltzmann collisions in this case play a vanishingly small role; generally speaking, one can also consider charged gravitating particles, then, possibly, additional diffusion terms should be considered on the left–hand side of the kinetic equation of the Fokker–Planck type, but in the first approximation we are not talking about them). The hydrodynamic technique is focused on the study of a disturbed flow in external fields, and interparticle interaction includes only through the viscosity coefficients and thermal conductivity, which can be influenced by other factors (in particular, in a numerical calculation, the inclusion of the so-called grid viscosity, or the presence of a local vortex motion, which is essential changes the mentioned coefficients by geometrical factors). At the same time, in the Vlasov equation the presence and importance of a self-consistent power member is clearly indicated, and the influence of that member not masked by external quasi–physical elements prevailing in modeling.

If we consider the situation with the formation of two-dimensional surfaces on cosmological scales, then, given the practically flat geometry of the modern Universe, there is no essential need from the very beginning to involve the gravitational equations of Einstein, and will be limited at first investigation of the possibility of the appearance of flat structures using the Vlasov–Poisson system of equations. In fact, we are looking here for an analogue of Bernstein–Green–Kruskal-type solutions for gravitational interaction. Carrying out a comparative analysis of the Poisson equation in the case of stationarity of the Vlasov equation, we obtain with one-dimensional mass motion of a system of particles, the appearance of quasiperiodic in the spatial variable along and parallel to the vector of the ordered motion of the planar sets (with exact accounting, the planes acquire structure with a density rapidly decreasing along the vector of mass motion).

Thus, the paper proposes a kinetic model for the occurrence of the periodicity of gravitational striations voids separated by two–dimensional surfaces due to the presence in the Hubble one–dimensional flow of Poisson structures associated with the quasi–oscillatory an equation that is a consequence of the Poisson equation proper. This approach is the development of models the appearance of flat structures on cosmological scales, and has certain advantages over the mentioned models: in accordance with the developed approach two–dimensional structures arise not depending on random perturbations of the density of the medium, and are causally conditioned by physical reality in the form of the presence of gravitational interaction between the particles, which makes during the evolution of the Hubble flow, they do not remain uniformly distributed over space, but to form a cellular macrostructure (this is due to the three–dimensionality of Euclidean space as the limit of space Minkowski, if we do not take into account the curvature of space –time in the Poisson equation).

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