Inequalities for m-Divisible Distributions

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Abstract

We state some inequalities for m-divisible and infinite divisible characteristic functions. Basing on them we propose corresponding inequalities for their moments of non-integer order.
Key words: infinite divisible distributions; characteristic functions; inequalities for characteristic functions.

1 Inequalities for characteristic functions. Estimates from below

Let $f(t)$ be the characteristic function of a distribution on a real line. We say that $f(t)$ is m-divisible (m is positive integer) if $f^{1/m}(t)$ is a characteristic function as well. The function $f(t)$ is infinitely divisible if it is m-divisible for all positive integers m. Properties of infinitely divisible characteristic functions were well-studied (see, for example, [3]). In particular, any probability distribution with a compact support is not infinitely divisible. However, this is not true for m-divisible distributions. Let us take arbitrary distribution with a compact support. Denote its characteristic function by $g(t)$ and define $f(t) = g^m(t)$. Clearly, $f(t)$ is m-divisible characteristic function of a distribution with compact support. Let us start with the case of arbitrary symmetric distribution having compact support.

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Theorem 1.1. *Let $g(t)$ be a characteristic function of an even non-degenerate distribution function having its support in interval $[-A, A]$, $A > 0$. Then the inequality*

$$\cos(\sigma \cdot t) \leq g(t) \tag{1.1}$$

holds for all $t \in (-4.49/A, 4.49/A)$. Here σ is standard deviation of the distribution with characteristic function $g(t)$.

Proof. Denote by $G(x)$ a distribution function corresponding to the characteristic function $g(t)$. We have

$$g(t) = \int_{-A}^{A} \cos(tx)dG(x) = \int_{0}^{A} \cos(tx)d\tilde{G}(x) = \int_{0}^{A^2} \cos(t\sqrt{y})dH(y), \tag{1.2}$$

where $\tilde{G}(x) = 2(G(x) - 1/2)$ and $H(y) = \tilde{G}(\sqrt{y})$. In view of the fact that $\cos(t)$ is an even function we can consider positive values of t only.

It is not difficult to see that the function $\cos(t\sqrt{y})$ is convex in y for the case when $0 \leq t\sqrt{y} \leq z_o$, where z_o is the first positive root of the equation $\sin z - z\cos z = 0$. Numerical calculations show that $z_o > 4.49$. For the case of $|t| \leq 4.49/A$ let us apply Jensen inequality to (1.2). We obtain

$$g(t) = \int_{0}^{A^2} \cos(t\sqrt{y})dH(y) \geq \cos\Big(t\sqrt{\int_{0}^{A^2} ydH(y)}\,\Big) = \cos(\sigma t).$$

□

The condition of support compactness may be changed by the restriction of absolute fifth-moment existence. Namely, the following result holds.

Theorem 1.2. *Let $G(x)$ be a symmetric probability distribution function. Denote by a_j its absolute moments and suppose that a_{10} is finite. Denote by $g(t)$ corresponding characteristic function and $\sigma^2 = a_2$. Then we have*

$$\cos(\sigma t) \leq g(t) \tag{1.3}$$

for all $|t| \leq 5(a_4 - \sigma^4)/(\sigma^{10} + 2\sigma^5 a_5 + a_{10})^{1/2}$.

Proof. Suppose that $g(t)$ is not identical to $\cos(\sigma t)$. Define

$$\varphi(t) = g(t) - \cos(\sigma t).$$

It is easy to verify that $\varphi(t)$ and its derivatives at the point $t = 0$ satisfy

$$\varphi^{(k)}(0) = 0, \quad \text{for} \quad k = 0, 1, 2, 3,$$

and $\varphi^{(4)}(0) = a_4 - a_2^2 > 0$. Therefore, $\varphi^{(4)}(t)$ is positive in some neighborhood of the point $t = 0$ and, consequently, $\varphi(t)$ is non-negative at least in some neighborhood obtained by means of forth times integration. Our aim now is to estimate the length of the interval for $\varphi^{(4)}(t)$ positiveness. For this consider derivative of $\varphi^{(4)}(t)$ that is $\varphi^{(5)}(t)$. We have

$$|\varphi^{(5)}(t)| = |\int_{-\infty}^{\infty} \big(\sigma^5 \sin(\sigma t) - \sin(tx)x^5\big) dG(x)| \leq$$

$$\leq \Big(\int_{-\infty}^{\infty} \big(\sigma^5 \sin(\sigma t) - \sin(tx)x^5\big)^2 dG(x)\Big)^{1/2} \leq \Big(\sigma^{10} + 2\sigma^5 a_5 + a_{10}\Big)^{1/2}.$$

In view of the facts that $\varphi^{(4)}(0) = a_4 - \sigma^4 > 0$ we see that $\varphi^{(4)}(t) \geq 0$ on the interval $[0, (a_4 - \sigma^4)/(\sigma^{10} + 2\sigma^5 a_5 + a_{10})^{1/2}]$, and, because of symmetry, on interval $-(a_4 - \sigma^4)/(\sigma^{10} + 2\sigma^5 a_5 + a_{10})^{1/2}, (a_4 - \sigma^4)/(\sigma^{10} + 2\sigma^5 a_5 + a_{10})^{1/2}$. This guarantees that

$$\varphi(t) \geq \frac{t^4}{24}\Big((a_4 - \sigma^4) - (\sigma^{10} + 2\sigma^5 a_5 + a_{10})^{1/2} t/5\Big).$$

From this and the symmetry of $\varphi(t)$ follows the result. □

Let us turn to the of m-divisible distribution with compact support.

Theorem 1.3. *Let $f(t)$ be a characteristic function of m-divisible symmetric distribution having compact support in $[-A, A]$, $A > 0$ and standard deviation σ. Then*

$$\cos^m(\sigma t/\sqrt{m}) \leq f(t) \tag{1.4}$$

for $|t| \leq \min(4.49m/A, \pi\sqrt{m}/(2\sigma))$.

Proof. Denote $g(t) = f^{1/m}(t)$. It is clear that:

1. $g(t)$ is a symmetric characteristic function;

2. $m\sigma^2(g) = \sigma^2(f) = \sigma^2$ (because variance of sum of independent random variables equals to the sum of their variances);

3. distribution with characteristic function g has compact support in $[-A/m, A/m]$ (see, for example, [2], Theorem 3.2.1).

Applying Theorem 1.1 to the function $g(t)$ we find

$$\cos(\sigma t/\sqrt{m}) \leq g(t) = f^{1/m}(t)$$

for $|t| \leq 4.49m/A$. However, for $|t| \leq \min(4.49m/A, \pi\sqrt{m}/(2\sigma))$ the left hand side of previous inequality is non-negative and we come to the conclusions of Theorem 1.3. □

Note that $\cos(\sigma t/\sqrt{m})$ is monotone increasing in m for $|t| \leq \pi\sqrt{m}/(2\sigma)$ and, therefore the estimator (1.5) is more precise than (1.1).

Let us give a little bit different result.

Theorem 1.4. *Let $f(t)$ be a characteristic function of m-divisible symmetric distribution having finite tenth moment a_{10}. Then*

$$\cos^m(\sigma t/\sqrt{m}) \leq f(t) \tag{1.5}$$

for $|t| \leq \min(C\sqrt{m}, \pi\sqrt{m}/(2\sigma))$, where positive C depends on absolute moments a_k, $(k = 1, \dots, 10)$ only.

Proof. Denote $g(t) = f^{1/m}(t)$. It is clear that:

1. $g(t)$ is a symmetric characteristic function;
2. $m\sigma^2(g) = \sigma^2(f) = \sigma^2$ (because variance of sum of independent random variables equals to the sum of their variances);
3. distribution with characteristic function g has finite absolute moments up to tenth order (see, for example, [2]).

It is not difficult to verify that

$$a_{k,m} \sim a_k/m \quad \text{as} \quad m \to \infty, \tag{1.6}$$

where $a_{k,m}$ is kth absolute moment of g. To finish the proof it is enough to apply Theorem 1.2 and the relation (1.6). □

Consider now the case of infinitely divisible distribution.

Theorem 1.5. *Let $f(t)$ be a symmetric infinite divisible characteristic function with finite second moment σ^2. Then*

$$\exp\{-\sigma^2 t^2/2\} \leq f(t) \tag{1.7}$$

for all $t \in \mathbb{R}^1$.

Proof. If $f(t)$ has finite tenth moment it is sufficient pass to limit in (1.5) as $m \to \infty$. In general case one can approximate f by infinitely divisible characteristic functions with finite tenth moment. □

Another proof. Kolmogorov representation formula (see, for example [3]) for $f(t)$ allows us to rewrite (1.7) in the form

$$\exp\{-\sigma^2 t^2/2\} \leq \exp\{-\int_{-\infty}^{\infty} \big(1-\cos(tx)\big)/x^2 dK(x)\}$$

or, equivalently,

$$\int_{-\infty}^{\infty} \big(1-\cos(tx)\big)/x^2 dK(x) \leq \sigma^2 t^2/2. \tag{1.8}$$

However,

$$2\big(1-\cos(tx)\big)/x^2 \leq 4\sin^2(\frac{tx}{2})/x^2 \leq t^2.$$

This leads to (1.8) with

$$\sigma^2 = \int_{-\infty}^{\infty} dK(x) = -f''(0).$$

□

From the last proof it follows that *if the equality in (1.7) attends in a point $t_o \neq 0$ then it holds for all $t \in \mathbb{R}^1$.*

Theorem 1.5 shows an extreme property of Gaussian distribution among the class of infinite divisible distributions with finite second moment. Another extreme property without any moment conditions was given in [1]. Let us give this result here.

Theorem 1.6. *Let $f(t)$ be a symmetric infinite divisible characteristic function. Then*

$$f(t) \geq f^4(t/2) \tag{1.9}$$

for all $t \in R^1$. If the equality in (1.9) attends in a point $t_o \neq 0$ then it holds for all $t \in \mathbb{R}^1$

Proof. From Lévy-Khinchin representation we have

$$f(t) = \exp\{-\int_{-\infty}^{\infty} \big(1-\cos(tx)\big)\frac{1+x^2}{x^2}d\Theta(x)\},$$

$$f^4(t/2) = \exp\{-4\int_{-\infty}^{\infty} \big(1-\cos(tx/2)\big)\frac{1+x^2}{x^2}d\Theta(x)\}.$$

However,

$$1-\cos(tx) = 2\sin^2(tx/2) = 8\sin^2(tx/4)\cos^2(tx/4) \leq$$
$$\leq 8\sin^2(tx/4) = 4\big(1-\cos(tx/2)\big)$$

□

Let us note that Theorem 1.5 may be obtained from Theorem 1.6. Really, assuming the existence of finite second moment we have

$$f(t) \geq f^4(t/2) \geq f^{4^2}(t/2^2) \geq \ldots \geq f^{4^k}(t/2^k) \to \exp\{-\sigma^2 t^2/2\}$$

as $k \to \infty$. This proves the inequality (1.7).

2 Inequalities for characteristic functions. Estimates from above

Our aim here is to proof the following result.

Theorem 2.1. *Let $g(t)$ be a characteristic function of an even non-degenerate function having its support in interval $[-A, A]$, $A > 0$. Then the inequality*

$$g(t) \leq \cos(a_{1/\gamma}^{\gamma} \cdot t) \tag{2.10}$$

holds for all $t \in (-\pi/(2A^\gamma), \pi/(2A^\gamma))$. Here $a_{1/\gamma}$ is absolute moment of the order $1/\gamma$ of the distribution with characteristic function $g(t)$, and $\gamma > 1$.

Proof. Denote by $G(x)$ probability distribution function corresponding to characteristic function $g(t)$. Set $\tilde{G}(x) = 2\big(G(xk) - 1/2\big)$, $H(y) = \tilde{G}(y^\gamma)$. We have

$$g(t) = \int_{-A}^{A} \cos(tx)dG(x) = \int_{0}^{A} \cos(tx)d\tilde{G}(x) =$$
$$= \int_{0}^{A} \cos(ty^\gamma)dH(y) \leq \cos(a_{1/\gamma}^{\gamma} \cdot t)$$

for all $|t| \leq \pi/(2A^\gamma)$. Here we used the fact that the function $\cos(ty^\gamma)$ is concave in y if $0 \leq y|t| \leq \pi/2$ and applied Jensen inequality. □

3 Inequalities for some moments of infinitely divisible distributions

Let us give some inequalities comparing the moments of infinitely divisible distributions with corresponding characteristics of Gaussian distribution.

Theorem 3.1. *Let X be a random variable having symmetric infinitely divisible distribution with finite second moment. Suppose that $0 < r < 2$. Then*

$$\mathbb{E}|X|^r \leq \mathbb{E}|Y|^r = \frac{2^{r/2}\sigma^r \Gamma((1+r)/2)}{\sqrt{\pi}}, \tag{3.1}$$

where random variable Y has symmetric Gaussian distribution with the same second moment σ^2 as X. The equality in (3.1) attends if and only if X has Gaussian distribution.

Proof. Recall that if Z is a random variable with characteristic function $h(t)$ then

$$\mathbb{E}|Z|^r = C_r \int_0^\infty \frac{1 - Re(h(t))}{t^{r+1}} dt,$$

where C_r depends on r only $(0 < r < 2)$. From (1.7) it follows that

$$1 - \exp\{-\sigma^2 t^2/2\} \geq 1 - f(t)$$

and

$$C_r \int_0^\infty \frac{1 - \exp(-\sigma^2 t^2/2)}{t^{r+1}} dt \geq C_r \int_0^\infty \frac{1 - f(t)}{t^{r+1}} dt.$$

□

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