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Multiplicity of solutions for nonlocal fractional equations with nonsmooth potentials[◇]

Ziqing Yuan* Lin Yu
Department of Mathematics, Shaoyang University, Shaoyang, Hunan, 422000, P.R. China

Abstract. This paper is concerned a specific category of nonlocal fractional Laplacian problems that involve nonsmooth potentials. By utilizing an abstract critical point theorem for nonsmooth functionals and combining it with the analytical framework on fractional Sobolev spaces developed by Servadei and Valdinoci, we are able to establish the existence of at least three weak solutions for nonlocal fractional problems. Moreover, this work also generalizes and improves upon certain results presented in the existing literature.

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1. Introduction

In this paper, we deal with the following nonlocal fractional Laplacian problem:

$$\begin{cases} -\mathcal{L}_K u \in \varepsilon\partial F(x,u) - \lambda\partial G(x,u) + \nu\partial H(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $(\mathbb{R}^n, |.|)$ with a C^2-boundary, $n > 2s$, $s \in (0,1)$, the maps $F, G, H : \Omega \times \mathbb{R}$ are measurable potential functionals, which are only locally Lipschitz and in general nonsmooth in the second variable. We denote by the generalized gradient of $\partial F(x,u)$, $\partial G(x,u)$ and $\partial H(x,u)$ to u. Furthermore $\mathcal{L}_K$ is the nonlocal operator defined as follows:

$$\mathcal{L}_K u(x) = \int_{\mathbb{R}^n} (u(x+y) + u(x-y) - 2u(x))K(y)\mathrm{d}y,\ x \in \mathbb{R}^n,$$

where $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ is a function which satisfies the following properties:

(K_1) $\gamma K \in L^1(\mathbb{R}^n)$, where $\gamma(x) = \min\{|x|^2, 1\}$;

(K_2) there exists $\beta > 0$ such that $K(x) \geq \beta|x|^{-(n+2s)}$;

(K_3) $K(x) = K(-x)$, for any $x \in \mathbb{R}^n \setminus \{0\}$.

A typical example for the Kernel K is given by $K(x) = |x|^{-(n+2s)}$. In this case $\mathcal{L}_K$ is the fractional Laplacian operator defined by

$$-(-\Delta)^s u(x) = \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}}\mathrm{d}y,\ x \in \mathbb{R}^n.$$

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E-mail: junjyuan@sina.com
* Corresponding author.

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These operators have various applications in different fields, including phase transitions, thin obstacles, finance, optimization, stratified materials, crystal dislocation, anomalous diffusion, semipermeable membranes, soft thin films, ultra-relativistic limits of quantum mechanics, multiple scattering, quasi-geostrophic flows, minimal surfaces, water waves, and materials science. For a basic introduction to this topic, we recommend referring to the references [1] and the monograph [2].

It is well-known that many free boundary problems and obstacle problems can be reduced to partial differential equations with nonsmooth potentials. The field of nonsmooth analysis is closely related to the development of critical points theory for nondifferentiable functions, particularly for locally Lipschitz continuous functionals based on Clarke's generalized gradient [18]. This theory provides a suitable mathematical framework to extend the classic critical point theory for C^1-functionals in a natural way, and to meet specific needs in applications such as nonsmooth mechanics and engineering. For a comprehensive understanding of this topic, we recommend referring to the monographs by [3,4,24] and references such as [8–13], among others.

If F, G and H are differentiable, then problem (1.1) becomes into the following form

$$\begin{cases} -\mathcal{L}_K u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.2}$$

In recent years, there have been many interesting results focusing on problem (1.2) using various methods. However, in our case, we only assume that the energy functional corresponding to problem 1.1 is locally Lipschitz instead of differentiable. This assumption poses certain difficulties and prevents us from applying classical variational methods to solve the problem. To overcome these difficulties, we need to utilize theories for locally Lipschitz functionals to establish existence results for this case. Fortunately, in [32, Theorem 3.3] (see Theorem 2.1 below), we have developed a nonsmooth three critical points theory that can be applied to prove that problem 1.1 has at least three critical points (see Theorem 3.1). One remarkable aspect of our results is that we do not impose any conditions on the behavior of the nonlinearities at the origin, which makes our results more interesting compared to most known results in the literature (e.g., [5–7] et. al.).

Recently, there has been significant attention focused on the study of fractional and nonlocal operators of elliptic type, both for pure mathematical research and with a view to concrete real-world applications. In [26], Servadel and Valdinoci proved the following fractional Laplacian equation:

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u = g & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{1.3}$$

They proved a maximum principle and used it to obtain their regularity results. Autuori and Pucci [21] discussed the elliptic problems involving the fractional Laplacian in $\mathbb{R}^N$ and derived three nontrivial critical values. Cabré and Sire [31] studied problem (1.3) and established necessary conditions on the nonlinearity f to admit certain types of solutions. In [22], Bisci, using variational methods, established three weak solutions via an abstract result by Ricceri about non-local equations. However, all of these works are based on the assumption that the potential functionals are smooth. To the best of our knowledge, there exist no results discussing problem (1.1) with nonsmooth potentials. For problems with nonsmooth potential functionals, most results focus on studying the Dirichlet problem involving the p-Laplacian or $p(x)$-Laplacian differential inclusion. For example, there exist some results studying the following problem with a nonsmooth potential in Sobolev spaces:

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) \in \partial F(x,u) & \text{for a.e. } x \in \Omega, \\ u\big|_{\partial\Omega} = 0. \end{cases} \tag{1.4}$$

Gansiński and Papageorgiou [24], using a variational approach combined with suitable truncation techniques and the method of upper-lower solutions, proved the existence of at least five nontrivial smooth solutions for problem (1.4). Iannizzotto and Marano [15], employing variational methods with truncation techniques, obtained at least three smooth solutions for problem (1.4) with $\partial F(x,u)$ given by $\lambda\partial F(x,u)$. Besides, Kyritsi and Papageorgiou [30], based on the nonsmooth critical point theory of Chang [23], derived two strictly positive solutions with $p \geq 2$. In [14], Kristály, employing a nonsmooth Ricceri-type variational principle, proved the existence of infinitely many, radially symmetric solutions of p-Laplacian differential inclusions in an unbounded domain. Results of $p(x)$-Laplacian differential inclusion can be found in [16,17,19,20].

However, we should mention that the variational method to deal with problem (1.1) is not often easy to perform. Variational approaches do not work when applied to these classes of equations due to the presence of the nonlocal term. Fortunately, our approach in this paper is realizable by checking that the associated energy functional satisfies the assumptions requested by a very recent and general nonsmooth three critical points theorem derived by Yuan and Huang [32, Theorem 3.3] (see Theorem 2.1 below) and thanks to a suitable framework developed in [27]. Furthermore, we observe a remarkable feature of our results: compared to most of the known results in the classical Laplacian case, no condition on the behavior of the involved nonlinearities at the origin is assumed. Therefore, our results are more interesting.

The rest of the paper is organized as follows. Section 2 contains the necessary preliminaries. In Section 3, we prove our main results.

2. Preliminaries

Some basic notations

- $\rightharpoonup$ means weak convergence, $\to$ strong convergence.
- C denotes all the embedding constants (the exact value may be different from line to line).

In this section, we briefly recall the definition of the functional space X_0, firstly introduced in [28]. The functional space X denotes the linear of Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that the restriction to Ω of any function g in X belongs to $L^2(\Omega)$ and

$$((x,y) \to (g(x)-g(y))\sqrt{K(x-y)}) \in L^2((\mathbb{R}^n \times \mathbb{R}^n) \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega), \mathrm{d}x\mathrm{d}y),$$

where $\mathcal{C}\Omega = \mathbb{R}^n \setminus \Omega$. We denote by X_0 the following linear subspace of X

$$X_0 = \{g \in X : g = 0 \text{ a.a. in } \mathbb{R}^n \setminus \Omega\}.$$

Note that X and X_0 are non-empty, since $C_0^2(\Omega) \subseteq X_0$ by Lemma 11 in [28]. Moreover, the space X is endowed with the norm defined as

$$\|g\|_X = \|g\|_{L^2(\Omega)} + \left(\int_Q |g(x)-g(y)|^2 K(x-y)\mathrm{d}x\mathrm{d}y\right)^{1/2},$$

where $Q = (\mathbb{R}^n \times \mathbb{R}^n) \setminus \mathcal{O}$ and $\mathcal{O} = (\mathcal{C}\Omega) \times (\mathcal{C}\Omega) \subseteq \mathbb{R}^n \times \mathbb{R}^n$. It is easy to see that $\|.\|_X$ is a norm on X (see, for instance [27]). By [27, Lemmas 6 and 7] we can take the function

$$X_0 \ni v \to \|v\|_{X_0} = \left(\int_Q |v(x) - v(y)|^2 K(x-y) \mathrm{d}x \mathrm{d}y \right)^{1/2} \tag{2.1}$$

as norm on X_0 in the sequel. Also $(X_0, \|.\|_{X_0})$ is a Hilbert space with scalar product

$$\langle u, v \rangle_{X_0} = \int_Q (u(x) - u(y))(v(x) - v(y)) K(x-y) \mathrm{d}x \mathrm{d}y,$$

see [27, Lemma 7].

Note that in (2.1) (and in the related scalar product) the integral can be extended to all $\mathbb{R}^n \times \mathbb{R}^n$, since $v \in X_0$ (and so $v = 0$ a.a. in $\mathbb{R}^n \setminus \Omega$). While for a general kernel K satisfying conditions from (K_1) to (K_3) we have that $X_0 \subset H^s(\mathbb{R}^n)$, in the model case $K(x) = |x|^{-(n+2s)}$ the space X_0 consists of all the functions of the usual fractional Sobolev space $H^s(\mathbb{R}^n)$ which vanish a.a. outside Ω (see [29, Lemma 7]).

Here $H^s(\mathbb{R}^n)$ denotes the fractional Sobolev space endowed with the norm (the so-called Gagliardo norm)

$$\|g\|_{H^s(\mathbb{R}^n)} = \|g\|_{L^2(\mathbb{R}^n)} + \left(\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|g(x) - g(y)|^2}{|x-y|^{n+2s}} \mathrm{d}x \mathrm{d}y \right)^{1/2}.$$

Recall the embedding properties of X_0 into the usual Lebesgue spaces (see [27, Lemma 8]). The embedding $j : X_0 \hookrightarrow L^q(\mathbb{R}^n)$ is continuous for any $q \in [1, 2^*]$, while it is compact when $q \in [1, 2^*)$, where $2^* = \frac{2n}{n-2s}$ denotes the fractional critical Sobolev exponent. Hence, for any $q \in [1, 2^*]$ there exists a positive constant c_q such that

$$\|v\|_{L^q(\mathbb{R}^n)} \leq c_q \|v\|_{X_0} \quad \text{for any } v \in X_0.$$

In what follows, let λ_1 be the 1-th eigenvalue of the operator $-\mathcal{L}_K$ with homogenous Dirichlet boundary data, namely the 1-th eigenvalue of the problem

$$\begin{cases} -\mathcal{L}_K u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{2.2}$$

Note that, as in the classical Laplacian case, the set of the eigenvalues of problem (2.2) consists of a sequence $\{\lambda_k\}_{k \in \mathbb{N}}$ with

$$0 < \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_k \leq \lambda_{k+1} \leq \ldots \tag{2.3}$$

and

$$\lambda_k \to +\infty \quad \text{as} \quad k \to \infty. \tag{2.4}$$

Definition 2.1 A function I: $X \to \mathbb{R}$ is locally Lipschitz if for every $u \in X$ there exist a neighborhood U of u and $L > 0$ such that for every ν, $\eta \in U$

$$|I(\nu) - I(\eta)| \leq L \|\nu - \eta\|.$$

Definition 2.2 Let $I : X \to \mathbb{R}$ be a locally Lipschitz function, $u, \nu \in X$: the generalized derivative of I in u along the direction ν,

$$I^0(u;\nu) = \limsup_{\eta \to u, \tau \to 0^+} \frac{I(\eta + \tau\nu) - I(\eta)}{\tau}.$$

It is easy to see that the function $\nu \mapsto I^0(u;\nu)$ is sublinear, continuous and so is the support function of a nonempty, convex and w^*- compact set $\partial I(u) \subset X^*$, defined by

$$\partial I(u) = \{u^* \in X^* : \langle u^*, \nu \rangle_X \leq I^0(u;\nu) \text{ for all } v \in X\}.$$

If $I \in C^1(X)$, then

$$\partial I(u) = \{I'(u)\}.$$

Clearly, these definitions extend those of the Gâteaux directional derivative and gradient.

A point $u \in X$ is a critical point of I, if $0 \in \partial I(u)$. It is easy to see that, if $u \in X$ is a local minimum of I, then $0 \in \partial I(u)$. For more details we refer the reader to Clarke [18].

Definition 2.3 We say that $u \in X$ is a solution of problem (1.1) if there exist $\xi(x,u) \in \partial F(x,u)$, $\zeta(x,u) \in \partial G(x,u)$ and $\eta(x,u) \in \partial H(x,u)$ for a.a. $x \in \Omega$ such that for all $v \in X$ we have

$$\begin{aligned}&\int_Q (u(x) - u(y))(v(x) - v(y))K(x-y)\mathrm{d}x\mathrm{d}y \\ &= \epsilon \int_\Omega \xi(x,u)v(x)\mathrm{d}x - \lambda \int_\Omega \zeta(x,u)v(x)\mathrm{d}x + \nu \int_\Omega \eta(x,u)v(x)\mathrm{d}x.\end{aligned}$$

Proposition 2.1([18]) Let $h : X \to \mathbb{R}$ be locally Lipschitz function. Then

(i) $(-h)^\circ(u;z) = h^\circ(u;-z)$ for all $u,\ z \in X$;

(ii) $h^\circ(u;z) = \max\{\langle u^*, z\rangle_X : u^* \in \partial h(u)\} \leq L\|z\|$ with L as in Definition 2.1, for all $u,\ z \in X$;

(iii) Let $j : X \to \mathbb{R}$ be a continuously differentiable function. Then $\partial j(u) = \{j'(u)\}$, $j^\circ(u;z)$ coincides with $\langle j'(u), z\rangle_X$ and $(h+j)^\circ(u;z) = h^\circ(u;z) + \langle j'(u), z\rangle_X$ for all $u,\ z \in X$;

(iv) (Lebourg's mean value theorem) Let u and v be two points in X. Then, there exists a point ω in the open segment between u and v, and $u^*_\omega \in \partial h(\omega)$ such that

$$h(u) - h(v) = \langle u^*_\omega, u - v\rangle_X;$$

(v) Let Y be a Banach space and $j : Y \to X$ a continuously differentiable function. Then $h \circ j$ is locally Lipschitz and

$$\partial(h \circ j)(u) \subseteq \partial h(j(y)) \circ j'(y) \text{ for all } y \in Y;$$

(vi) If $h_1,\ h_2 : X \to \mathbb{R}$ are locally Lipschitz, then

$$\partial(h_1 + h_2)(u) \subseteq \partial h_1(u) + \partial h_2(u);$$

(vii) $\partial h(u)$ is convex and weakly* compact and the set-valued mapping $\partial h : X \to 2^{X^*}$ is weakly* upper semicontinuous;

(viii) $\partial(\lambda h)(u) = \lambda \partial h(u)$ for every $\lambda \in \mathbb{R}$.

Let I, Ψ, $\Phi : X \to \mathbb{R}$ be three given functions, for each $\mu > 0$ and $r \in]\inf_X \Phi, \sup_X \Phi[$, we set

$$h_1(\mu I + \Psi, \Phi, r) = \inf_{u \in \Phi^{-1}(]-\infty, r[)} \frac{\mu I(u) + \Psi(u) - \inf_{u \in \Phi^{-1}(]-\infty, r])}(\mu I + \Psi)}{r - \Phi(u)}$$

and

$$h_2(\mu I + \Psi, \Phi, r) = \sup_{u \in \Phi^{-1}(]r, +\infty[)} \frac{\mu I(u) + \Psi(u) - \inf_{u \in \Phi^{-1}(]-\infty, r])}(\mu I + \Psi)}{r - \Phi(u)}.$$

When $\Psi + \Phi$ is bounded below, for each $r \in]\inf_X \Phi, \sup_X \Phi[$ such that

$$\inf_{u \in \Phi^{-1}(]-\infty, r])} I(u) < \inf_{u \in \Phi^{-1}(r)} I(u).$$

Set

$$h_3(I, \Psi, \Phi, r) = \inf \left\{ \frac{\Psi(u) - \gamma + r}{\eta_r - I(u)} : u \in X, \Phi(u) < r, I(u) < \eta_r \right\},$$

where

$$\gamma = \inf_{u \in X} (\Psi(u) + \Phi(u))$$

and

$$\eta_r = \inf_{u \in \Phi^{-1}(r)} I(u).$$

With the above notations, our abstract tool for proving the main result of our paper is [32, Theorem 3.3] and we recall here for the readers' convenience.

Theorem 2.1 *Let $(X, \|\cdot\|)$ be a reflexive Banach space, $I \in C^1(X, \mathbb{R})$ a sequentially weakly lower semicontinuous function, bounded on any bounded subset of X, such that I' is of type $(S)_+$. Ψ and $\Phi : X \to \mathbb{R}$ are two locally Lipschitz functions with compact gradient. Assume also that the function $\Psi + \lambda\Phi$ is bounded below for all $\lambda > 0$ and that*

$$\liminf_{\|u\| \to +\infty} \frac{\Psi(u)}{I(u)} = -\infty. \tag{2.5}$$

Then, for each $r > \sup_N \Phi$, where N is the set of all global minima of I, for each $\mu > \max\{0, h_3(I, \Psi, \Phi, r)\}$ and each compact interval $[a, b] \subset]0, h_2(\mu I + \Psi, \Phi, r)[$, there exists a number $\rho > 0$ with the following property: for every $\lambda \in [a, b]$ and every locally Lipschitz function $H : X \to \mathbb{R}$ with compact gradient, there exists $\delta > 0$ such that, for each $\nu \in [0, \delta]$, the function $\mu I(u) + \Psi(u) + \lambda\Phi(u) + \nu H(u)$ has at least three critical points in X whose norms are less than ρ.

3. The main results

Firstly, we define $I(u), \Psi(u), \Phi(u), \tilde{H}(u) : X_0 \mapsto \mathbb{R}$ by

$$I(u) = \frac{\|u\|_{X_0}^2}{2}, \quad \Psi(u) = -\mathscr{F}(u), \quad \mathscr{F}(u) = \int_\Omega F(x,u)\mathrm{d}x,$$
$$\Phi(u) = \int_\Omega G(x,u)\mathrm{d}x, \quad \tilde{H}(u) = \int_\Omega H(x,u)\mathrm{d}x$$

for all $u \in X$. It is easy to see that the functional I is a continuously Gâteaux differentiable whose Gâteaux derivative at the point $u \in X_0$ is the functional $I'(u) \in X_0^*$ given by

$$\langle I'(u), v\rangle = \int_Q (u(x)-u(y))(v(x)-v(y))K(x-y)\mathrm{d}x\mathrm{d}y$$

for all $v \in X_0$. For each $r \in]\inf_X \Phi, \sup_X \Phi[$, set

$$h_3^*(I,\Psi,\Phi,r) = \inf\left\{\frac{\Psi(u)-\hat{\gamma}+r}{\hat{\eta}_r - I(u)} : u \in X, \Phi(u) < r, I(u) < \hat{\eta}_r\right\},$$

where

$$\hat{\gamma} = \int_\Omega \inf_{u\in\mathbb{R}} (G(x,u) - F(x,u))\mathrm{d}x,$$

and

$$\hat{\eta}_r = \inf_{u \in \Phi^{-1}(r)} I(u).$$

For each $\epsilon \in \left]0, \frac{1}{\max\{0, h_3^*(I,\Psi,\Phi,r)\}}\right[$, let

$$h_2^*(I+\Psi,\Phi,r) = \sup_{u\in\Phi^{-1}(]r,+\infty[)} \frac{I(u)+\epsilon\Psi(u) - \inf_{\Phi^{-1}(]-\infty,r])}(I+\epsilon\Psi)}{r-\Phi(u)}.$$

In order to discuss problem (1.1), we need the following hypotheses:

(F_1) for all $u \in \mathbb{R}$, $\Omega \ni x \mapsto F(x,u)$ is measurable;

(F_2) for a.a. $x \in \Omega$, $\mathbb{R} \ni u \mapsto F(x,u)$ is locally Lipschitz;

(F_3) $|\xi| \le k_1(1+|u|^{q_1-1})$ for a.a. $x \in \Omega$ and every $u \in \mathbb{R}$, $\xi(x,u) \in \partial F(x,u)$ $(k_1 > 0,\ q_1 \in (2, 2^*))$;

(F_4)

$$\lim_{|u|\to+\infty} \frac{\inf_{x\in\Omega} F(x,u)}{u^2} = +\infty \text{ and } \lim_{|u|\to+\infty} \frac{\sup_{x\in\Omega} F(x,u)}{|u|^\alpha} < +\infty,$$

where $\alpha \in (2, 2^*)$;

(G_1) for all $u \in \mathbb{R}$, $\Omega \ni x \mapsto G(x,u)$ is measurable;

(G_2) for a.a. $x \in \Omega$, $\mathbb{R} \ni u \mapsto G(x,u)$ is locally Lipschitz;

(G_3) $|\zeta| \le k_2(1+|u|^{q_2-1})$ for a.a. $x \in \Omega$ and every $u \in \mathbb{R}$, $\zeta(x,u) \in \partial G(x,u)$ $(k_2 > 0,\ q_2 \in (2, 2^*))$;

(G_4)

$$\lim_{|u|\to+\infty}\frac{\inf_{x\in\Omega}G(x,u)}{|u|^{\alpha}}=+\infty,$$

where $\alpha\in(2,2^*)$;

(H_1) for all $u\in\mathbb{R}$, $\Omega\ni x\mapsto H(x,u)$ is measurable;

(H_2) for a.a. $x\in\Omega$, $\mathbb{R}\ni u\mapsto H(x,u)$ is locally Lipschitz;

(H_3) $|\eta|\le k_3(1+|u|^{q_3-1})$ for a.a. $x\in\Omega$ and every $u\in\mathbb{R}$, $\eta(x,u)\in\partial H(x,u)$ ($k_3>0,\ q_3\in(2,2^*)$).

Remark 3.1 *It is easy to see that there exist lots of functions which satisfy hypotheses* (F_1)*-*(F_4)*,* (G_1)*-*(G_4) *and* (H_1)*-*(H_4)*. For example, for simplicity, we drop the* x*-dependence.*

$$F(u)=\begin{cases}|u|, & |u|<1,\\ |u|^{2+a_1}, & |u|\ge 1,\end{cases}\qquad G(u)=\begin{cases}|u|, & |u|<1,\\ |u|^{\alpha+a_2}, & |u|\ge 1,\end{cases}\qquad \text{and } H(u)=|u|,$$

where $0<a_1<\alpha-2$*,* $0<a_2<2^*-\alpha$*.*

Lemma 3.1 *If hypotheses* $(K_1)-(K_3)$ *hold, then*

(i) $I':X_0\to X_0^*$ *is a continuous, bounded and strictly monotone operator;*

(ii) I' *is a mapping of type* (S_+)*, i.e., if* $u_n\rightharpoonup u$ *in* X_0 *and* $\overline{\lim}_{n\to+\infty}\langle I'(u_n)-I'(u),u_n-u\rangle\le 0$*, then* $u_n\to u$ *in* X_0*.*

Proof. (i) By virtue of the properties of $(K_1)-(K_3)$, it is obvious that I' is continuous and bounded. Note that

$$\langle I'(u),u\rangle=\int_Q|u(x)-u(y)|^2K(x-y)\mathrm{d}x\mathrm{d}y,$$
$$\langle I'(u),v\rangle=\langle I'(v),u\rangle=\int_Q(u(x)-u(y))(v(x)-v(y))K(x-y)\mathrm{d}x\mathrm{d}y,$$
$$\langle I'(v),v\rangle=\int_Q|v(x)-v(y)|^2K(x-y)\mathrm{d}x\mathrm{d}y,$$

then, we have

$$\begin{aligned}&\langle I'(u)-I'(v),u-v\rangle\\ &=\langle I'(u),u\rangle-\langle I'(u),v\rangle-\langle I'(v),u\rangle+\langle I'(v),v\rangle\\ &=\int_Q[(u(x)-u(y))^2-2(u(x)-u(y))(v(x)-v(y))+(v(x)-v(y))^2]K(x,y)\mathrm{d}x\mathrm{d}y \qquad (3.1)\\ &=\int_Q[(u(x)-u(y))-(v(x)-v(y))]^2K(x,y)\mathrm{d}x\mathrm{d}y\ge 0,\end{aligned}$$

i.e., I' is monotone. In fact I' is strictly monotone. Indeed, if $\langle I'(u)-I'(v),u-v\rangle=0$, then we have

$$\int_Q[(u(x)-u(y))-(v(x)-v(y))]^2K(x,y)\mathrm{d}x\mathrm{d}y=0,$$

so $u \equiv v$. Therefore, $\langle I'(u)-I'(v), u-v\rangle > 0$ if $u \neq v$. This means that I' is a strictly monotone operator in X.

(ii) From (i), if $u_n \rightharpoonup u$ and $\overline{\lim}_{n\to+\infty}\langle I'(u) - I'(v), u - v\rangle \leq 0$, then $\lim_{n\to+\infty}\langle I'(u) - I'(v), u - v\rangle = 0$. According to (3.1), $u_n \to u$ in Ω, so we obtain a subsequence (which we still denoted by u_n) satisfying $u_n \to u$ a.a. $x \in \Omega$. From Fadou's lemma, we have

$$\underline{\lim}_{n\to+\infty}\int_Q |u_n(x) - u_n(y)|^2 K(x, y)\mathrm{d}x\mathrm{d}y \geq \int_Q |u(x) - u(y)|^2 K(x, y)\mathrm{d}x\mathrm{d}y. \tag{3.2}$$

By $u_n \rightharpoonup u$ we derive

$$\lim_{n\to+\infty}\langle I'(u_n), u_n - u\rangle = \lim_{n\to+\infty}\langle I'(u_n) - I'(u), u_n - u\rangle = 0. \tag{3.3}$$

On the other hand, we also have

$$\begin{aligned}
&\langle I'(u_n), u_n - u\rangle \\
&= \int_Q [(u_n(x) - u_n(y)) - (u(x) - u(y))](u_n(x) - u_n(y))K(x - y)\mathrm{d}x\mathrm{d}y \\
&= \int_Q [(u_n(x) - u_n(y))^2 - (u_n(x) - u_n(y))(u(x) - u(y))]K(x - y)\mathrm{d}x\mathrm{d}y \\
&\geq \int_Q \frac{K(x - y)}{2}[|u_n(x) - u_n(y)|^2 - |u(x) - u(y)|^2]\mathrm{d}x\mathrm{d}y.
\end{aligned} \tag{3.4}$$

In view of (3.2), (3.3) and (3.4), we have

$$\lim_{n\to+\infty}\int_Q K(x - y)|u_n(x) - u_n(y)|^2\mathrm{d}x\mathrm{d}y = \int_Q K(x - y)|u(x) - u(y)|^2\mathrm{d}x\mathrm{d}y.$$

Therefore, $u_n \to u$ in X_0, i.e., I' is of type (S_+). □

The next Lemma displays some properties of $\mathscr{F}(u)$.

Lemma 3.2 *If hypotheses $(F_1) - (F_3)$ hold, then $\mathscr{F} : X \to \mathbb{R}$ is a locally Lipschitz function with compact gradient.*

Proof. Firstly we show that $\mathscr{F}$ is locally Lipschitz. Let $u, v \in X_0$. According to the Lebourg's mean value theorem, we have

$$\begin{aligned}
|\mathscr{F}(u) - \mathscr{F}(v)| &\leq \int_\Omega |F(x, u(x)) - F(x, v(x))|\mathrm{d}x \\
&\leq \int_\Omega k_1(1 + |u(x)|^{q_1-1} + 1 + |v(x)|^{q_1-1})|u(x) - v(x)|\mathrm{d}x \\
&\leq k_1C\|u - v\|_{L^2(\Omega)} + k_1(\|u\|^{q_1-1}_{L^{q_1}(\Omega)} + \|v\|^{q_1-1}_{L^{q_1}(\Omega)})\|u - v\|_{L^{q_1}(\Omega)} \\
&\leq k_1C\|u - v\|_{X_0} + k_1C(\|u\|^{q_1-1}_{X_0} + \|v\|^{q_1-1}_{X_0})\|u - v\|_{X_0}.
\end{aligned}$$

Then it is easy to see that $\mathscr{F}$ is locally Lipschitz.

Next, we prove that $\partial\mathscr{F}$ is compact. Choosing $u \in X_0$, $u^* \in \partial\mathscr{F}(u)$ we obtain that for every $v \in X_0$

$$\langle u^*, v\rangle \le \mathscr{F}^\circ(u; v) \tag{3.5}$$

and $\mathscr{F}^\circ(u;\cdot) : L^r(\Omega) \to \mathbb{R}$ is a subadditive function (see Proposition 2.1). Furthermore, $u^* \in X_0^*$ is continuous also with respect to the topology induced on X_0 by the norm $\|\cdot\|_{L^r(\Omega)}$. Indeed, setting $L > 0$ a Lipschitz constant for $\mathscr{F}$ in a neighborhood of u, for all $z \in X_0$ we derive from Proposition 2.1 (ii)

$$\langle u^*, z\rangle \le L\|z\|_{L^r(\Omega)}, \qquad \langle u^*, -z\rangle \le L\|-z\|_{L^r(\Omega)}.$$

So

$$\langle u^*, z\rangle \le L\|z\|_{L^r(\Omega)}.$$

Hence, by Hahn-Banach Theorem, u^* can be extended to an element of the dual $L^r(\Omega)$ (complying with (3.5)) for all $v \in L^r(\Omega)$, this means that we can represent u^* as an element of $L^{r'}(\Omega)$ and write for every $v \in L^r(\Omega)$

$$\langle u^*, v\rangle = \int_\Omega u^*(x)v(x)\mathrm{d}x. \tag{3.6}$$

Set $\{u_n\}$ be a sequence in X_0 such that $\|u_n\| \le M$ for all $n \in \mathbb{N}$ ($M > 0$) and take $\xi_n \in \partial\mathscr{F}(u_n)$ for all $n \in \mathbb{N}$. It follows from (F_3) and (3.6) that

$$\begin{aligned}
\langle \xi_n, v\rangle &= \int_\Omega \xi_n v(x)\mathrm{d}x \\
&\le \int_\Omega |\xi_n||v(x)|\mathrm{d}x \\
&\le \int_\Omega k_1(1 + |u_n(x)|^{q_1-1})|v(x)|\mathrm{d}x \\
&\le k_1C(1 + \|u_n\|_0^{q_1-1})\|v\|_{X_0} \\
&\le k_1C(1 + M^{q_1-1})\|v\|_{X_0}
\end{aligned}$$

for all $n \in \mathbb{N},\ u \in X_0$. So

$$\|\xi_n\|_{X_0^*} \le k_1C(1 + M^{q_1-1}),$$

i.e., the sequence $\{\xi_n\}$ is bounded. Hence, passing to a subsequence, we have $\xi_n \rightharpoonup \xi \in X_0^*$. We will prove that $\{\xi_n\} \subset X_0^*$ has a strong convergence. We proceed by contradiction. Assume that there exists some $\varepsilon > 0$ such that

$$\|\xi_n - \xi\|_{X_0^*} > \varepsilon$$

for all $n \in \mathbb{N}$ and hence for all $n \in \mathbb{N}$ there exists $v_n \in B(0, 1)$ such that

$$\langle \xi_n - \xi, v_n\rangle > \varepsilon. \tag{3.7}$$

Recall that $\{v_n\}$ is a bounded sequence and passing to a subsequence, one has

$$v_n \rightharpoonup v \in X_0, \quad \|v_n - v\|_{L^2(\Omega)} \to 0, \quad \|v_n - v\|_{L^{q_1}(\Omega)} \to 0.$$

Hence, for n large enough, we have

$$|\langle \xi_n - \xi, v\rangle| < \frac{\varepsilon}{4}, \qquad |\langle \xi, v_n - v\rangle| < \frac{\varepsilon}{4},$$
$$\|v_n - v\|_{L^2(\Omega)} < \frac{\varepsilon}{4k_1C}, \quad \|v_n - v\|_{L^{q_1}(\Omega)} < \frac{\varepsilon}{4k_1M^{q-1}}.$$

Then,

$$\begin{aligned}\langle \xi_n - \xi, v_n\rangle &= \langle \xi_n - \xi, v\rangle + \langle \xi_n, v_n - v\rangle - \langle \xi, v_n - v\rangle \\ &\le \frac{\varepsilon}{2} + \int_\Omega |\xi_n||v_n(x) - v(x)|\mathrm{d}x \\ &\le \frac{\varepsilon}{2} + k_1\int_\Omega (1+|u_n|^{q_1-1})|v_n(x)-v(x)|\mathrm{d}x \\ &\le \frac{\varepsilon}{2} + k_1C\|v_n - v\|_{L^2(\Omega)} + k_1\|u_n\|^{q_1-1}_{L^{q_1-1}}\|v_n - v\|_{L^{q_1}(\Omega)} \\ &\le \frac{\varepsilon}{2} + k_1C\|v_n - v\|_{L^2(\Omega)} + k_1M^{q_1-1}\|v_n - v\|_{L^{q_1}(\Omega)} \\ &\le \varepsilon,\end{aligned}$$

which contradicts to (3.7). $\square$

Similar, we have the following properties of the functions $\Phi(u)$ and $\tilde{H}(u)$.

Lemma 3.3 *If $(G_1)-(G_3)$ and $(H_1)-(H_3)$ hold, then $\Phi(u)$, $\tilde{H}(u) : X_0 \to \mathbb{R}$ are locally Lipschitz functionals with compact gradient.*

With the above lemmas, our main result reads as follows.

Theorem 3.1 *Let $s \in (0,1)$, $n > 2s$. If hypotheses $(K_1)-(K_3)$, $(F_1)-(F_4)$, $(G_1)-(G_4)$ and $(H_1)-(H_3)$ hold, then for all $r > 0$, $\epsilon \in \left]0, \frac{1}{\max\{0, h_3^*(I,\Psi,\Phi,r)\}}\right[$ and all compact interval $[a,b] \subset]0, h_2^*(I+\Psi,\Phi,r)[$, there exist numbers $\rho > 0$ and $\delta > 0$ such that for all $\lambda \in [a,b]$ and all $\nu \in [0,\delta]$, problem (1.1) has at least three weak solutions whose norms in X are less than ρ.*

Proof. We will employ Theorem 2.1 to prove it. Observe that X_0 is a reflexive Banach space. $I \in C^1(X_0,\mathbb{R})$ is continuous and convex, and hence weakly l.s.c. and obviously bounded on any bounded subset of X_0. From Lemma 3.1, I' is of type (S_+). Moreover, it follows from Lemmas 3.2 and 3.3 that Φ, Ψ and $\tilde{H}$ are locally Lipschitz functionals with compact gradient. Hence we only need to prove that the functional $\Psi + \lambda\Phi$ is bounded below for all $\lambda > 0$ and $\liminf_{\|u\|\to+\infty} \frac{\Psi(u)}{I(u)} = -\infty$. Firstly, we prove that $\Psi + \lambda\Phi$ is bounded below for all $\lambda > 0$. From hypotheses (F_3) and (F_4) there exists $c_1 > 0$ such that

$$F(x,u) \le c_1(1+|u|^\alpha). \tag{3.8}$$

Moreover, from (G_3) and (G_4), we also have that for all $c_2 > 0$ there exists a constant $c_3 > 0$ such that

$$G(x,u) \ge c_2|u|^\alpha - c_3 \tag{3.9}$$

for a.a. $x \in \Omega$. By virtue of (3.7) and (3.8), for each $\lambda > 0$, choosing $c_2 > \frac{c_1}{\lambda}$ we derive that

$$\begin{aligned}\Psi + \lambda\Phi &= \int_\Omega [\lambda G(x,u) - F(x,u)]\mathrm{d}x \\ &\ge \int_\Omega [\lambda(c_2|u|^\alpha - c_3) - c_1(1+|u|^\alpha)]\mathrm{d}x\end{aligned}$$

$$= \int_\Omega [(\lambda c_2 - c_1)|u|^\alpha - \lambda c_3 - c_1]\mathrm{d}x$$
$$\geq -(\lambda c_3 + c_1)|\Omega|,$$

which means that $\Psi + \lambda\Phi$ is bounded below.

Next, we prove that

$$\liminf_{\|u\|\to+\infty} \frac{\Psi(u)}{I(u)} = -\infty. \tag{3.10}$$

From [25, Proposition 9 and Appendix A] we have the following characterization of the following eigenvalue λ_1:

$$\lambda_1 = \min_{u\in X_0\setminus\{0_{X_0}\}} \frac{\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)\mathrm{d}x\mathrm{d}y}{\int_\Omega u(x)^2 \mathrm{d}x\mathrm{d}x}. \tag{3.11}$$

Furthermore, the first eigenfunction $u_1 \in X_0$ is nonnegative in Ω (see [25, Proposition 9 and Appendix A], or [26, Corollary 8]). It follows from (3.11) that

$$\|u\|_{X_0}^2 = \lambda \int_\Omega u_1(x)^2 \mathrm{d}x.$$

In order to prove (3.10) it is enough to show that

$$\lim_{k\to+\infty} \frac{\Psi(ku_1)}{\|ku_1\|_{X_0}^2} = -\infty. \tag{3.12}$$

For this purpose, fix two positive numbers $M_1,\ M_2$ with $0 < 2M_1 < M_2$. Note that

$$\lim_{|u|\to+\infty} \frac{\inf_{x\in\Omega} F(x,u)}{u^2} = +\infty,$$

there exists large constant $m_1 > 0$ when $|u| > m_1$, we have

$$F(x,u) \geq \lambda_1 M_2 u^2$$

for a.a. $x \in \Omega$. For each $k \in \mathbb{N}$, put

$$\Omega_k = \left\{ x \in \Omega : u_1(x) \geq \frac{m_1}{k} \right\}.$$

It is obvious that for every $k \in \mathbb{N}$, one has $\Omega_k \subseteq \Omega_{k+1}$, the sequence $\left\{ \int_{\Omega_k} u_1(x)^2 \mathrm{d}x \right\}$ is non-decreasing, i.e.,

$$\int_{\Omega_k} u_1(x)^2 \mathrm{d}x \leq \int_{\Omega_{k+1}} u_1(x)^2 \mathrm{d}x$$

for every $k \in \mathbb{N}$ and $\int_{\Omega_k} u_1(x)^2 \mathrm{d}x \to \int_\Omega u_1(x)^2 \mathrm{d}x$. Based on this point, set $\hat{k} \in \mathbb{N}$ such that

$$\int_{\Omega_k} u_1(x)^2 \mathrm{d}x > \frac{2M_1}{M_2} \int_\Omega u_1(x)^2 \mathrm{d}x.$$

By virtue of hypotheses $(F_1) - (F_3)$, there exists a constant $c_4 > 0$ such that

$$\sup_{\Omega\times[0,m_1]} |F(x,u)| < c_4.$$

For all $k \in \mathbb{N}$ satisfying

$$k > \max\left\{\hat{k}, \left(\frac{|\Omega| \sup_{\Omega\times[0,m_1]} |F(x,u)|}{M\|u_1\|^2_{X_0}}\right)\right\},$$

we have

$$\begin{aligned}
\frac{\mathscr{F}(ku_1)}{\|ku_1\|^2_{X_0}} &= \frac{\int_{\Omega_k} F(x, ku_1(x))\mathrm{d}x}{k^2\|u_1\|^2_{X_0}} + \frac{\int_{\Omega\setminus\Omega_k} F(x, ku_1(x))\mathrm{d}x}{k^2\|u_1\|^2_{X_0}} \\
&\geq \frac{\lambda_1 M_2 \int_{\Omega_k} u_1(x)^2\mathrm{d}x}{\|u_1\|^2_{X_0}} + \frac{\int_{\Omega\setminus\Omega_k} F(x, ku_1(x))\mathrm{d}x}{k^2\|u_1\|^2_{X_0}} \\
&> \frac{2\lambda_1 M_1 \int_{\Omega_k} u_1(x)^2\mathrm{d}x}{\|u_1\|^2_{X_0}} - \frac{|\Omega| \sup_{(x,u)\in[0,m_1]} |F(x,u)|}{k^2\|u_1\|^2_{X_0}} \\
&> 2M_1 - M_1 = M_1 \to +\infty \quad (\text{as} \quad M_1 \to +\infty),
\end{aligned}$$

i.e.,

$$\lim_{k\to+\infty} \frac{\Psi(ku_1)}{\|ku_1\|^2{}_{X_0}} = -\infty.$$

Hence, the proof is completed. □

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