A Simple Argument Proves the Riemann Hypothesis

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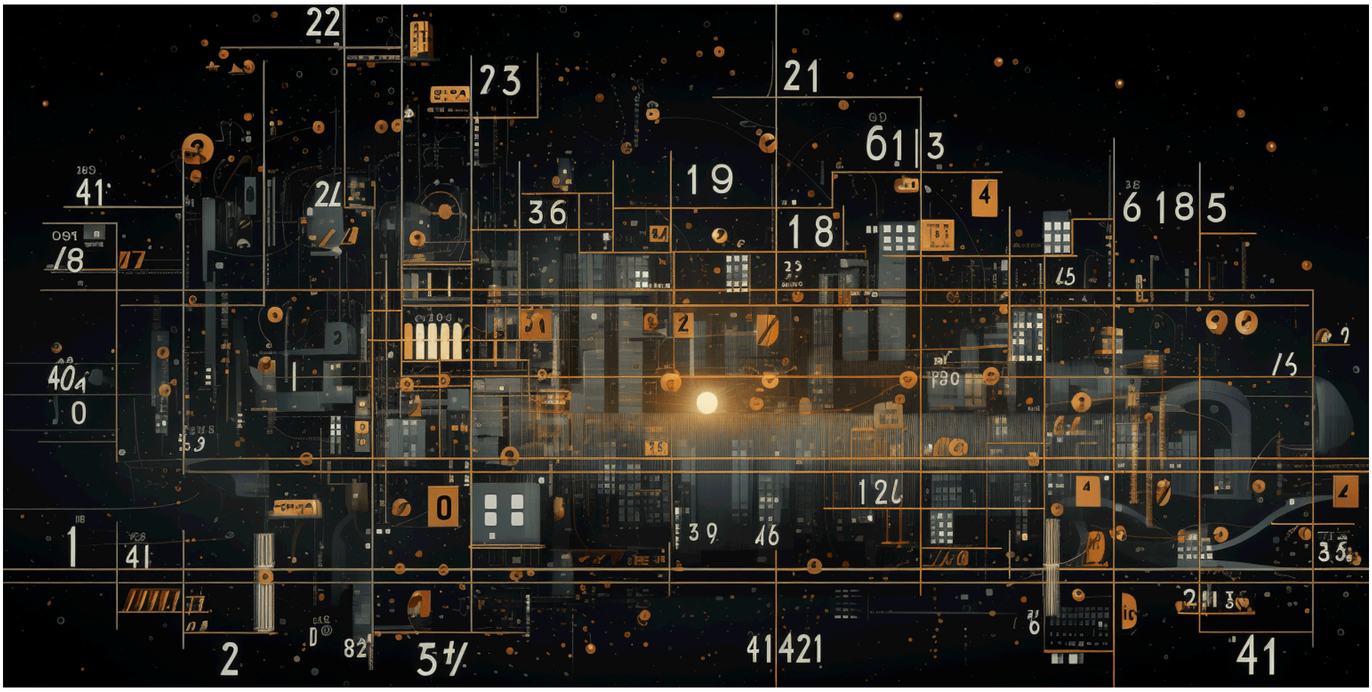

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A SIMPLE ARGUMENT PROVES THE RIEMANN HYPOTHESIS.

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Abstract. I have written a new proof of Riemann Hypothesis.
MSC Class: 11M26, 11M06.

There is a vivid interest to Riemann Hypothesis, and there are no reasons to doubt the Riemann Hypothesis: [1, 2].

Guy Robin gives the following definition:

Definition.
A number y is called "colossally abundant" if, for some $\epsilon > 0$, one has

$$\frac{\sigma(z)}{z^{1+\epsilon}} \leq \frac{\sigma(y)}{y^{1+\epsilon}} \tag{1}$$

for all values of z [4]. $\sigma(z)$ denotes the sum-of-divisors function [5]. For example, if z is a prime number, then $\sigma(z) = 1 + z$.

Grönwall's theorem in Ref. [3] is the following.

Theorem 1.
For the Grönwall function $G(n) = \sigma(n)/(n \log(\log n))$, one has

$$\limsup G(n \to \infty) = \exp(\gamma_E)\,, \tag{2}$$

where $\gamma_E = 0.577\ldots$ is the Euler–Mascheroni constant. The proof is found in Ref. [3]. I am using Eq. (2) in another shape, namely

$$G(n \to \infty) \leq \exp(\gamma_E)\,, \tag{3}$$

which reads $G(n) \leq X(n)$, where $X(n)$ is a function for any n with a single known property: $X(n) = \exp(\gamma_E)$ at $n \to \infty$. So, written in a short form (without the $X(n)$), I have Eq. (3).

Theorem 2.
There exist infinitely many colossally abundant numbers [6].

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Theorem 3.
The Riemann Hypothesis, if false, implies an infinitude of numbers n of the type $G(n) > \exp(\gamma_E)$ [4], page 188.

1. Proof of the Riemann Hypothesis

Eq. (3) of Theorem 1 implies $G(B \to \infty) \leq \exp(\gamma_E) \approx 1.78107$. In the following, due to Theorem 2, B will be seen as a very large colossally abundant number. And, in the following, $A = 55440$. It holds that $G(A) = 232128/(55440 \log(\log 55440)) \approx 1.75125 < \exp(\gamma_E)$.

Holds $3 < A < B$,

$$G(n) \leq \max(G(A), G(B))\,, \tag{4}$$

where $A \leq n \leq B$.

Dr. Robin has claimed that A and B have to be consecutive in addition to $A < B$, to get

$$\frac{\sigma(n)}{n^{1+d}} \leq \frac{\sigma(A)}{A^{1+d}} = \frac{\sigma(B)}{B^{1+d}} \tag{5}$$

for some $d > 0$. But I am not seeing any proof of Eq. (5) in his paper. After this formula, the proof of Dr. Robin's Proposition 1 continues on page 192 without references to consecutivity, and the final result is in Eq. (4). But let me derive the formula (5) without usage of consecutivity.

$$\frac{\sigma(A)}{A^{1+b}} \geq \frac{\sigma(B)}{B^{1+b}} \tag{6}$$

for some $b > 0$ because A is colossally abundant number. On the other hand,

$$\frac{\sigma(B)}{B^{1+d}} \geq \frac{\sigma(A)}{A^{1+d}} \tag{7}$$

for some $d > 0$ because B is colossally abundant number.

Then

$$\frac{\sigma(A)}{A} \geq \frac{\sigma(B)}{B}\,(A/B)^b\,, \tag{8}$$

$$\frac{\sigma(A)}{A} \leq \frac{\sigma(B)}{B}\,(A/B)^d\,. \tag{9}$$

Holds $A < B$, then $A/B < 1$; so, the b and d can be arbitrary numbers within the ranges $b_0 \leq b < \infty$, and $0 \leq d < d_0$. Here b_0 and

d_0 are satisfying

$$\frac{\sigma(A)}{A} = \frac{\sigma(B)}{B} (A/B)^{b_0} , \tag{10}$$

$$\frac{\sigma(A)}{A} = \frac{\sigma(B)}{B} (A/B)^{d_0} . \tag{11}$$

Latter two equations imply $b_0 = d_0$. Hence, $b = d$ situation will be exploit in the following.

Hence, I have either Eq. (5) holding, or

$$\frac{\sigma(n)}{n^{1+d}} \geq \frac{\sigma(A)}{A^{1+d}} = \frac{\sigma(B)}{B^{1+d}} . \tag{12}$$

In the latter case, Dr. Robin's page 192 would give

$$G(n) \geq \min(G(A), G(B)) \tag{13}$$

for every n from $A \leq n \leq B$. Recall that there are unlimitedly large prime numbers p with $G(p) = (1+p)/(p \operatorname{loglog} p) = 0$ in the limit $p \to \infty$. So, the formula (13), if true, implies existence of the limit $G(B) = 0$ at $B \to \infty$. But $G(B) > \sigma(B)/B^{1+x}$, where $x \neq 0$ is a finite number, and B is large. So, if $G(B) = 0$, then $\sigma(B)/B^{1+x} = 0$ in the limit $B \to \infty$. But the definition of colossally abundant number B says that $\sigma(B)/B^{1+x}$ cannot turn itself to zero:

$$\frac{\sigma(B)}{B^{1+x}} \geq \frac{\sigma(A)}{A^{1+x}} . \tag{14}$$

Therefore, Eq. (13) cannot be true for my selection of $A = 55440$ and $B \to \infty$. The true formula is in Eq. (4).

Values of Grönwall function $G(B \to \infty) \leq \exp(\gamma_E) \approx 1.78107$, $G(A) = 232128/(55440 \log(\log 55440)) \approx 1.75125 < \exp(\gamma_E)$ in the Eq. (4) imply that one has $G(n) \leq \exp(\gamma_E)$ for every value of n within $55440 \leq n \leq B$. Therefore, Eq. (4) implies that only a finite amount of numbers are of the type $G(n) > \exp(\gamma_E)$. Notably, such numbers are showing $n < A$. Finally, Theorem 3 implies that Riemann Hypothesis cannot be false.

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