The Change of Basis Groupoid

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***Abstract*—Change of basis in finite-dimensional vector spaces has numerous significant applications. This research explores the algebraic structure of change of basis matrices within a set of m bases of a finite vector space using category theory. The investigation reveals a connected groupoid of order m^2 whose morphisms correspond to change of basis matrices. Subgroupoids within this structure correspond to upper and lower triangular matrices and matrices with alternating elements of 0. We identify bases leading to triangular change of basis matrices. For univariate polynomial families, they occur where the minimum degree of the polynomials increases, or the maximum degree decreases. Similarly, we find that alternating bases lead to alternating change of basis matrices. These bases occur with basis polynomials that have definite parity such as classical orthogonal polynomials. A commutative diagram elucidates the subgroupoids with morphisms corresponding to triangular and alternating change of basis matrices. This study enhances our understanding of algebraic properties of change of basis.**

I. Background and Related Work

This paper investigates the algebraic structure of changes of bases of a finite-dimensional vector space. It shows that this structure is a connected groupoid [1], [2] that has subgroupoids. These subgroupoids directly occur for bases formed from polynomial sequences such as Chebyshev polynomials and Zernike radial polynomials. The bases defined with these sequences can have descending maximum degree, ascending minimum degree, or definite parity.

Change of basis in a finite vector space has numerous significant and widespread applications in numerical computing, statistics, and engineering. These applications include spectral methods for solving differential equations numerically, e.g., [3]; image, video, and data compression, e.g., [4] and implementing DCT-II in JPEG compression [5]; in Principal Component Analysis [6], and computer graphics [7].

For example, in spectral methods, change of basis can result in better convergence [3], lower computational complexity [8], and better numerical stability [8]. Much work has been done on numerical solutions for finding the coefficients in Legendre expansions, e.g., see the summary in Hale [9]. From equation (1.2) of Hale and Townsend [9], these coefficients are c_n^{leg} where.

$$p_N(x) = \sum_{n=0}^{N} c_n^{leg} P_n(x)$$

and $x \in [-1, 1]$. The terms $P_n(x)$ on the right side of the equation denote Legendre polynomials. Floating-point numbers represent the coefficients c_n^{leg} . A linear combination of Chebyshev polynomials of the first kind can also express $p_N(x)$:

$$p_N(x) = \sum_{n=0}^{N} c_n^{cheb} T_n(x)$$

and $x \in [-1, 1]$. They provide an algorithm for implementing the transform between the coefficients c_n^{leg} and c_n^{cheb} with an accuracy of fifteen decimal places [9].

These transforms form scaled changes of bases between Legendre and Chebyshev polynomials of the first kind. For example, let M_{TP} denote the change of basis matrix from $P_n(x)$ to $T_n(x)$ where $0 \leq n \leq 4$,

$$M_{TP} = \begin{bmatrix} 1 & 0 & \frac{1}{4} & 0 & \frac{9}{64} \\ 0 & 1 & 0 & \frac{3}{8} & 0 \\ 0 & 0 & \frac{3}{4} & 0 & \frac{5}{16} \\ 0 & 0 & 0 & \frac{5}{8} & 0 \\ 0 & 0 & 0 & 0 & \frac{35}{64} \end{bmatrix} \text{ and } M_{TP} \begin{bmatrix} c_0^{leg} \\ c_1^{leg} \\ c_2^{leg} \\ c_3^{leg} \\ c_4^{leg} \end{bmatrix} = \begin{bmatrix} c_0^{cheb} \\ c_1^{cheb} \\ c_2^{cheb} \\ c_3^{cheb} \\ c_4^{cheb} \end{bmatrix}$$

II. Bases

Defining a basis of a finite vector space can involve a change of basis. For example, consider the basis of the Chebyshev polynomials of the first kind:

$$\begin{aligned} T_0(x) =& 1 \\ T_1(x) =& x \\ T_2(x) =& 2x^2 - 1 \\ T_3(x) =& 4x^3 - 3x \end{aligned}$$

The coordinate vector of $T_3(x)$ is $(0, 0, 0, 1)$ with respect to the basis

$$\{T_0(x), T_1(x), T_2(x), T_3(x)\}.$$

It is also $(0, -3, 0, 4)$ with respect to the basis of monomials $\{1, x, x^2, x^3\}$.

In matrix form, we have a change of basis matrix from Chebyshev polynomials T up to $T_3(x)$ to the monomials M where the transposes of the coefficient vectors of the domain basis vectors with respect to $\{1, x, x^2, x^3\}$ form its columns.

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

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This defines a mapping from the coordinates vectors defined using the basis $\{T_0(x), \dots, T_3(x)\}$ to the coordinate vectors of the same vector using the basis $\{1, x, x^2, x^3\}$. For example

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ -2 \\ 4 \end{bmatrix}$$

and

$$T_3(x) - T_2(x) + 2T_0(x) = 4x^3 - 2x^2 - 3x + 3$$

so that $(2, 0, -1, 1)$ is mapped to $(3, -3, -2, 4)$.

Instead of the monomials, we can use the shifted Legendre polynomials for example to represent the basis of Chebyshev polynomials of the first kind. The first four shifted Legendre polynomials represented using monomials give a descending basis.

$$\begin{aligned} \tilde{P}_0(x) =& 1 \\ \tilde{P}_1(x) =& 2x - 1 \\ \tilde{P}_2(x) =& 6x^2 - 6x + 1 \\ \tilde{P}_3(x) =& 20x^3 - 30x^2 + 12x - 1. \end{aligned}$$

We can express $T_3(x)$ uniquely in terms of the shifted Legendre polynomials by

$$T_3(x) = \frac{1}{5}\tilde{P}_3(x) + \tilde{P}_2(x) + \frac{3}{10}\tilde{P}_1(x) - \frac{1}{2}\tilde{P}_0(x)$$

and similarly for all other basis vectors in $\{T_0(x), T_1(x), \dots\}$.

A change of basis matrix maps the coordinate vector of a vector v defined using the basis of Chebyshev polynomials T to its coordinate vector when the definition of v uses the basis of shifted Legendre polynomials. For example, when $n = 4$, this matrix has the following form.

$$\begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{3} & -\frac{1}{2} \\ 0 & \frac{1}{2} & 1 & \frac{3}{10} \\ 0 & 0 & \frac{1}{3} & 1 \\ 0 & 0 & 0 & \frac{1}{5} \end{bmatrix}$$

Generally, defining a set of basis vectors of a vector space V requires using a known basis of V.

Definition 1. *Let V be a vector space of finite dimension with coefficients over a given field F, and $s = \{s_0, \dots, s_{n-1}\}$ and $v = \{v_0, \dots, v_{n-1}\}$ be bases of V.*

The coordinate vector of a basis vector s_i with respect to the basis u has the form $(c_0, \dots, c_{n-1})$ where $c_i \in F$ and $s_i = \sum_{k=0}^{n-1} c_k v_k$.

For every vector $w \in V$ there exists a unique sum $w = \sum_{k=0}^{n-1} a_k v_k$ where $a_k \in F$. The coordinate vector of w with respect to the basis v has the form $(a_0, a_1, \dots, a_n)$.

We note that if $s = v$ in Definition 1, then the coordinate vectors of s with respect to v are the elements of the standard basis. Sometimes, e.g. [10, Chapter 6], v is not stated explicitly and it is assumed to be the standard basis such as $\{(-1, 2), (2, -1)\}$ for the vector space $\mathbb{R}^2$. However, in this more abstract context, the basis to which a coordinate vector refers to is stated explicitly.

Assumption 1. We assume that a basis of the form

$$\{s_0, s_1, \dots, s_{n-1}\}$$

of a vector space forms an ordered basis [10, §6.2] so that there is a total ordering $<$ for which $s_0 < s_1 < \dots < s_{n-1}$.

A coordinate vector $(c_0, c_1, \dots, c_{n-1})$ with respect to this basis means

$$(c_0, c_1, \dots, c_{n-1}) = c_0 s_0 + c_1 s_1 + \dots + c_{n-1} s_{n-1}.$$

We also assume that the standard basis has the ordering $(1, 0, \dots, 0) < (0, 1, \dots, 0) \dots < (0, 0, \dots, 1)$.

A. Ascending, Descending and Alternating Bases

Definition 2. *Let V be a vector space of finite dimension $n > 0$ and $s = \{s_0, s_1, \dots, s_{n-1}\}$ be a basis of V where each element of s is a coordinate vector of the form $r = (a_0, a_1, \dots, a_{n-1})$ and each coordinate is an element of a field F. These coordinates implicitly refer to an ordered basis $v = \{v_0, v_1, \dots, v_{n-1}\}$ of V such that*

$$s_i = \sum_{k=0}^{n-1} a_k v_k.$$

We define $\min r = \min\{i \mid a_i \neq 0\}$ and $\max r = \max\{i \mid a_i \neq 0\}$.

The basis s is an ascending basis, if

$$\{\min s_0, \min s_1, \dots, \min s_{n-1}\} = \{0, 1, \dots n-1\}.$$

The basis s is a descending basis, if

$$\{\max s_0, \max s_1, \dots, \max s_{n-1}\} = \{0, 1, \dots n-1\}.$$

The basis s is an alternating basis, if for every $0 \leq i, j \leq n-1$, s_i with the form $(a_0, a_1, \dots, a_{n-1})$, and i and j have different parity then $a_j = 0$.

Assumption 2. We assume, without loss of generality, that the ordering of s in Definition 2 is $s_0 < s_1 < \dots < s_{n-1}$ so that for each s_i where $0 \leq i \leq n-1$, of the form $(a_0, a_1, \dots, a_{n-1})$, $a_j = 0$ where $i < j \leq n-1$.

We now give five examples that use Definition 2.

- An example of a descending basis is Chebyshev polynomials of the first kind of degrees 1, 3 and 5. This basis is $\{x, \ \ 4x^3 - 3x, \ \ 16x^5 - 20x^3 + 5x\}$.
  The basis vectors of this subspace of $\mathbb{R}[X]$ have the following basis of coordinate vectors:
  $s = \{(1, 0, 0), (-3, 4, 0), (5, -20, 16)\}$ with respect to the basis $v = \{x, x^3, x^5\}$.
  The change of basis matrix M_{vs} is the following upper triangular matrix.

$$\begin{bmatrix} 1 & -3 & 5 \\ 0 & 4 & -20 \\ 0 & 0 & 16 \end{bmatrix}$$

- A set of coordinate vectors of the standard basis for $\mathbb{R}^4$ is the same as the standard basis for $\mathbb{R}^4$. This basis is both an ascending and descending basis.
- An example of an ascending basis is the set of Zernike Radial polynomials

$$\{R_9^3(x), R_9^5(x), R_9^7(x), R_9^9(x)\}.$$

For example,

$$R_9^5(x) = 36x^9 - 56x^7 + 21x^5.$$

The basis s of coordinate vectors with respect to the basis $v = \{x^3, x^5, x^7, x^9\}$ is

$$s = \{(-20, 105, -168, 84), (0, 21, -56, 36), (0, 0, -8, 9), (0, 0, 0, 1)\}.$$

The change of basis matrix M_{vs} is the following lower triangular matrix.

$$\begin{bmatrix} -20 & 0 & 0 & 0 \\ 105 & 21 & 0 & 0 \\ -168 & -56 & -8 & 0 \\ 84 & 36 & 9 & 1 \end{bmatrix}$$

- An example of basis in $\mathbb{R}^8$ that is neither an ascending nor descending basis is given by the Haar function wavelet matrix [4]:

$$W_8 = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & -1 \end{bmatrix}$$

From the first two columns of W_8 the coordinate vectors $(1,1,1,1,1,1,1,1)$ and $(1,1,1,1,-1,-1,-1,-1)$ that are defined with respect to the standard basis both have non-zero coordinates in the first and eighth dimensions.
- The basis $\{T_0(x), \ldots, T_3(x)\}$ with respect to the basis $\{1, x, x^2, x^3\}$ is an alternating basis. Its coordinate vectors are

$$\{(1,0,0,0), (0,1,0,0), (-1,0,2,0), (0,-3,0,4)\}.$$

The basis vector s_3 has coordinate vector $(0,-3,0,4)$ and a_0 and a_2 have even indices, so that $a_0 = 0$ and $a_2 = 0$.

B. Change of Basis

Let V be a vector space that has bases s and t. The change of basis matrix M_{ts} defines a linear transformation that maps the coordinates b_s of a vector with respect to s to its coordinates a_t with respect to t. It satisfies the following equation.

$$M_{ts} b_s = a_t \tag{1}$$

The subscripts are simple types [11], i.e., ts is the type of a function whose domain has elements of type s and whose range has elements of type t.

We define a change of basis matrix as follows. It is equivalent to other definitions, e.g., [10, §6.3].

Definition 3. *Let V be a vector space that has bases $s = \{s_0, s_1, \ldots, s_{n-1}\}$ and $t = \{t_0, t_1, \ldots, t_{n-1}\}$. The change of basis matrix M_{ts} is the $n \times n$ matrix whose i^{th} column is equal to s_i^T, the transpose of the coordinate vector s_i with respect to t of the basis vectors in s where $0 \le i \le n-1$.*

The following lemma is a property of change of basis matrices, e.g., [10, Theorem 6.12 (c)].

Lemma 1. *For every change of basis matrix M_{ts}, its inverse M_{ts}^{-1} exists and $M_{ts}^{-1} = M_{st}$.*

Definition 4. *Two change of basis matrices M_{ts} and M_{vu} are equal if and only if they have the same elements, $t = v$ and $s = u$.*

Lemma 2. *Every change of basis matrix is unique.*

Proof. Suppose M_{ts} equals a change of basis matrix and that there exists another change of basis matrix N_{ts} that has the same dimensions as M_{ts}. The matrix M_{ts}^{-1} exists from Lemma 1, and N_{ts} is its right inverse. We have $M_{ts} = M_{ts}(M_{st}N_{ts})$ so that $M_{ts} = N_{ts}$ by using the associativity of matrix multiplication, so that M_{ts} is unique. □

III. Change of Basis Groupoid

We use category theory, e.g., [12], to analyse the algebraic structure of change of basis and show that it is a groupoid [1], [2]. Bases of a vector space are the objects of a groupoid and change of basis matrices are its morphisms.

Definition 5. *The category CB its set of objects* obj CB *is a finite set of bases that span the same vector space. The identifier of a basis in* obj CB *is its type identifier.*

The morphisms hom CB *correspond to change of basis matrices. For every $s, t \in$* obj CB *there is a morphism $s \to t \in$* hom CB *that corresponds to the change of basis matrix M_{ts}.*

We show that CB is a groupoid.

Theorem 1. *Given a vector space V with a set S of m bases of V. The set of change of basis matrices M_{ts} where $s, t \in S$ is a connected groupoid CB of order m^2.*

Proof. From Definition 5, CB has a finite set of objects obj CB, and for every pair $s, t \in$ obj CB, there is a unique morphism $s \to t \in$ hom CB that corresponds to M_{ts}. The existence of this morphism follows from Definition 3 of a change of basis matrix. Its uniqueness follows from Lemma 2.

For every object $s \in$ obj CB, there is an identity morphism $s \to s$. This morphism corresponds to the identity change of basis matrix M_{ss}.

For every triple of objects $s, t, u \in$ obj CB there is a function $\circ$ such that $(s \to t) \circ (t \to u) \to (s \to u)$. This corresponds to change of basis matrix product $M_{ut} M_{ts} = M_{us}$

so that $\circ$ is an associative function. The function $\circ$ has the polymorphic type $\gamma\alpha(\gamma\beta)(\beta\alpha)$.

For every pair of objects $s, t \in$ obj CB there is an inverse function that maps the morphism $s \to t$ to $t \to s$. The inverse function corresponds to matrix inversion which, from Lemma 1, is always defined .

Composition satisfies the properties of a groupoid. For every $s \to t \in$ hom CB,

- $(s \to s) \circ (s \to t) = (s \to t)$
- $(s \to t) \circ (t \to t) = (s \to t)$
- $(s \to t) \circ (t \to s) = (s \to s)$
- $(t \to s) \circ (s \to t) = (t \to t)$.

The groupoid has the connected or transitive property, because for any two bases in $s, t \in$ obj CB, there exists a morphism $s \to t \in$ hom CB.

The set obj CB contains m bases, so that there are m^2 permutations of them taken two at a time with repetitions, which is the order of the groupoid. □

A. Example

Suppose CB is a particular groupoid for which obj CB comprises three bases: monomials M; Legendre polynomials P expressed in terms of monomials; and Chebyshev polynomials of the first kind T expressed in terms of monomials. All basis vectors have even degree from 0 to 6. The change of basis matrix

$$M_{TP} = \begin{bmatrix} 1 & \frac{1}{4} & \frac{9}{64} & \frac{25}{256} \\ 0 & \frac{3}{4} & \frac{5}{16} & \frac{105}{512} \\ 0 & 0 & \frac{35}{64} & \frac{63}{256} \\ 0 & 0 & 0 & \frac{231}{512} \end{bmatrix}$$

This gives

$$\begin{bmatrix} 1 & \frac{1}{4} & \frac{9}{64} & \frac{25}{256} \\ 0 & \frac{3}{4} & \frac{5}{16} & \frac{105}{512} \\ 0 & 0 & \frac{35}{64} & \frac{63}{256} \\ 0 & 0 & 0 & \frac{231}{512} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{57}{128} \\ \frac{297}{256} \\ \frac{63}{128} \\ \frac{231}{256} \end{bmatrix}$$

In terms of polynomials, we have $2P_6(x) + P_2(x)$

$$= \frac{231}{256}T_6(x) + \frac{63}{128}T_4(x) + \frac{297}{256}T_2(x) + \frac{57}{128}T_0(x)$$
$$= \frac{1}{8}(231x^6 - 315x^4 + 117x^2 - 9).$$

From Boyd [13], the matrix

$$M_{PT} = \begin{bmatrix} 1 & -\frac{1}{3} & -\frac{1}{15} & -\frac{1}{35} \\ 0 & \frac{4}{3} & -\frac{16}{21} & -\frac{4}{21} \\ 0 & 0 & \frac{64}{35} & -\frac{384}{385} \\ 0 & 0 & 0 & \frac{512}{231} \end{bmatrix}$$

which is the inverse of M_{TP}. The matrix

$$M_{MP} = \begin{bmatrix} 1 & -\frac{1}{2} & \frac{3}{8} & -\frac{5}{16} \\ 0 & \frac{3}{2} & -\frac{15}{4} & \frac{105}{16} \\ 0 & 0 & \frac{35}{8} & -\frac{315}{16} \\ 0 & 0 & 0 & \frac{231}{16} \end{bmatrix}$$

These matrices correspond to morphisms in hom CB. From them we can find the other six groupoid morphisms, i.e., $M \to P$, $M \to T$, $T \to M$, $M \to M$, $P \to P$ and $T \to T$. The three identity matrices have different types: MM, PP and TT.

IV. Subgroupoids

The change of basis matrices that correspond to the morphisms of a subgroupoid can share a property such as being upper triangular matrices, or matrices whose alternate elements are zero. Matrix multiplication and inversion preserve these properties of the matrices.

Triangular change of basis matrices can occur when the vector space is $\mathbb{R}[X]$ and the bases are families of orthogonal polynomials such the Chebyshev polynomials and Zernike radial polynomials.

In general, other properties of square matrices are also preserved under matrix multiplication and inversion, e.g., orthogonality and unitriangularity. However, they do not seem to be as relevant to the change of basis groupoid as triangularity and alternation.

We use the following definition of subgroupoid.

Definition 6. *Let G be a groupoid with a set E of elements and an operation $\circ$. A subgroupoid H of G is a groupoid that has a set $E' \subseteq E$ and the same operation $\circ$ as G.*

A. Triangular Subgroupoids

There are proper change of basis subgroupoids whose morphisms correspond to triangular change of basis matrices. For example, the groupoid for change of basis between Chebyshev polynomials of the second kind (U), Legendre polynomials (L) and Bernstein polynomials (B) up to degree $n > 0$ contains two subgroupoids whose morphisms correspond to triangular change of basis matrices: one with the elements $M_{UL}, M_{LU}, M_{LL}, M_{UU}$; and the other with the element M_{BB}.

These subgroupoids of a change of matrix groupoids occur when the bases of the groupoid are ascending or descending bases. Firstly, we need a lemma about the preservation of triangularity.

Lemma 3. *Matrix multiplication, and matrix inversion when it is defined preserve the property of matrices being lower or upper triangular ones.*

Proof. Let matrices M and N be both lower or upper triangular $n \times n$ matrices. The dot product of the i^{th} row of M with the j^{th} column of N where $0 \le i, j \le n-1$ is 0 if $i < j$

when they are both lower triangular matrices. It is 0 if $i > j$ when they are both upper triangular matrices.

Similarly, we can show that if M and MN are both lower or upper triangular matrices, then so is N, respectively. It follows that matrix inversion, when it is defined, preserves upper or lower triangularity. □

Theorem 2. *Let V be a vector space of finite dimension $n > 0$ with bases s and t that are defined with respect to a basis v of V.*

- *M_{st} is a lower triangular matrix if and only if s and t are ascending bases with respect to v.*
- *M_{st} is an upper triangular matrix if and only if s and t are descending bases with respect to v.*

Proof. We show this for ascending bases. The case for descending ones is similar. Suppose that s and t are ascending bases with respect to v. From Definition 2, the basis s is an ordered set of coordinate vectors with respect to v.

From Definition 3, there is a change of basis matrix M_{vs} whose columns are the respective transposed coordinate vectors in s. From Definition 2, M_{vs} is a lower triangular matrix. In an analogous way, we can form the lower triangular change of basis matrix M_{vt}.

From Lemma 1, $M_{sv} = M_{vs}^{-1}$. Therefore, we have from Lemma 3 that M_{sv} is a lower triangular matrix, and so is $M_{sv}M_{vt}$. From Theorem 1, we have $M_{st} = M_{sv}M_{vt}$.

In the converse direction, if M_{st} is a lower triangular matrix then s and t can be defined with respect to a basis v of V. From Theorem 1 it is the product $M_{st} = M_{sv}M_{vt}$. from Lemma 3, we have M_{sv} and M_{vt} are also lower triangular matrices, and M_{vs} is a lower triangular matrix. It follows that the bases s and t are ascending bases with respect to v from Definition 2. □

An example of Theorem 2 is the change of basis matrix from Bernstein to Zernike radial polynomials in the vector space with basis $\{x^3, x^4, x^5, x^6\}$ over $\mathbb{R}$.

The set of coordinate vectors of the basis s of Bernstein polynomials $\{b_{3,6}(x), b_{4,6}(x), b_{5,6}(x), b_{6,6}(x)\}$ with respect to $v = \{x^3, x^4, x^5, x^6\}$ is

$$\{(20, -60, 60, -20), (0, 15, -30, 15), (0, 0, 6, -6),\\ (0, 0, 0, 1)\}.$$

The set of coordinate vectors of the basis t of Zernike radial polynomials $\{r_5^3(x), r_6^4(x), r_5^5(x), r_6^6(x)\}$ with respect to $\{x^3, x^4, x^5, x^6\}$ is

$$\{(-4, 0, 5, 0), (0, -5, 0, 6), (0, 0, 1, 0), (0, 0, 0, 1\}.$$

These are ascending bases and their change of basis matrices M_{vs} and M_{vt} are lower triangular matrices.

The change of basis matrix $M_{vt}^{-1}M_{vs} = M_{ts}$ is

$$\begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 5 & 0 & 1 & 0 \\ 0 & 6 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 20 & 0 & 0 & 0 \\ -60 & 15 & 0 & 0 \\ 60 & -30 & 6 & 0 \\ -20 & 15 & -6 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} -5 & 0 & 0 & 0 \\ 12 & -3 & 0 & 0 \\ 85 & -30 & 6 & 0 \\ -92 & 33 & -6 & 1 \end{bmatrix}$$

From the first column, we have

$$b_{3,6}(x) = -92r_6^6(x) + 85r_5^5(x) + 12r_6^4(x) - 5r_5^3(x).$$

Theorem 3. *Let V be a vector space with a set S of m bases of V that are either all ascending bases or all descending bases with respect to a basis v of V. It is not necessary that $v \in S$. There is a unique change of basis groupoid LTr or UTr that is a connected groupoid of order m^2, whose objects are S and whose morphisms all correspond to lower triangular matrices, or to upper triangular matrices, respectively.*

Proof. It follows from Theorem 1 that LTr or UTr is a connected groupoid of order m^2. The morphisms of LTr or UTr all correspond to lower or upper triangular matrices depending on whether S is a set of ascending or descending bases with respect to v, respectively, from Theorem 2.

Matrix multiplication and inversion preserve lower or upper triangularity of matrices from Lemma 3. □

The product of a lower triangular matrix with an upper triangular matrix is not triangular in general. As an example, the change of basis matrix from monomials to Bernstein polynomials is a lower triangular matrix, and that from shifted Legendre polynomials to monomials is an upper triangular but their product is not triangular. When $n = 5$, their product is the change of basis matrix from shifted Legendre polynomials to Bernstein polynomials, e.g., [14]:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{4} & 0 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{6} & 0 & 0 \\ 1 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 0 & 2 & -6 & 12 & -20 \\ 0 & 0 & 6 & -30 & 90 \\ 0 & 0 & 0 & 20 & -140 \\ 0 & 0 & 0 & 0 & 70 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 1 & -\frac{1}{2} & -\frac{1}{2} & 2 & -4 \\ 1 & 0 & -1 & 0 & 6 \\ 1 & \frac{1}{2} & -\frac{1}{2} & -2 & -4 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

B. Alternating Subgroupoids

These subgroupoids occur, for example, when the vector space is $\mathbb{R}[X]$ and the bases are polynomials that have definite parity. An example from §1, is the change of basis matrix from

Chebyshev polynomials T up to $T_3(x)$ to the monomials M which is the upper triangular alternating matrix

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Definition 7. *A $n \times n$ matrix M is an alternating matrix if for every element $m_{i,j}$ of M, if i and j have unequal parities then $m_{i,j} = 0$ where $0 \le i, j \le n-1$.*

If the morphisms of a change of basis groupoid correspond to alternating matrices, then the groupoid operation preserves this property.

Lemma 4. *Let Alt be a change of basis groupoid. If M_{ut} and M_{ts} are alternating matrices that correspond to morphisms of Alt, then M_{us} is an alternating matrix and corresponds to the morphism $s \to u$ of Alt. The inverse of M_{ut} is an alternating matrix and corresponds to the morphism of $u \to t$ of Alt.*

Proof. This follows directly from Definition 7, and the observation that the dot product of the i^{th} row of M_{ut} with the j^{th} column of M_{ts} is 0 when i and j have unequal parities where $0 \le i, j \le n-1$. □

In a similar way to the proof of Theorem 2, we can show the following.

Theorem 4. *Let V be a vector space of finite dimension $n > 0$ with bases s and t that are defined with respect to a basis v of V.*

M_{st} is an alternating matrix if and only if s and t are alternating bases with respect to v.

Definition 8. *A change of basis groupoid is an alternating groupoid if and only if all its objects are alternating matrices.*

It follows immediately from Lemma 4 that an alternating groupoid is well defined. Triangularity and alternation are independent properties of change of basis matrices.

An example of an upper triangular change of basis matrix without alternation occurs for the Laguerre polynomials. They form a descending basis and do not have definite parity. For example, the change of basis matrix from $\{L_0(x), L_1(x), L_2(x), L_3(x), L_4(x)\}$ to $\{1, x, x^2, x^3, x^4\}$:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -3 & -4 \\ 0 & 0 & \frac{1}{2} & \frac{3}{2} & 3 \\ 0 & 0 & 0 & -\frac{1}{6} & -\frac{2}{3} \\ 0 & 0 & 0 & 0 & \frac{1}{24} \end{bmatrix}$$

We could use Bernstein polynomials to give a similar example of a non-alternating matrix. It would be a lower triangular matrix because these polynomials form an ascending basis with respect to the monomials.

V. Summary with Categories

Change of basis in finite vector spaces has significant and widespread applications. In this context, we explore the algebraic structure of change of basis.

Theorem 1 states that in general, the change of basis matrices of a set of m bases of a finite vector space forms a connected groupoid of order m^2. Its objects are bases of a vector space and its morphisms correspond to change of basis matrices that represent typed linear transformations between the bases. The groupoid function corresponds to composition of changes of basis found by matrix multiplication of change of basis matrices.

Lemma 5. *Give a category CB, it has full subcategories*

- LTr *where* obj LTr *are ascending bases and* hom LTr *correspond to lower triangular matrices*
- UTr *where* obj UTr *are descending bases and* hom UTr *correspond to upper triangular matrices*
- Alt *where* obj Alt *are alternating bases and* hom Alt *correspond to alternating matrices*
- LAlt *where* obj LAlt *are ascending alternating bases and* hom LAlt *correspond to lower triangular alternating matrices*
- UAlt *where* obj UAlt *are descending alternating bases and* hom UAlt *correspond to upper triangular alternating matrices*

The subcategory relations are given in the following commutative diagram.

$$\begin{array}{ccccc} \mathrm{LTr} & \longleftarrow & \mathrm{CB} & \longrightarrow & \mathrm{UTr} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{LAlt} & \longleftarrow & \mathrm{Alt} & \longrightarrow & \mathrm{UAlt} \end{array}$$

Fig. 1. Triangularity and alternation commute.

Proof. The objects of the categories are supersets of the objects of their subcategories. Change of basis matrices between ascending bases and descending bases have lower and upper triangular change of basis matrices from Theorem 2. Morphism inversion and composition preserve upper and lower triangularity from Theorem 3. They preserve alternation from Lemma 4.

Let G be any category and H be a subcategory of G in Figure 1. The category H is a full subcategory of G because we can verify that obj G $\supseteq$ obj H, every morphism in hom H is a morphism in hom G, and every identity in H occurs in G. □

Lemma 5 shows that there exist subgroupoids of the change of basis groupoid, that stem from the properties of triangularity and alternation. These properties occur in the change of basis matrices that correspond to the morphisms of the groupoid, and the groupoid function preserves them. Triangularity and alternation occur independently and Figure 1 is commutative.

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