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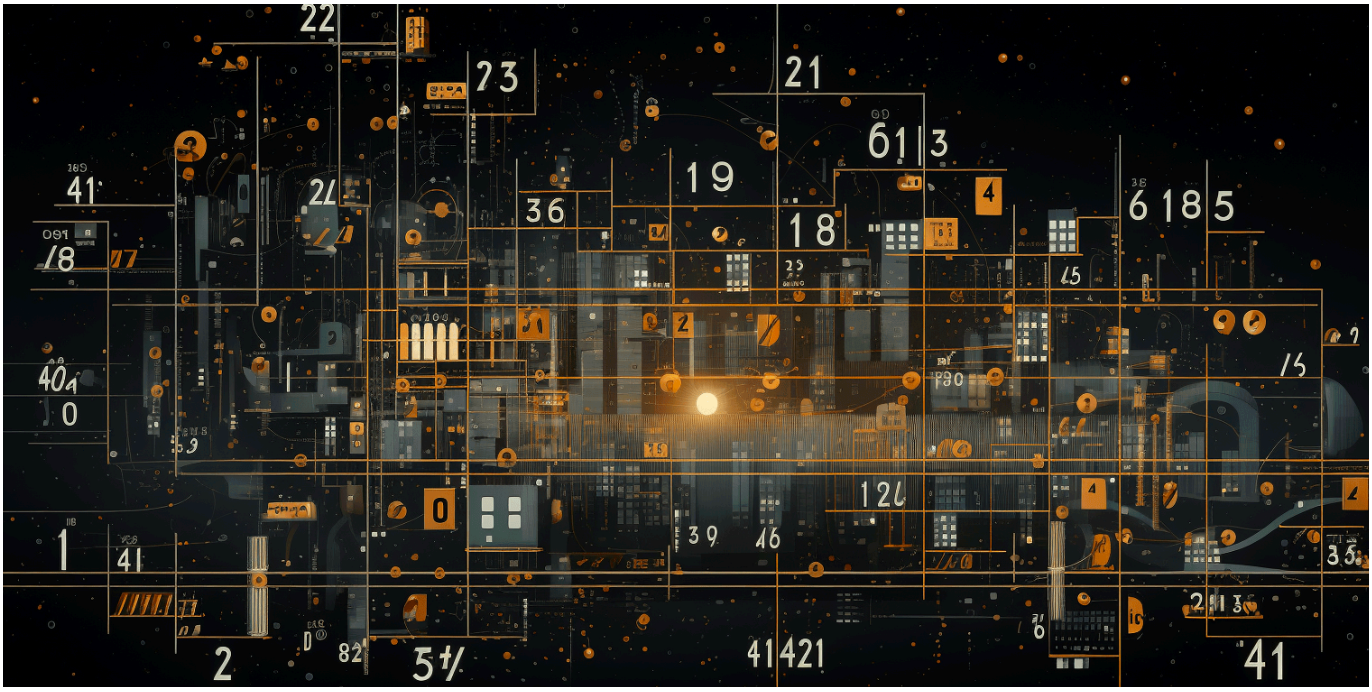
A Simple Argument Proves the Riemann Hypothesis

Dmitri Martila



Preprint v2

July 18, 2023

https://doi.org/10.32388/ZTG0NQ.2

A SIMPLE ARGUMENT PROVES THE RIEMANN HYPOTHESIS.

DMITRI MARTILA
INDEPENDENT RESEARCHER
J. V. JANNSENI 6–7, PÄRNU 80032, ESTONIA

Abstract. I have written a new proof of Riemann Hypothesis.
MSC Class: 11M26, 11M06.

There is a vivid interest to Riemann Hypothesis, and there are no reasons to doubt the Riemann Hypothesis: [1, 2].

Guy Robin gives the following definition:

Definition.
A number y is called "colossally abundant" if, for some $\epsilon > 0$, one has

$$\frac{\sigma(z)}{z^{1+\epsilon}} \leq \frac{\sigma(y)}{y^{1+\epsilon}} \tag{1}$$

for all values of z [4]. $\sigma(z)$ denotes the sum-of-divisors function [5]. For example, if z is a prime number, then $\sigma(z) = 1 + z$.

Grönwall's theorem in Ref. [3] is the following.

Theorem 1.
For the Grönwall function $G(n) = \sigma(n)/(n \log(\log n))$, one has

$$\lim\sup G(n \to \infty) = \exp(\gamma_E)\,, \tag{2}$$

where $\gamma_E = 0.577\ldots$ is the Euler–Mascheroni constant. The proof is found in Ref. [3]. I am using Eq. (2) in another shape, namely

$$G(n \to \infty) \leq \exp(\gamma_E)\,, \tag{3}$$

which reads $G(n) \leq X(n)$, where $X(n)$ is a function for any n with a single known property: $X(n) = \exp(\gamma_E)$ at $n \to \infty$. So, written in a short form (without the $X(n)$), I have Eq. (3).

Theorem 2.
There exist infinitely many colossally abundant numbers [6].

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eestidima@gmail.com.

Theorem 3.
The Riemann Hypothesis, if false, implies an infinitude of numbers n of the type $G(n) > \exp(\gamma_E)$ [4], page 188.

1. Proof of the Riemann Hypothesis

In this part of the proof, I am demonstrating that for any colossally abundant numbers A and B, holds

$$G(n) \leq \max(G(A), G(B))\,, \tag{4}$$

where n is any number from $6 \leq A \leq n \leq B$.

Dr. Robin has claimed [4] that A and B have to be consecutive in addition to $A < B$, to get

$$\frac{\sigma(n)}{n^{1+d}} \leq \frac{\sigma(A)}{A^{1+d}} = \frac{\sigma(B)}{B^{1+d}} \tag{5}$$

for some $d > 0$. But I am not seeing any proof of Eq. (5) in his paper. After this formula, the proof of Dr. Robin's Proposition 1 continues on page 192 without references to consecutivity, and the final result is in Eq. (4). But let me derive the formula (5) without usage of consecutivity.

$$\frac{\sigma(A)}{A^{1+b}} \geq \frac{\sigma(B)}{B^{1+b}} \tag{6}$$

for some $b > 0$ because A is colossally abundant number. On the other hand,

$$\frac{\sigma(B)}{B^{1+d}} \geq \frac{\sigma(A)}{A^{1+d}} \tag{7}$$

for some $d > 0$ because B is colossally abundant number.

Then

$$\frac{\sigma(A)}{A} \geq \frac{\sigma(B)}{B}\,(A/B)^b\,, \tag{8}$$

$$\frac{\sigma(A)}{A} \leq \frac{\sigma(B)}{B}\,(A/B)^d\,. \tag{9}$$

Holds $A < B$, then $A/B < 1$; so, the b and d can be arbitrary numbers within the ranges $b_0 \leq b < \infty$, and $0 \leq d < d_0$. Here b_0 and d_0 are satisfying

$$\frac{\sigma(A)}{A} = \frac{\sigma(B)}{B}\,(A/B)^{b_0}\,, \tag{10}$$

$$\frac{\sigma(A)}{A} = \frac{\sigma(B)}{B}\,(A/B)^{d_0}\,. \tag{11}$$

Latter two equations imply $b_0 = d_0$. Hence, $b = d$ situation will be exploit in the following. Therefore,

$$\frac{\sigma(A)}{A^{1+d}} = \frac{\sigma(B)}{B^{1+d}}\,. \tag{12}$$

Take a look at Eq. (5). The only chance for inequality to become violated is that n is a superabundant number. So, in the following part of the proof I assume that n is a superabundant number. Any colossally abundant number is superabundant. [7] Then from the definition of a superabundant number B,

$$\frac{\sigma(A)}{A} \le \frac{\sigma(n)}{n} \le \frac{\sigma(B)}{B}\,. \tag{13}$$

Holds

$$\frac{\sigma(A)}{A^{1+x}} = \frac{\sigma(n)}{n^{1+x}}\,, \tag{14}$$

$$\frac{\sigma(B)}{B^{1+y}} = \frac{\sigma(n)}{n^{1+y}}\,. \tag{15}$$

for some $x > 0$ and $y > 0$. Then, from Eqs. (12), (13), (14), and (15), $x \le d \le y$ has to hold for Eq. (5) to take place. Let me insert the $\sigma(n)/n$ from Eq. (14) into Eq. (15),

$$\frac{\sigma(A)}{A^{1+x}}\, n^{x-y} = \frac{\sigma(B)}{B^{1+y}}\,. \tag{16}$$

Let me insert the $\sigma(B)/B$ from Eq. (12) into Eq. (16), I get

$$n^{x-y}\, A^{d-x} = B^{d-y}\,. \tag{17}$$

This can be seen as a function $d = d(n)$, which can vary from $d = x$ up to $d = y$. In case $d = x$, Eq. (17) has $n = B$ as the solution; and in case $d = y$, Eq. (17) has $n = A$ as the solution. This coincided with the domain of n, which was $A \le n \le B$.

So, Eq. (5) is proven; and in the following, n is an arbitrary number again. It means that, it is not necessarily a superabundant number; and it is not necessarily a colossally abundant number.

Eq. (3) of Theorem 1 implies $G(B \to \infty) \le \exp(\gamma_E) \approx 1.78107$. In the following, due to Theorem 2, B will be seen as a very large colossally abundant number. And, in the following, $A = 55440$ is my chosen colossally abundant number [7]. It holds that $G(A) = 232128/(55440 \log(\log 55440)) \approx 1.75125 < \exp(\gamma_E)$. These values of Grönwall function in the Eq. (4) imply that one has $G(n) \le \exp(\gamma_E)$ for every value of n within $55440 \le n \le B$. Therefore, Eq. (4) implies that only a finite amount of numbers are of the type $G(n) > \exp(\gamma_E)$.

Notably, such numbers are showing $n < A$. Finally, Theorem 3 implies that Riemann Hypothesis cannot be false.

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